Chapter 8 A simple model of the unpredictability of weather: The Lorenz Equations

1. Objectives

In this chapter, we will investigate the transition to chaos in the Lorenz equations – a system of non-linear ordinary differential equations. Using interactive examples, and analytical and numerical techniques, you will determine the stability of the solutions to the system, and discover a rich variability in their behavior. You will program Runge-Kuttan code for the problem, and determine the relative merits of each.

2. Readings

There is no required reading for this chapter, beyond the contents of this chapter itself. However if you would like additional background on any of the following topics, the refer to the sections indicated below.

Easy Reading:

- Gleick [Gleick(1987), pp. 9-31], an interesting overview of the science of chaos (with no mathematical details), and a look at its history.
- Palmer [Palmer(1993)] has a short article on Lorenz' work and concentrating on its consequences for weather prediction.

Mathematical Details:

- Sparrow [Sparrow(1982)], an in-depth treatment of the mathematics behind the Lorenz equations, including some discussion of numerical methods.

Related Sites on the Web: The following are just a few of the **many** sites on the Web that have information related to the Lorenz attractor, and chaos in general.

- http://www.ncsa.uiuc.edu/SCMS/DigLib/text/chaos/Chaos.html: A sequence of images for various values of the parameter r. It also includes an example of the behaviour of the "Duffing oscillator".
- http://www.mindspring.com/~pcoleman/pjchomem.html: A PC-based package called **STRANGE** that demonstrates dynamical systems concepts, including the Lorenz and Rössler attractors.
- http://www.interactive.net/~mizrach/SNDE/snde.html: The "Society for Nonlinear Dynamics and Econometrics," with lots of links to information sources at other sites.

3 Introduction

For many people working in the physical sciences, the *butterfly effect* is a well-known phrase. But even if you are unacquainted with the term, its consequences are something you are intimately familiar with. Edward Lorenz investigated the feasibility of performing accurate, long-term weather forecasts, and came to the conclusion that *even something as seemingly insignificant as the flap of a butterfly's wings can have an influence on the weather on the other side of the globe.* This implies that global climate modelers must take into account even the tiniest of variations in weather conditions in order to have even a hope of being accurate. Some of the models used today in weather forecasting have up to *a million unknown variables!*

With the advent of modern computers, many people believed that accurate predictions of systems as complicated as the global weather were possible. Lorenz' studies [Lorenz(1963)], both analytical and numerical, were concerned with simplified models for the flow of air in the atmosphere. He found that even for systems with considerably fewer variables than the weather, the long-term behaviour of solutions is intrinsically unpredictable. He found that this type of non-periodic, or *chaotic* behaviour, appears in systems that are described by non-linear differential equations.

The atmosphere is just one of many hydrodynamical systems, which exhibit a variety of solution behaviour: some flows are steady; others oscillate between two or more states; and still others vary in an irregular or haphazard manner. This last class of behaviour in a fluid is known as *turbulence*, or in more general systems as *chaos*. Examples of chaotic behaviour in physical systems include

- thermal convection in a tank of fluid, driven by a heated plate on the bottom, which displays an irregular patter of "convection rolls" for certain ranges of the temperature gradient;
- a rotating cylinder, filled with fluid, that exhibits regularly-spaced waves or irregular, nonperiodic flow patterns under different conditions;
- the Lorenzian water wheel, a mechanical system, described in Appendix A.1.

One of the simplest systems to exhibit chaotic behaviour is a system of three ordinary differential equations, studied by Lorenz, and which are now known as the *Lorenz equations* (see equations (4.1)). They are an idealization of a more complex hydrodynamical system of twelve equations describing turbulent flow in the atmosphere, but which are still able to capture many of the important aspects of the behaviour of atmospheric flows. The Lorenz equations determine the evolution of a system described by three time-dependent state variables, $x(t)$, $y(t)$ and $z(t)$. The state in Lorenz' idealized climate at any time, t, can be given by a single point, (x, y, z), in *phase space*. As time varies, this point moves around in the phase space, and traces out a curve, which is also called an *orbit* or *trajectory*. The plot in Figure 1 illustrates a sample orbit in phase space (with initial value $(5, 5, 5)$). Notice that the orbit appears to be lying in a surface composed of two "wings". In fact, for the parameter values used here, all orbits, no matter the initial conditions, are eventually attracted to this surface; such a surface is called an *attractor*, and this specific one is termed the *butterfly attractor* ... a very fitting name, both for its appearance, and for the fact that it is a visualization of solutions that exhibit the "butterfly effect." The individual variables are plotted versus time in Figure 2.

