GENERALIZED NON-SYMMETRIC DIVERGENCE MEASURES AND INEQUALITIES

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Abstract. In this paper we shall consider one parametric generalization of some non-symmetric divergence measures. The *non-symmetric divergence measures* are such as: *relative information*, χ^2-*divergence*, *relative J-divergence*, *relative Jensen-Shannon divergence* and *relative arithmetic and geometric divergence.* All the generalizations considered can be written as particular cases of Csiszár *f-divergence.* By putting some conditions on the probability distributions, the aim here is to obtain relationships among the *relative divergence measures of type s.* The particular cases of these bounds lead us to very interesting cases.

1. Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \; n \geqslant 2,$$

be the set of all complete finite discrete probability distributions. There are many information and divergence measures existing in the literature on information theory and statistics. Some of them are symmetric with respect to probability distributions, while others are not. Here, in this paper, we shall work only with non-symmetric measures. Through out the paper it is under stood that the probability distributions $P, Q \in \Gamma_n$.

1.1. **Non-Symmetric Divergence Measures.** Here we shall give some non-symmetric measures of information. The most famous among them are χ^2-*divergence* and Kullback-Leibler *relative information.*

- χ^2-**Divergence** (Pearson [15])

$$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1 \tag{1.1}$$

and

$$\chi^2(Q||P) = \sum_{i=1}^{n} \frac{(q_i - p_i)^2}{p_i} = \sum_{i=1}^{n} \frac{q_i^2}{p_i} - 1. \tag{1.2}$$

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• **Relative Information** (Kullback and Leiber [12])

$$K(P||Q) = \sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}) \tag{1.3}$$

and

$$K(Q||P) = \sum_{i=1}^{n} q_i \ln(\frac{q_i}{p_i}). \tag{1.4}$$

• **Relative Jensen-Shannon Divergence** (Sibson [17], Sgarro [16])

$$F(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{2p_i}{p_i + q_i}\right) \tag{1.5}$$

and

$$F(Q||P) = \sum_{i=1}^{n} q_i \ln\left(\frac{2q_i}{q_i + p_i}\right). \tag{1.6}$$

• **Relative Arithmetic-Geometric Divergence** (Taneja [23])

$$G(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \ln\left(\frac{p_i + q_i}{2p_i}\right) \tag{1.7}$$

and

$$G(Q||P) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \ln\left(\frac{p_i + q_i}{2q_i}\right). \tag{1.8}$$

• **Relative J-Divergence** (Dragomir et al. [8])

$$D(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i + q_i}{2q_i}\right) \tag{1.9}$$

and

$$D(Q||P) = \sum_{i=1}^{n} (q_i - p_i) \ln\left(\frac{p_i + q_i}{2p_i}\right). \tag{1.10}$$

The above measures are written in pairs, and generally one is *adjoint* of another and *vice-versa*, for example, $K(Q||P)$ is the *adjoint* of $K(P||Q)$ and *vice-versa*. Here we understand by *non-symmetric measures*, when P is changed by Q and *vice-versa* we get the different expression, that is, for example, $K(P||Q) \neq K(Q||P)$. The same is true with the other measures. It is not so in case of symmetric measures. The symmetric versions of the above measures are given by

$$\Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P), \tag{1.11}$$

$$\begin{aligned} J(P||Q) &= K(P||Q) + K(Q||P) \\ &= D(P||Q) + D(Q||P), \end{aligned} \tag{1.12}$$

(1.13) $$I(P||Q) = \frac{1}{2}\left[F(P||Q) + F(Q||P)\right]$$

and

(1.14) $$T(P||Q) = \frac{1}{2}\left[G(P||Q) + G(Q||P)\right].$$

After simplification, we can write

(1.15) $$J(P||Q) = 4\left[I(P||Q) + T(Q||P)\right]$$

and

(1.16) $$D(Q||P) = \frac{1}{2}\left[F(P||Q) + G(P||Q)\right].$$

Dragomir et al. [9] studied the measures (1.11). We call it [22] by *symmetric chi-square divergence*. The measure (1.12) is well known Jeffreys-Kullback-Leiber [11, 12] *J-divergence*. The measure (1.13) is *Jensen-Shannon divergence* studied by Sibson [17] and Burbea and Rao [2, 3]). The measure (1.14) is *arithmetic and geometric mean divergence* studied by Taneja [19]. More details on some of these divergence measures can be seen in Taneja [18, 19] and in on line book by Taneja [21].

In this paper our aim is to work with one parametric generalizations of *non symmetric divergence measures* given by (1.1)-(1.10). Some of these generalizations are introduced for the first time in this paper. We shall call these generalizations by *non-symmetric divergence measures of type s*.

2. Non-Symmetric Divergence Measures of Type s

In this section we shall introduce one parametric generalization of the measures given by (1.1)-(1.10). The generalization of the measures (1.1)-(1.4) is already known in the literature and is studied by many authors. Here we shall call it by *relative information of type s* and is given by

- **Relative Information of Type s**

(2.1) $$\Phi_s(P||Q) = \begin{cases} K_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], & s \neq 0, 1 \\ K(Q||P) = \sum_{i=1}^{n} q_i \ln\left(\frac{q_i}{p_i}\right), & s = 0 \\ K(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right), & s = 1 \end{cases},$$

for all $s \in \mathbb{R}$.

The measure (2.1) admits the following particular cases:

(i) $\Phi_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P)$.
(ii) $\Phi_0(P||Q) = K(Q||P)$.
(iii) $\Phi_{1/2}(P||Q) = 4\left[1 - B(P||Q)\right] = 4h(P||Q)$.
(iv) $\Phi_1(P||Q) = K(P||Q)$.
(v) $\Phi_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.

The measures $B(P||Q)$ and $h(P||Q)$ appearing in part (iii) are given by

$$B(P||Q) = \sqrt{p_i q_i} \tag{2.2}$$

and

$$h(P||Q) = 1 - B(P||Q) = \frac{1}{2}\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2 \tag{2.3}$$

respectively.

The measure $B(P||Q)$ is famous as Bhattacharyya [1] *coefficient* and the measure $h(P||Q)$ is known as Hellinger [10] *discrimination.*

From the above expression (2.1), we observe that $\Phi_2(P||Q) = \Phi_{-1}(Q||P)$ and $\Phi_1(P||Q) = \Phi_0(Q||P)$.

Now we shall present new one parametric generalization for the measures given by (1.5)-(1.8). These generalizations unify the measures given by (1.5) and (1.7) and (1.6) and (1.8) respectively.

- **Unified Relative JS and AG – Divergence of Type s**

Let us consider the following unified one parametric generalization of the measures (1.5) and (1.7) simultaneously.

$$\Omega_s(P||Q) = \begin{cases} FG_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i \left(\frac{p_i+q_i}{2p_i}\right)^s - 1\right], & s \neq 0, 1 \\ F(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{2p_i}{p_i+q_i}\right), & s = 0 \\ G(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i+q_i}{2}\right) \ln\left(\frac{p_i+q_i}{2p_i}\right), & s = 1 \end{cases}. \tag{2.4}$$

The measure (2.4) admits the following particular cases:

(i) $\Omega_{-1}(P||Q) = \frac{1}{4}\Delta(P||Q)$.
(ii) $\Omega_0(P||Q) = F(P||Q)$.
(iii) $\Omega_{1/2}(P||Q) = 4\left[1 - B\left(P||\frac{P+Q}{2}\right)\right] = 4h\left(P||\frac{P+Q}{2}\right)$.
(iv) $\Omega_1(P||Q) = G(P||Q)$.
(v) $\Omega_2(P||Q) = \frac{1}{8}\chi^2(Q||P)$.

The expression $\Delta(P||Q)$ appearing in part (i) is the well known *triangular discrimination*, and is given by

$$\Delta(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}. \tag{2.5}$$

The adjoint of $\Omega_s(P||Q)$ is given by

$$\Omega_s(Q||P) = \begin{cases} FG_s(Q||P) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} q_i \left(\frac{p_i+q_i}{2q_i}\right)^s - 1\right], & s \neq 0,1 \\ F(Q||P) = \sum_{i=1}^{n} q_i \ln\left(\frac{2q_i}{p_i+q_i}\right), & s = 0 \\ G(Q||P) = \sum_{i=1}^{n} \left(\frac{p_i+q_i}{2}\right) \ln\left(\frac{p_i+q_i}{2q_i}\right), & s = 1 \end{cases}. \tag{2.6}$$

The measure (2.6) can also be obtained from (2.1) by replacing p_i by $\frac{p_i+q_i}{2}$. It admits the following particular cases:

(i) $\Omega_{-1}(Q||P) = \frac{1}{4}\Delta(P||Q)$.
(ii) $\Omega_0(Q||P) = F(Q||P)$.
(iii) $\Omega_{1/2}(Q||P) = 4\left[1 - B\left(\frac{P+Q}{2}||Q\right)\right]$.
(iv) $\Omega_1(Q||P) = G(Q||P)$.
(v) $\Omega_2(Q||P) = \frac{1}{8}\chi^2(P||Q)$.