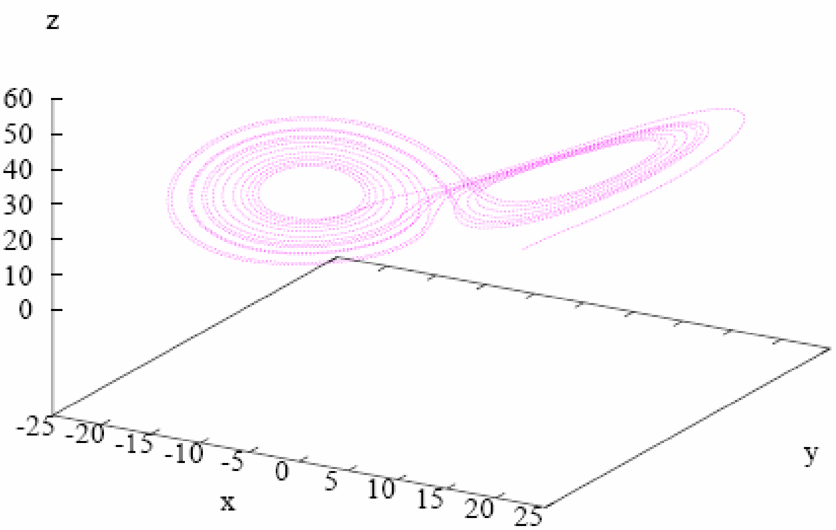

Figure 1: A plot of the solution to the Lorenz equations as an orbit in phase space. Parameters: $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$; initial values: $(x, y, z) = (5, 5, 5)$.

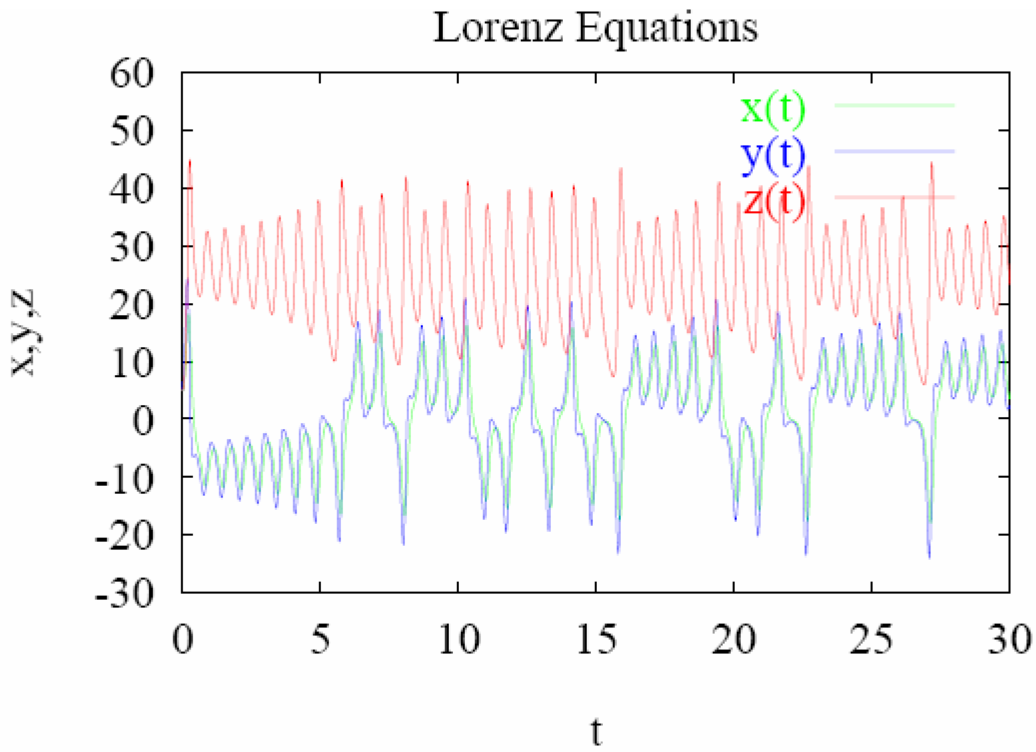

Figure 2: A plot of the solution to the Lorenz equations versus time. Parameters: $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$; initial values: $(x, y, z) = (5, 5, 5)$.