• Relative J-Divergence of Type s

We shall propose one parametric generalization of the *relative J-divergence* measures given by (1.9) and (1.10). These generalizations are given by

$$\zeta_s(P||Q) = \begin{cases} D_s(P||Q) = (s-1)^{-1} \sum_{i=1}^{n} (p_i - q_i)\left(\frac{p_i+q_i}{2q_i}\right)^{s-1}, & s \neq 1 \\ D(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i+q_i}{2q_i}\right), & s = 1 \end{cases} \tag{2.7}$$

The adjoint of $\zeta_s(P||Q)$ is given by

$$\zeta_s(Q||P) = \begin{cases} D_s(Q||P) = (s-1)^{-1} \sum_{i=1}^{n} (q_i - p_i)\left(\frac{p_i+q_i}{2p_i}\right)^{s-1}, & s \neq 1 \\ D(Q||P) = \sum_{i=1}^{n} (q_i - p_i) \ln\left(\frac{p_i+q_i}{2p_i}\right), & s = 1 \end{cases} \tag{2.8}$$

for all $s \in \mathbb{R}$.

The measures (2.7) and (2.8) admit the following particular cases:

(i) $\zeta_0(P||Q) = \zeta_0(Q||P) = \Delta(P||Q)$.
(ii) $\zeta_1(P||Q) = D(P||Q)$.
(iii) $\zeta_1(Q||P) = D(Q||P)$.
(iv) $\zeta_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.
(v) $\zeta_2(Q||P) = \frac{1}{2}\chi^2(Q||P)$.

Thus we observe that the measure *relative information of type s*, $\Phi_s(P||Q)$ contains in particular the classical measures such as: Bhattacharyya *coefficient*, χ^2-*divergence* and Hellingar *discrimination*. The *unified relative JS and AG – divergences of type s*, $\Omega_s(P||Q)$ and $\Omega_s(Q||P)$ contains in particular the *triangular discrimination* and χ^2-*divergence*, while the *relative J-divergences of type s*, $\zeta_s(P||Q)$ and $\zeta_s(Q||P)$ yield in particular the *triangular discrimination* and χ^2-*divergence*.

The measure (2.1) can be seen in various papers. The measures (2.4) and (2.5) are new and are introduced for the first time here. The measure (2.7) can be seen in the work of Dragomir et al. The measure (2.8) is the *adjoint* of (2.7) and is written for the first time here. Some interesting properties of the *generalized relative divergence measures* can be seen in the recent work by authors [13, 14].

In this paper our aim is to relate these generalized measures of type s with each other. In order to do so, we shall make use of the Csiszár *f-divergence* and its property recently studied by Taneja [24].

3. Csiszár $f-$Divergence and its Particular Cases

Given a function $f : [0, \infty) \to \mathbb{R}$, the *f-divergence* measure introduced by Csiszár [4] is given by

$$C_f(P||Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{3.1}$$

for all $P, Q \in \Gamma_n$.

It is well known in the literature [4, 5] that *if the function f is convex and normalized, i.e., $f(1) = 0$, then the Csiszár $f-$divergence $C_f(P||Q)$ and its adjoint $C_f(Q||P)$ are nonnegative and convex in the pair of probability distribution $(P, Q) \in \Gamma_n \times \Gamma_n$.*

The generalized measures given in Section 2 can be written as particular cases of Csiszár *f-divergence* (3.1). These particular cases are given by the following examples.

Example 3.1. (*Relative information of type s*)**.** Let us consider

$$\phi_s(x) = \begin{cases} [s(s-1)]^{-1}\left[x^s - 1 - s(x-1)\right], & s \neq 0, 1 \\ x - 1 - \ln x, & s = 0 \\ 1 - x + x \ln x, & s = 1 \end{cases}, \tag{3.2}$$

for all $x > 0$ in (3.1). Then $C_f(P||Q) = \Phi_s(P||Q)$, where $\Phi_s(P||Q)$ is as given by (2.1).

Moreover,

$$\phi'_s(x) = \begin{cases} (s-1)^{-1}(x^{s-1} - 1), & s \neq 1 \\ \ln x, & s = 1 \end{cases}, \tag{3.3}$$

and

$$\phi''_s(x) = x^{s-2}. \tag{3.4}$$

Thus we have $\phi''_s(x) > 0$ for all $x > 0$, and hence, $\phi_s(x)$ is convex for all $x > 0$. Also, we have $\phi_s(1) = 0$. In view of this we can say that *relative information of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Example 3.2. (*Relative JS and AG - divergence of type s*)**.** Let us consider

$$\psi_s(x) = \begin{cases} [s(s-1)]^{-1}\left[x\left(\frac{x+1}{2x}\right)^s - x - s\left(\frac{1-x}{2}\right)\right], & s \neq 0, 1 \\ \frac{1-x}{2} - x \ln\left(\frac{x+1}{2x}\right), & s = 0 \\ \frac{x-1}{2} + \left(\frac{x+1}{2}\right) \ln\left(\frac{x+1}{2x}\right), & s = 1 \end{cases}, \tag{3.5}$$

for all $x > 0$ in (3.1). Then $C_f(P||Q) = \Omega_s(P||Q)$, where $\Omega_s(P||Q)$ is as given by (2.4).

Moreover,

$$\psi_s'(x) = \begin{cases} (s-1)^{-1}\left\{\frac{1}{s}\left[\left(\frac{x+1}{2x}\right)^s - 1\right] + \frac{1}{2}\left[1 - \frac{1}{x}\left(\frac{x+1}{2x}\right)^{s-1}\right]\right\}, & s \neq 0,1 \\ \frac{1-x}{2(1+x)} - \ln\left(\frac{x+1}{2x}\right), & s = 0 \\ \frac{1}{2}\left[1 - x^{-1} + \ln\left(\frac{x+1}{2x}\right)\right], & s = 1 \end{cases} \tag{3.6}$$

and

$$\psi_s''(x) = \frac{1}{4x^3}\left(\frac{x+1}{2x}\right)^{s-2}. \tag{3.7}$$

Thus we have $\psi_s''(x) > 0$ for all $x > 0$, and hence, $\psi_s(x)$ is convex for all $x > 0$. Also, we have $\psi_s(1) = 0$. In view of this we can say that *relative JS and AG – divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Example 3.3. (*Adjoint of Relative JS and AG – divergence of type s*)**.** Let us consider

$$\upsilon_s(x) = \begin{cases} [s(s-1)]^{-1}\left[\left(\frac{x+1}{2}\right)^s - 1 - s\left(\frac{x-1}{2}\right)\right], & s \neq 0,1 \\ \frac{x-1}{2} + \ln\left(\frac{2}{x+1}\right), & s = 0 \\ \frac{1-x}{2} + \frac{x+1}{2}\ln\left(\frac{x+1}{2}\right), & s = 1 \end{cases}, \tag{3.8}$$

for all $x > 0$ in (3.1). Then $C_f(P||Q) = \Omega_s(Q||P)$, where $\Omega_s(Q||P)$ is as given by (2.5).

Moreover,

$$\upsilon_s'(x) = \frac{1}{2}\begin{cases} (s-1)^{-1}\left[\left(\frac{x+1}{2}\right)^{s-1} - 1\right], & s \neq 1 \\ \ln\left(\frac{x+1}{2}\right), & s = 1 \end{cases}, \tag{3.9}$$

and

$$\upsilon_s''(x) = \frac{1}{4}\left(\frac{x+1}{2}\right)^{s-2}. \tag{3.10}$$

Thus we have $\upsilon_s''(x) > 0$ for all $x > 0$, and hence, $\upsilon_s(x)$ is convex for all $x > 0$. Also, we have $\upsilon_s(1) = 0$. In view of this we can say that *adjoint of relative JS and AG – divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Example 3.4. (*Relative J-divergence of type s*). Let us consider

$$\xi_s(x) = \begin{cases} (s-1)^{-1}(x-1)\left[\left(\frac{x+1}{2}\right)^{s-1} - 1\right], & s \neq 1 \\ (x-1)\ln\left(\frac{x+1}{2}\right), & s = 1 \end{cases}, \tag{3.11}$$

for all $x > 0$ in (3.1). Then $C_f(P||Q) = \zeta_s(P||Q)$, where $\zeta_s(P||Q)$ is by (2.6).

Moreover,

$$\xi_s'(x) = \begin{cases} \frac{1}{2}(x-1)\left(\frac{x+1}{2}\right)^{s-2} + (s-1)^{-1}\left[\left(\frac{x+1}{2}\right)^{s-1} - 1\right], & s \neq 1 \\ \frac{x-1}{x+1} + \ln\left(\frac{x+1}{2}\right), & s = 1 \end{cases} \tag{3.12}$$

and

$$\xi_s''(x) = \left(\frac{x+1}{2}\right)^{s-3}\left[\frac{sx+(4-s)}{4}\right]. \tag{3.13}$$

Thus we have $\xi_s''(x) > 0$ for all $x > 0$ and $0 \leqslant s \leqslant 4$, and hence, $\xi_s(x)$ is convex for all $x > 0$ and $0 \leqslant s \leqslant 4$. Also, we have $\xi_s(1) = 0$. In view of this we can say that *relative J-divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$ for all and $0 \leqslant s \leqslant 4$.

Example 3.5. (*Adjoint of relative J-divergence of type s*). Let us consider

$$\varsigma_s(x) = \begin{cases} (s-1)^{-1}(1-x)\left[\left(\frac{x+1}{2x}\right)^{s-1} - 1\right], & s \neq 1 \\ (1-x)\ln\left(\frac{x+1}{2x}\right), & s = 1 \end{cases}, \tag{3.14}$$

for all $x > 0$ in (3.1). Then $C_f(P||Q) = \zeta_s(Q||P)$, where $\zeta_s(Q||P)$ is as given by (2.7).