The solution of the Lorenz equations have several very important characteristics**:**

1. The solution remains within a bounded region (that is, none of the values of the solution "blow up"), which means that the solution will always be physically reasonable.

2. The solution flips back and forth between the two wings of the butterfly diagram, with no apparent pattern. This "strange" way that the solution is attracted towards the wings gives rise to the name *strange attractor*.

3. The resulting solution depends very heavily on the given initial conditions. Even a very tiny change in one of the initial values can lead to a solution which follows a totally different trajectory, if the system is integrated over a long enough time interval.

4. The solution is irregular or *chaotic*, meaning that it is impossible, based on parameter values and initial conditions (which may contain small measurement errors), to predict the solution at any future time.

4 The Lorenz Equations

As mentioned in the previous section, the equations we will be considering in this lab model an idealized hydrodynamical system: two-dimensional convection in a tank of water which is heated at the bottom (as pictured in Figure 3).

Lorenz wrote the equations in the form

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x)\\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= xy - bz\end{aligned} \tag{4.1}$$

where σ, r and b are real, positive parameters. The variables in the problem can be interpreted as follows:

- x is proportional to the intensity of the convective motion (positive for clockwise motion, and a larger magnitude indicating more vigorous circulation),
- y is proportional to the temperature difference between the ascending and descending currents (it's positive if the warm water is on the bottom),
- z is proportional to the distortion of the vertical temperature profile from linearity (a value of 0 corresponds to a linear gradient in temperature, while a positive value indicates that the temperature is more uniformly mixed in the middle of the tank and the strongest gradients occur near the boundaries),

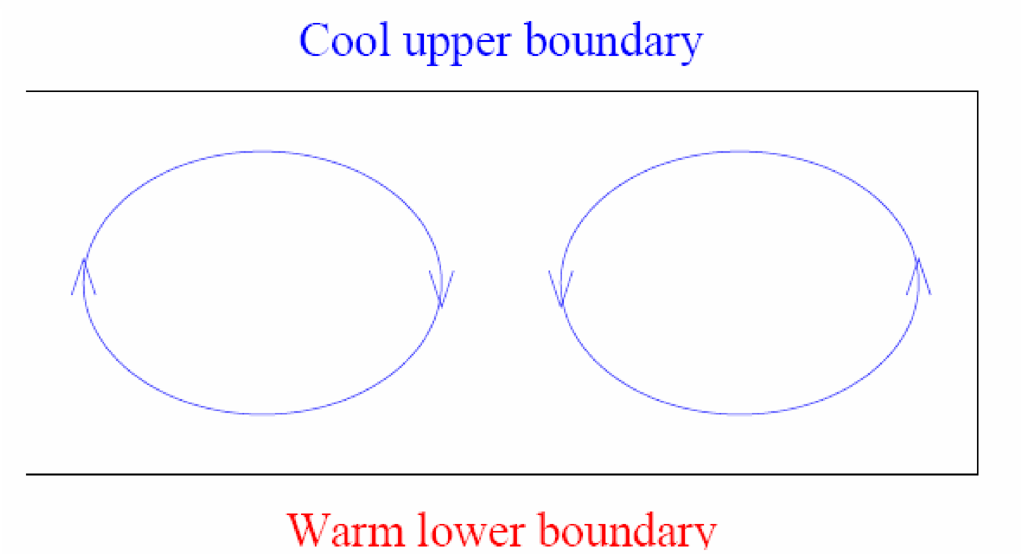

Figure 3: Lorenz studied the flow of fluid in a tank heated at the bottom, which results in "convection rolls", where the warm fluid rises, and the cold fluid is drops to the bottom.

t is the dimensionless time,

σ is called the Prandtl number (it involves the viscosity and thermal conductivity of the fluid),

r is a control parameter, representing the temperature difference between the top and bottom of the tank, and

b measures the width-to-height ratio of the convection layer.

Notice that these equations are *non-linear* in x, y and z, which is a result of the non-linearity of the fluid flow equations from which this simplified system is obtained.

> **Mathematical Note:** This system of equations is derived by Saltzman [Saltzman(1962)] for the thermal convection problem. However, the same equations (4.1) arise in other physical systems as well.

Remember from Section 3 that the Lorenz equations exhibit nonperiodic solutions which behave in a chaotic manner. Using analytical techniques, it is actually possible to make some qualitative predictions about the behaviour of the solution before doing any computations. However, before we move on to a discussion of the stability of the problem in Section 4.4, you should do the following exercise, which will give you a hands-on introduction to the behaviour of solutions to the Lorenz equations.