Moreover,

$$\varsigma_s'(x) = \begin{cases} \frac{x-1}{x(x+1)}\left(\frac{x+1}{2x}\right)^{s-1} - (s-1)^{-1}\left[\left(\frac{x+1}{2x}\right)^{s-1} - 1\right], & s \neq 1 \\ \frac{x-1}{x(x+1)} - \ln\left(\frac{x+1}{2x}\right), & s = 1 \end{cases} \tag{3.15}$$

and

$$\varsigma_s''(x) = \left(\frac{x+1}{2x}\right)^{s-1}\left[\frac{(4-s)x+s}{x^2(x+1)^2}\right] \tag{3.16}$$

Thus we have $\varsigma_s''(x) > 0$ for all $x > 0$ and $0 \leqslant s \leqslant 4$, and hence, $\varsigma_s(x)$ is convex for all $x > 0$ and $0 \leqslant s \leqslant 4$. Also, we have $\varsigma_s(1) = 0$. In view of this we can say that *adjoint of relative J-divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$ for all $0 \leqslant s \leqslant 4$.

4. Inequalities Among Generalized Relative Divergences

In this section we shall relate the *relative divergence measures of type s* given by (2.1), (2.4), (2.6)-(2.8). This comparison is based on the following theorem recently developed by Taneja [24].

Theorem 4.1. *Let $f_1, f_2 : I \subset \mathbb{R}_+ \to \mathbb{R}$ two generating mapping are normalized, i.e., $f_1(1) = f_2(1) = 0$ and suppose the assumptions:*

(i) *f_1 and f_2 are twice differentiable on (r, R);*
(ii) *there exists the real constants m, M such that $0 \leqslant m < M$ and*

$$m \leqslant \frac{f_1''(x)}{f_2''(x)} \leqslant M, \;\; f_2''(x) > 0, \;\; \forall x \in (r, R) \tag{4.1}$$

then we have the inequalities:

$$m\, C_{f_2}(P||Q) \leqslant C_{f_1}(P||Q) \leqslant M\, C_{f_2}(P||Q). \tag{4.2}$$

We shall apply the above theorem to establish inequalities among the *relative divergences of type s*. These inequalities are given in the following propositions.

Proposition 4.1. *We have the following inequalities among the measures $\Omega_s(P||Q)$ and $\Phi_t(P||Q)$:*

$$\frac{1}{4r^{t+1}}\left(\frac{r+1}{2r}\right)^{s-2}\Phi_t(P||Q) \leqslant \Omega_s(P||Q) \leqslant \frac{1}{4R^{t+1}}\left(\frac{R+1}{2R}\right)^{s-2}\Phi_t(P||Q),\ s+t\leqslant 1,\ t\leqslant -1 \tag{4.3}$$

and

$$\frac{1}{4R^{t+1}}\left(\frac{R+1}{2R}\right)^{s-2}\Phi_t(P||Q) \leqslant \Omega_s(P||Q) \leqslant \frac{1}{4r^{t+1}}\left(\frac{r+1}{2r}\right)^{s-2}\Phi_t(P||Q),\ s+t\geqslant 1,\ t\geqslant -1. \tag{4.4}$$

Proof. Let us consider

$$g_{(\psi_s,\phi_t)}(x) = \frac{\psi_s''(x)}{\phi_t''(x)} = \frac{1}{4x^{t+1}}\left(\frac{x+1}{2x}\right)^{s-2},\ x\in(0,\infty) \tag{4.5}$$

where $\psi_s''(x)$ and $\varphi_t''(x)$ are as given by (3.7) and (3.4) respectively.

From (4.5) one has

$$g'_{(\psi_s,\phi_t)}(x) = -\left(\frac{x+1}{2x}\right)^{s-2}\frac{x(t+1)+t+s-1}{4x^{t+2}(x+1)} \begin{cases} \geqslant 0, & t\leqslant -1,\ s+t\leqslant 1 \\ \leqslant 0, & t\geqslant -1,\ s+t\geqslant 1 \end{cases}. \tag{4.6}$$

In view of (4.6) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\psi_s,\phi_t)}(x) = \begin{cases} \frac{1}{4r^{t+1}}\left(\frac{r+1}{2r}\right)^{s-2}, & s+t\leqslant 1,\ t\leqslant -1 \\ \frac{1}{4R^{t+1}}\left(\frac{R+1}{2R}\right)^{s-2}, & s+t\geqslant 1,\ t\geqslant -1 \end{cases} \tag{4.7}$$

and

$$M = \sup_{x\in[r,R]} g_{(\psi_s,\phi_t)}(x) = \begin{cases} \frac{1}{4R^{t+1}}\left(\frac{R+1}{2R}\right)^{s-2}, & s+t\leqslant 1,\ t\leqslant -1 \\ \frac{1}{4r^{t+1}}\left(\frac{r+1}{2r}\right)^{s-2}, & s+t\geqslant 1,\ t\geqslant -1 \end{cases}. \tag{4.8}$$

Now (4.7) and (4.8) together with (4.2) give the inequalities (4.3) and (4.4). □

Proposition 4.2. *We have the following inequalities among the measures $\Omega_s(Q||P)$ and $\Phi_t(P||Q)$:*

$$\frac{1}{4r^{t-2}}\left(\frac{r+1}{2}\right)^{s-2}\Phi_t(P||Q) \leqslant \Omega_s(Q||P) \leqslant \frac{1}{4R^{t-2}}\left(\frac{R+1}{2}\right)^{s-2}\Phi_t(P||Q),\ \ s\geqslant t,\, t\leqslant 2 \tag{4.9}$$

and

$$\frac{1}{4R^{t-2}}\left(\frac{R+1}{2}\right)^{s-2}\Phi_t(P||Q) \leqslant \Omega_s(Q||P) \leqslant \frac{1}{4r^{t-2}}\left(\frac{r+1}{2}\right)^{s-2}\Phi_t(P||Q),\ \ s\leqslant t,\, t\geqslant 2. \tag{4.10}$$

Proof. Let us consider

$$g_{(\upsilon_s,\phi_t)}(x) = \frac{\upsilon_s''(x)}{\phi_t''(x)} = \frac{1}{4x^{t-2}}\left(\frac{x+1}{2}\right)^{s-2},\ \ x\in(0,\infty) \tag{4.11}$$

where $\upsilon_s''(x)$ and $\varphi_s''(x)$ are as given by (3.10) and (3.4) respectively.

From (4.11) one has

$$g'_{(\psi_s,\phi_t)}(x) = \left(\frac{x+1}{2}\right)^{s-2}\frac{x(s-t)+2-t}{4x^{t-1}(x+1)}\begin{cases}\geqslant 0, & s\geqslant t,\, t\leqslant 2\\ \leqslant 0, & s\leqslant t,\, t\geqslant 2\end{cases}. \tag{4.12}$$

In view of (4.12) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\upsilon_s,\phi_t)}(x) = \begin{cases}\frac{1}{4r^{t-2}}\left(\frac{r+1}{2}\right)^{s-2}, & s\geqslant t,\, t\leqslant 2\\ \frac{1}{4R^{t-2}}\left(\frac{R+1}{2}\right)^{s-2}, & s\leqslant t,\, t\geqslant 2\end{cases} \tag{4.13}$$

and

$$M = \sup_{x\in[r,R]} g_{(\upsilon_s,\phi_t)}(x) = \begin{cases}\frac{1}{4R^{t-2}}\left(\frac{R+1}{2}\right)^{s-2}, & s\geqslant t,\, t\leqslant 2\\ \frac{1}{4r^{t-2}}\left(\frac{r+1}{2}\right)^{s-2}, & s\leqslant t,\, t\geqslant 2\end{cases} \tag{4.14}$$

Now (4.13) and (4.14) together with (4.2) give the inequalities (4.9) and (4.10). □

Proposition 4.3. *We have the following inequalities among the measures $\zeta_s(P||Q)$ and $\Phi_t(P||Q)$:*

$$\left(\frac{r+1}{2}\right)^{s-3}\left(\frac{sr+4-s}{4r^{t-2}}\right)\Phi_t(P||Q) \leqslant \zeta_s(P||Q) \leqslant \left(\frac{R+1}{2}\right)^{s-3}\left(\frac{sR+4-s}{4R^{t-2}}\right)\Phi_t(P||Q),$$
$$0\leqslant s\leqslant 4,\ t\leqslant 2,\ s\geqslant t+1 \tag{4.15}$$

and

$$\left(\frac{R+1}{2}\right)^{s-3}\left(\frac{sR+4-s}{4R^{t-2}}\right)\Phi_t(P||Q) \leqslant \zeta_s(P||Q) \leqslant \left(\frac{r+1}{2}\right)^{s-3}\left(\frac{sr+4-s}{4r^{t-2}}\right)\Phi_t(P||Q), \quad 0 \leqslant s \leqslant 4,\ t \geqslant 2,\ s \leqslant t+1. \tag{4.16}$$

Proof. Let us consider

$$g_{(\xi_s,\phi_t)}(x) = \frac{\xi_s''(x)}{\phi_t''(x)} = \left(\frac{x+1}{2}\right)^{s-3}\frac{sx+4-s}{4x^{t-2}},\ x \in (0,\infty) \tag{4.17}$$

where $\xi_s''(x)$ and $\phi_t''(x)$ are as given by (3.13) and (3.4) respectively.