Problem 1: Lorenz' results are based on the following values of the physical parameters taken from Saltzman's paper [Saltzman(1962)]:

$$\sigma = 10 \quad \text{and} \quad b = \frac{8}{3}.$$

As you will see in Section 4.4, there is a *critical value of the parameter* r, $r^* = 470/19 \approx 24.74$ (for these values of σ and b); it is *critical* in the sense that for any value of $r > r^*$, the flow is unstable.

To allow you to investigate the behaviour of the solution to the Lorenz equations, you can try out various parameter values in the following interactive example. *Initially, leave σ and b alone, and modify only r and the initial conditions.* If you have time, you can try varying the other two parameters, and see what happens. Here are some suggestions:

- Fix the initial conditions at $(5, 5, 5)$ and vary r between 0 and 100.
- Fix $r = 28$, and vary the initial conditions; for example, try $(0, 0, 0)$, $(0.1, 0.1, 0.1)$, $(0, 0, 20)$, $(100, 100, 100)$, $(8.5, 8.5, 27)$, etc.
- Anything else you can think of ...

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- Anything else you can think of ...

1. Describe the different types of behaviour you see and compare them to what you saw in Figure 1. Also, discuss the results in terms of what you read in Section 3 regarding the four properties of the solution.
2. One question you should be sure to ask yourself is: *Does changing the initial condition affect where the solution ends up?* The answer to this question will indicate whether there really is an attractor which solutions approach as $t \to \infty$.
3. Finally, for the different types of solution behaviour, can you interpret the results physically in terms of the thermal convection problem?

Now, we're ready to find out why the solution behaves as it does. In Section 3, you were told about four properties of solutions to the Lorenz equations that are also exhibited by the atmosphere, and in the problem you just worked though, you saw that these were also exhibited by solutions to the Lorenz equations. In the remainder of this section, you will see mathematical reasons for two of those characteristics, namely the boundedness and stability (or instability) of solutions.

4.1 Boundedness of the Solution

The easiest way to see that the solution is bounded in time is by looking at the motion of the solution in phase space, (x, y, z), as the flow of a fluid, with velocity $(\dot{x}, \dot{y}, \dot{z})$ (the "dot" is used to represent a time derivative, in order to simplify notation in what follows). The *divergence of this flow* is given by

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z},$$

and measures how the volume of a fluid particle or parcel changes – a positive divergence means that the fluid volume is increasing locally, and a negative volume means that the fluid volume is shrinking locally (zero divergence signifies an incompressible fluid). If you look back to the Lorenz equations (4.1), and take partial derivatives, it is clear that the divergence of this flow is given by

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(\sigma + b + 1).$$

Since σ and b are both positive, real constants, the divergence is a negative number, which is always less than -1. Therefore, each small volume shrinks to zero as the time $t \to \infty$, at a rate which is independent of x, y and z. The consequence for the solution, (x, y, z), is that every trajectory in phase space is eventually confined to a region of zero volume. As you saw in Problem 1, this region, or *attractor*, need not be a point – in fact, the two wings of the "butterfly diagram" are a surface with zero volume.

The most important consequence of the solution being bounded is that none of the physical variables, x, y, or z "blows up." Consequently, we can expect that the solution will remain with physically reasonable limits.

4.2 Steady States

A *steady state* of a system is a point in phase space from which the system will not change in time, once that state has been reached. In other words, it is a point, (x, y, z), such that the solution does not change, or where

$$\frac{dx}{dt} = 0 \quad \text{and} \quad \frac{dy}{dt} = 0 \quad \text{and} \quad \frac{dz}{dt} = 0.$$

This point is usually referred to as a *stationary point* of the system.

Problem 2: Set the time derivatives equal to zero in the Lorenz equations (4.1), and solve the resulting system to show that there are three possible steady states, namely the points

$$(0, 0, 0),$$

$$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1), \text{ and}$$

$$(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1).$$

Remember that r is a positive real number, so that that there is *only one* stationary point when $0 \leq r \leq 1$, but all three stationary points are present when $r > 1$.

While working through Problem 1, did you notice the change in behaviour of the solution as r passes through the value 1? If not, then go back to the interactive example and try out some values of r both less than and greater than 1 to see how the solution changes.

A steady state tells us the behaviour of the solution only at a single point. *But what happens to the solution if it is perturbed slightly away from a stationary point? Will it return to the stationary point; or will it tend to move away from the point; or will it oscillate about the steady state; or something else ... ?* All of these questions are related to the long-term, *asymptotic* behaviour or *stability* of the solution near a given point. You already should have seen some examples of different asymptotic solution behaviour in the Lorenz equations for different parameter values. The next section describes a general method for determining the stability of a solution near a given stationary point.

4.3 Linearization about the Steady States

The difficult part of doing any theoretical analysis of the Lorenz equations is that they are *non-linear. So, why not approximate the non-linear problem by a linear one?*

This idea should remind you of what you need about Taylor series in chapter 2. There, we were approximating a function, $f(x)$, around a point by expanding the function in a Taylor series, and the first order Taylor approximation was simply a linear function in x. The approach we will take here is similar, but will get into Taylor series of functions of more than one variable: $f(x, y, z, \ldots)$.

The basic idea is to replace the right hand side functions in (4.1) with a linear approximation about a stationary point, and then solve the resulting system of *linear ODE's*. Hopefully, we can then say something about the non-linear system at values of the solution *close to the stationary point* (remember that the Taylor series is only accurate close to the point we're expanding about).

So, let us first consider the stationary point $(0, 0, 0)$. If we linearize a function $f(x, y, z)$ about $(0, 0, 0)$ we obtain the approximation:

$$f(x,y,z) \approx f(0,0,0) + f_x(0,0,0)\cdot(x-0) + f_y(0,0,0)\cdot(y-0) + f_z(0,0,0)\cdot(z-0).$$

If we apply this formula to the right hand side function for each of the ODE's in (4.1), then we obtain the following linearized system about $(0, 0, 0)$:

$$\begin{aligned} \frac{dx}{dt} &= -\sigma x + \sigma y \\ \frac{dy}{dt} &= rx - y \\ \frac{dz}{dt} &= -bz. \end{aligned} \tag{4.2}$$

(note that each right hand side is now a linear function of x, y and z). It is helpful to write this system in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4.3}$$

the reason for this being that the *eigenvalues* of the matrix give us valuable information about the solution to the linear system. In fact, it is a well-known result from the study of dynamical systems is that if the matrix in (4.3) has *distinct* eigenvalues λ_1, λ_2 and λ_3, then the solution to this equation is given by

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + c_3 e^{\lambda_3 t}, \tag{4.4}$$

and similarly for the other two solution components, $y(t)$ and $z(t)$ (the c_i's are constants that are determined by the initial conditions of the problem). This should not seem too surprising, if you think that the solution to the scalar equation $dx/dt = \lambda x$ is $x(t) = e^{\lambda t}$.

Problem 3: Remember from Lab #3 that the eigenvalues of a matrix, A, are given by the roots of the characteristic equation, $det(A-\lambda I) = 0$. Determine the characteristic equation of the matrix in (4.3), and show that the eigenvalues of the linearized problem are

$$\lambda_1 = -b, \quad \text{and} \quad \lambda_2, \lambda_3 = \frac{1}{2}\left(-\sigma - 1 \pm \sqrt{(\sigma-1)^2 + 4\sigma r}\right). \tag{4.5}$$

When $r > 1$, the same linearization process can be applied at the remaining two stationary points, which have eigenvalues that satisfy another characteristic equation:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r + \sigma)b\lambda + 2\sigma b(r-1) = 0. \tag{4.6}$$

4.4 Stability of the Linearized Problem

Now that we know the eigenvalues of the system around each stationary point, we can write down the solution to the linearized problem. However, it is not the exact form of the linearized solution that we're interested in, but rather its *stability*. In fact, the eigenvalues give us all the information we need to know about how the linearized solution behaves in time, and so we'll only talk about the eigenvalues from now on.

It is possible that two of the eigenvalues in (4.5) or in (4.6) can be complex numbers – *what does this mean for the solution in (4.4)?* The details are a bit involved, but the important thing to realize is that if $\lambda_2, \lambda_3 = a \pm ib$ are complex (remember that complex roots always occur in conjugate pairs) then the solutions can be rearranged so that they are of the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{at}\cos(bt) + c_3 e^{at}\sin(bt). \tag{4.7}$$

In terms of the asymptotic stability of the problem, we need to look at the asymptotic behaviour of the soution (4.4) or (4.7), as $t \to \infty$, from which several conclusions can be drawn:

1. If the eigenvalues are *real and negative*, then the solution will go to zero as $t \to \infty$. In this case the linearized solution is *stable*.