From (4.17) one has

$$g'_{(\xi_s,\phi_t)}(x) = \frac{1}{4x^{t-1}(x+1)}\left(\frac{x+1}{2}\right)^{s-2}\left[s(s-t)x^2 + (-s^2+8s-4t-4)x + (2-t)(4-s)\right]. \tag{4.18}$$

Reorganizing (4.18), we can write

$$g'_{(\xi_s,\phi_t)}(x) = \frac{1}{4x^{t-1}(x+1)}\left(\frac{x+1}{2}\right)^{s-2}\left[s(s-t)x^2 + \left((4(s-t)-(s-2)^2\right)x + (2-t)(4-s)\right] \leqslant 0,\ 0 \leqslant s \leqslant 4,\ t \geqslant 2,\ s \leqslant t \tag{4.19}$$

Again reorganizing (4.18), we can write

$$g'_{(\xi_s,\phi_t)}(x) = \frac{1}{4x^{t-1}(x+1)}\left(\frac{x+1}{2}\right)^{s-2}\left[s(s-t)x^2 + \left(s(4-s)+4(s-t-1)\right)x + (2-t)(4-s)\right] \geqslant 0,\ 0 \leqslant s \leqslant 4,\ t \leqslant 2,\ s \geqslant t+1. \tag{4.20}$$

In view of (4.19) and (4.20) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\xi_s,\phi_t)}(x) = \begin{cases} \left(\frac{r+1}{2}\right)^{s-3}\frac{sr+4-s}{4r^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \leqslant 2,\ s \geqslant t+1 \\ \left(\frac{R+1}{2}\right)^{s-3}\frac{sR+4-s}{4R^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \geqslant 2,\ s \leqslant t+1 \end{cases} \tag{4.21}$$

and

$$M = \sup_{x\in[r,R]} g_{(\xi_s,\phi_t)}(x) = \begin{cases} \left(\frac{R+1}{2}\right)^{s-3}\frac{sR+4-s}{4R^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \geqslant 2,\ s \leqslant t+1 \\ \left(\frac{r+1}{2}\right)^{s-3}\frac{sr+4-s}{4r^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \leqslant 2,\ s \geqslant t+1 \end{cases} \tag{4.22}$$

Now (4.21) and (4.22) together with (4.2) give the inequalities (4.15) and (4.16). □

Proposition 4.4. *We have the following inequalities among the measures $\zeta_s(Q||P)$ and $\Phi_t(P||Q)$:*

$$\left(\frac{r+1}{2r}\right)^{s-3}\left(\frac{(4-s)r+s}{4r^{t+2}}\right)\Phi_t(P||Q) \leqslant \zeta_s(Q||P) \leqslant \left(\frac{R+1}{2R}\right)^{s-3}\left(\frac{(4-s)R+s}{4R^{t+2}}\right)\Phi_t(P||Q), \quad 0\leqslant s\leqslant 4,\ t\leqslant -1,\ s+t\leqslant 1 \tag{4.23}$$

and

$$\left(\frac{R+1}{2R}\right)^{s-3}\left(\frac{(4-s)R+s}{4R^{t+2}}\right)\Phi_t(P||Q) \leqslant \zeta_s(Q||P) \leqslant \left(\frac{r+1}{2r}\right)^{s-3}\left(\frac{(4-s)r+s}{4r^{t+2}}\right)\Phi_t(P||Q), \quad 0\leqslant s\leqslant 4,\ t\geqslant -1,\ s+t\geqslant 2. \tag{4.24}$$

Proof. Let us consider

$$g_{(\varsigma_s,\phi_t)}(x) = \frac{\varsigma_s''(x)}{\phi_t''(x)} = \left(\frac{x+1}{2x}\right)^{s-3}\frac{(4-s)x+s}{4x^{t+2}},\ x\in(0,\infty) \tag{4.25}$$

where $\varsigma_s''(x)$ and $\phi_t''(x)$ are as given by (3.16) and (3.4) respectively.

From (4.25) one has

$$g'_{(\varsigma_s,\phi_t)}(x) = \frac{1}{x^{t+1}(x+1)^3}\left(\frac{x+1}{2x}\right)^{s-1}\big[(s-4)(t+1)x^2 + (s^2-8s+8-4t)x + s(1-s-t)\big]. \tag{4.26}$$

Reorganizing (4.26) we can write

$$g'_{(\varsigma_s,\phi_t)}(x) = \frac{1}{x^{t+1}(x+1)^3}\left(\frac{x+1}{2x}\right)^{s-1}\big[(s-4)(t+1)x^2 + \left(s(s-4)+4(2-s-t)\right)x + s(1-s-t)\big] \leqslant 0,\ 0\leqslant s\leqslant 4,\ t\geqslant -1,\ s+t\geqslant 2. \tag{4.27}$$