2. If the eigenvalues are real, and *at least one* is positive, then the solution will blow up as $t \to \infty$. In this case the linearized solution is *unstable*.

3. If there is a complex conjugate pair of eigenvalues, $a \pm ib$, then the solution exhibits oscillatory behaviour (with the appearance of the terms $\sin bt$ and $\cos bt$). If the real part, a, of all eigenvalues is negative, the oscillations will decay in time and the solution is *stable*; if the real part is positive, then the oscillations will grow, and the solution is *unstable*. If the complex eigenvalues have zero real part, then the oscillations will neither decay nor increase in time – the resulting linearized problem is periodic, and we say the solution is *marginally stable*.

Now, the million dollar question:

> *Does the stability of the non-linear system parallel that of the linearized systems near the stationary points?*

The answer is "almost always". We won't go into why, or why not, but just remember that you can usually expect the non-linear system to behave just as the linearized system near the stationary states.

The discussion of stability of the stationary points for the Lorenz equations will be divided up based on values of the parameter r (assuming $\sigma = 10$ and $b = \frac{8}{3}$). You've already seen that the behaviour of the solution changes significantly, by the appearance of two additional stationary points, when r passes through the value 1. You'll also see an explanation for the rest of the behaviour you observed:

$0 < r < 1$: there is only one stationary state, namely the point $(0, 0, 0)$. You can see from (4.5) that for these values of r, there are three, real, negative roots. The origin is a *stable* stationary point; that is, it attracts nearby solutions to itself.

$r > 1$: The origin has one positive, and two negative, real eigenvalues. Hence, the origin is *unstable*. Now, we need only look at the other two stationary points, whose behaviour is governed by the roots of (4.6) ...

$1 < r < \frac{470}{19}$: The other two stationary points have eigenvalues that have negative real parts. So these two points are *stable*.

It's also possible to show that two of these eigenvalues are real when $r < 1.346$, and they are complex otherwise (see Sparrow [Sparrow(1982)] for a more complete discussion). Therefore, the solution begins to exhibit oscillatory behaviour beyond a value of r greater than 1.346.

$r > \frac{470}{19}$: The other two stationary points have one real, negative eigenvalue, and two complex eigenvalues with positive real part. Therefore, these two points are *unstable*. In fact, all three stationary points are unstable for these values of r.

The stability of the stationary points is summarized in Table 1.

	$(0, 0, 0)$	$(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, b-1)$
$0 < r < 1$	stable	–
$1 < r < \frac{470}{19}$	unstable	stable
$r > \frac{470}{19}$	unstable	unstable

Table 1: Summary of the stability of the stationary points for the Lorenz equations; parameters $\sigma = 10$, $b = \frac{8}{3}$.

Note: This "critical value" of $r^* = \frac{470}{19}$ is actually found using the formula

$$r^* = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}.$$

See Sparrow [Sparrow(1982)] for more details.

Note: A qualitative change in behaviour of in the solution when a parameter is varied is called a *bifurcation*. Bifurcations occur at:

- $r = 1$, when the origin switches from stable to unstable, and two more stationary points appear.
- $r = r^*$, where the remaining two stationary points switch from being stable to unstable.

Remember that the linear results apply only near the stationary points, and do not apply to all of the phase space. Nevertheless, the behaviour of the orbits near these points can still say quite a lot about the behaviour of the solutions.

Problem 4: Based on the analytical results from this section, you can now go back to your results from Problem 1 and look at them in a new light. Write a short summary of your results (including a few plots or sketches), describing how the solution changes with r in terms of the existence and stability of the stationary points.

There have already been hints at problems with the linear stability analysis. One difficulty that hasn't been mentioned yet is that for values of $r > r^*$, the problem has oscillatory solutions, which are unstable. *Linear theory does not reveal what happens when these oscillations become large!* In order to study more closely the long-time behaviour of the solution, we must turn to numerical integration (in fact, all of the plots you produced in Problem 1 were generated using a numerical code).

5 Numerical Integration

In Lorenz' original paper, he discusses the application of the forward Euler and leap frog time-stepping schemes, but his actual computations are done using the

Since we already have a lot of experience with Runge-Kutta methods for systems of ODE's from earlier labs, you'll be using this approach to solve the Lorenz equations as well.

5.0 Adaptive Stepsize in Runge-Kutta

As a rule of thumb, accuracy increases in Runge-Kutta methods as stepsize decreases. At the same time, the number of function evaluations performed increases. This trade-off between accuracy of the solution and computational cost always exists, but in the ODE solution algorithms presented earlier it often appears to be unnecessarily large. To see this, consider the solution to a problem in two different time intervals. In the first one, the solution is close to steady, whereas in the second one it changes quickly. For acceptable accuracy with a non-adaptive method the step size will have to be adjusted so that the approximate solution is close to the actual solution in the second interval. The stepsize will be fairly small, so that the approximate solution is able to follow the changes in the solution here. However, as there is no change in stepsize throughout the solution process, the same step size will be applied to approximate the solution in the first time interval, where clearly a much larger stepsize would suffice to achieve the same accuracy. Thus, in a region where the solution behaves nicely a lot of function evaluations are wasted because the stepsize is chosen in accordance with the most quickly changing part of the solution. The way to address this problem is the use of adaptive stepsize control. This class of algorithms adjusts the stepsize taken in a time interval according to the properties of the solution in that interval, making it useful for producing a solution that has a given accuracy in the minimum number of steps.

1) Designing Adaptive Stepsize Control

Now that the goal is clear, the question remains of how to close in on it. As mentioned above, an adaptive algorithm is usually asked to solve a problem to a desired accuracy. To be able to adjust the stepsize in Runge-Kutta the algorithm must therefore calculate some estimate of how far its solution deviates from the actual solution. If with its initial stepsize this estimate is already well within the desired accuracy, the algorithm can proceed with a larger stepsize. If the error estimate is larger than the desired accuracy, the algorithm decreases the stepsize at this point and attempts to take a smaller step. Calculating this error estimate will always increase the amount of work done at a step compared to non-adaptive methods. Thus, the remaining problem is to devise a method of calculating this error estimate that is both inexpensive and accurate.

2) Error Estimate by Step Doubling

The first and simple approach to arriving at an error estimate is to simply take every step twice. The second time the step is divided up into two steps, producing a different estimate of the solution. The difference in the two solutions can be used to produce an estimate of the truncation error for this step. How expensive is this method to estimate the error? A single step of fourth order Runge-Kutta always takes four function evaluations. As the second time the step is taken in half-steps, it will take 8 evaluations. However, the first function evaluation in taking a step twice is identical to both steps, and thus the overall cost for one step with step doubling is 12 - 1 = 11 function evaluations. This should be compared to taking two normal half-steps as this corresponds to the overall accuracy achieved. So we are looking at 3 function evaluations more per step, or an increase of computational cost by a factor of 1:375.

Step doubling works in practice, but the next section presents a slicker way of arriving at an error estimate that is less computationally expensive. It is the commonly used one today.

3) Error Estimate using Embedded Runge-Kutta

Another way of estimating the truncation error of a step is due to the existence of the special fifth order Runge-Kutta methods discussed earlier. These methods use six function evaluations which can be recombined to produce a fourth-order method . Again, the difference between the fifth and the fourth order solution is used to calculate an estimate of the truncation error. Obviously this method requires less function evaluations than step doubling, as the two estimates use the same evaluation points. Originally this method was found by Fehlberg, and later Cash and Karp produced the set of constants presented earlier that produce an efficient and accurate error estimate.

4) Using Error to Adjust the Stepsize

Both step doubling and embedded methods leave us with the difference between two different order solutions to the same step. Provided is a desired accuracy, Δ_{des}. The way this accuracy is specified depends on the problem. It can be relative to the solution at step i,

$$\Delta_{des}(i) = RTOL \cdot |y(i)|$$

where $RTOL$ is the relative tolerance desired. An absolute part should be added to this so that the desired accuracy does not become zero. There are more ways to adjust the error specification to the problem, but the overall goal of the algorithm always is to make $\Delta_{est}(i)$, the estimated error for a step, satisfy

$$|\Delta_{est}(i)| \leq \Delta_{des}(i)|$$

Note also that for a system of ODEs Δ_{des} is of course a vector, and it is wise to replace the componentwise comparison by a vector norm.

Note now that the calculated error term is $O(h^5)$ as it was found as an error estimate to fourth-order Runge-Kutta methods. This makes it possible to scale the stepsize as

$$h_{new} = h_{old}[\frac{\Delta_{des}}{\Delta_{est}}]^{1/5} \tag{4.12}$$

or, to give an example of the suggested use of vector norms above, the new stepsize is given by

$$h_{new} = Sh_{old}\{[\frac{1}{N}\sum_{i=1}^{N}(\frac{\Delta_{est}(i)}{\Delta_{des}(i)})^2]^{1/2}\}^{-1/5}\} \tag{4.13}$$

using the root-mean-square norm. S appears as a safety factor $(0 < S < 1)$ to counteract the inaccuracy in the use of estimates.

Problem 5: In this problem, you will be investigated whether or not an adaptive Runge-Kutta code is the best choice for the Lorenz equations.

Write code to integrate the Lorenz equations, and use the code to compute in both adaptive and non-adaptive modes. Compare the number of time steps taken (plot the time step vs the integration time for both methods). Which method is more efficient?

To answer this last question, you will have to consider the cost of the adaptive scheme, compared to the non-adaptive one. The adaptive scheme is obviously more expensive, but by how much? You should think in terms of the number of multiplicative operations that are required in every time step for each method. You don't have to give an exact operation count, round figures will do.

Problem 6: One property of chaotic systems such as the Lorenz equations is their *sensitivity to initial conditions* – a consequence of the "butterfly effect." Modify your code from Problem 5 to compute two trajectories (in the chaotic regime $r > r^*$) with different initial conditions *simultaneously*. Use two initial conditions that are very close to each other, say $(1, 1, 20)$ and $(1, 1, 20.001)$. Use your "method of choice" (adaptive/non-adaptive), and plot the distance between the two trajectories as a function of time. What do you see?

One important limitation of numerical methods is immediately evident when approximating non-periodic dynamical systems such as the Lorenz equations: namely, *every computed solution is periodic*. That is, when we're working in floating point arithmetic, there are only finitely many numbers that can be represented, and the solution must eventually repeat itself. When using single precision arithmetic, a typical computer can represent many more floating point numbers than we could ever perform integration steps in a numerical scheme. However, it is still possible that round-off error might introduce a periodic orbit in the numerical solution where one does not really exist. In our computations, this will not be a factor, but it is something to keep in mind.

5.1 Other Chaotic Systems

There are many other ODE systems that exhibit chaos. An example is one studied by Rössler, which obeys a similar-looking system of three ODE's:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned} \tag{5.8}$$

Suppose that $b = 2$, $c = 4$, and consider the behaviour of the attractor as a is varied. When a is small, the attractor is a simple closed curve. As a is increased, however, this splits into a double loop, then a quadruple loop, and so on. Thus, a type of *period-doubling* takes place, and when a reaches about 0.375, there is a fractal attractor in the form of a band, that looks something like what is known in mathematical circles as a *Möbius strip*.

Note: If you're really keen on this topic, you might be interested in using your code to investigate the behaviour of this system of equations, *though you are not required to hand anything in for this!*

First, you could perform a stability analysis for (5.8), like you saw above for the Lorenz equations. Then, modify your code to study the Rössler attractor. Use the code to compare your analytical stability results to what you actually see in the computations.

6 Summary

In this chapter, you have had the chance to investigate the solutions to the Lorenz equations and their stability in quite some detail. You saw that for certain parameter values, the solution exhibits non-periodic, chaotic behavior. The question to ask ourselves now is: What does this system tell us about the dynamics of flows in the atmosphere? In fact, this system has been simplified so much that it is no longer an accurate model of the physics in the atmosphere. However, we have seen that the four characteristics of flows in the atmosphere (mentioned in Section 3) are also present in the Lorenz equations.

Each state in Lorenz' idealized "climate" is represented by a single point in phase space. For a given set of initial conditions, the evolution of a trajectory describes how the weather varies in time. The buttery attractor embodies all possible weather conditions that can be attained in the Lorenzian climate. By changing the value of the parameter r (and, for that matter, σ or b), the shape of the attractor changes. Physically, we can interpret this as a change in some global property of the weather system resulting in a modification of the possible weather states.

The same methods of analysis can be applied to more complicated models of the weather. One can imagine a model where the depletion of ozone and the increased concentration of greenhouse gases in the atmosphere might be represented by certain parameters. Changes in these parameters result in changes in the shape of the global climate attractor for the system. By studying the attractor, we could determine whether any new, and possibly devastating, weather states are present in this new ozone-deficient atmosphere.

We began by saying in the Introduction that the buttery effect made accurate long-term forecasting impossible. Nevertheless, it is still possible to derive meaningful qualitative information from such a chaotic dynamical system.

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