Again, reorganizing (4.26) we can write

$$g'_{(\varsigma_s,\phi_t)}(x) = \frac{1}{x^{t+1}(x+1)^3}\left(\frac{x+1}{2x}\right)^{s-1}\big[(s-4)(t+1)x^2 + \big((s-2)^2+4(1-s-t)\big)x + s(1-s-t)\big] \geqslant 0,\ 0\leqslant s\leqslant 4,\ t\leqslant -1,\ s+t\leqslant 1. \tag{4.28}$$

In view of (4.27) and (4.28) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\varsigma_s,\phi_t)}(x) = \begin{cases} \left(\frac{r+1}{2r}\right)^{s-3} \frac{(4-s)r+s}{4r^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \leqslant -1,\ s+t \leqslant 1 \\ \left(\frac{R+1}{2R}\right)^{s-3} \frac{(4-s)R+s}{4R^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \geqslant -1,\ s+t \geqslant 2 \end{cases} \tag{4.29}$$

and

$$M = \sup_{x\in[r,R]} g_{(\varsigma_s,\phi_t)}(x) = \begin{cases} \left(\frac{R+1}{2R}\right)^{s-3} \frac{(4-s)R+s}{4R^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \leqslant -1,\ s+t \leqslant 1 \\ \left(\frac{r+1}{2r}\right)^{s-3} \frac{(4-s)r+s}{4r^{t-2}}, & 0 \leqslant s \leqslant 4,\ t \geqslant -1,\ s+t \geqslant 2 \end{cases} \tag{4.30}$$

Now (4.29) and (4.30) together with (4.2) give the inequalities (4.23) and (4.24). □

Proposition 4.5. *We have the following inequalities among the measures $\Omega_s(Q||P)$ and $\Omega_t(P||Q)$:*

$$r^{t+1}\left(\frac{r+1}{2}\right)^{s-t}\Omega_t(P||Q) \leqslant \Omega_s(Q||P) \leqslant R^{t+1}\left(\frac{R+1}{2}\right)^{s-t}\Omega_t(P||Q),\ s \geqslant -1,\ t \geqslant -1. \tag{4.31}$$

and

$$R^{t+1}\left(\frac{R+1}{2}\right)^{s-t}\Omega_t(P||Q) \leqslant \Omega_s(Q||P) \leqslant r^{t+1}\left(\frac{r+1}{2}\right)^{s-t}\Omega_t(P||Q),\ s \leqslant -1,\ t \leqslant -1. \tag{4.32}$$

Proof. Let us consider

$$g_{(\upsilon_s,\psi_t)}(x) = \frac{\upsilon_s''(x)}{\psi_t''(x)} = x^{t+1}\left(\frac{x+1}{2}\right)^{s-t},\ x \in (0,\infty) \tag{4.33}$$

where $\upsilon_s''(x)$ and $\psi_t''(x)$ are as given by (3.10) and (3.7) respectively.

From (4.33) one has

$$g'_{(\upsilon_s,\psi_t)}(x) = \left(\frac{x+1}{2}\right)^{s-t-1}\left(\frac{x^t\left(x(1+s)+(1+t)\right)}{2}\right) \begin{cases} \geqslant 0, & s \geqslant -1,\ t \geqslant -1 \\ \leqslant 0, & s \leqslant -1,\ t \leqslant -1 \end{cases} \tag{4.34}$$

In view of (4.34) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\upsilon_s,\psi_t)}(x) = \begin{cases} r^{t+1}\left(\frac{r+1}{2}\right)^{s-t}, & s \geqslant -1,\ t \geqslant -1 \\ R^{t+1}\left(\frac{R+1}{2}\right)^{s-t} & s \leqslant -1,\ t \leqslant -1 \end{cases} \tag{4.35}$$

and

$$M = \sup_{x\in[r,R]} g_{(\upsilon_s,\psi_t)}(x) = \begin{cases} R^{t+1}\left(\frac{R+1}{2}\right)^{s-t}, & s \geqslant -1,\ t \geqslant -1 \\ r^{t+1}\left(\frac{r+1}{2}\right)^{s-t} & s \leqslant -1,\ t \leqslant -1 \end{cases} \tag{4.36}$$

Now (4.35) and (4.36) together with (4.2) give the inequalities (4.31) and (4.32). □

Proposition 4.6. *We have the following inequalities among the measures $\zeta_s(P||Q)$ and $\Omega_t(P||Q)$:*

$$r^{t+1}\left(\frac{r+1}{2}\right)^{s-t-1}(sr+4-s)\,\Omega_t(P||Q) \leqslant \zeta_s(P||Q) \leqslant R^{t+1}\left(\frac{R+1}{2}\right)^{s-t-1}(sR+4-s)\,\Omega_t(P||Q),\ 0 \leqslant s \leqslant 4,\ t \geqslant -1. \tag{4.37}$$

Proof. Let us consider

$$g_{(\xi_s,\psi_t)}(x) = \frac{\xi_s''(x)}{\psi_t''(x)} = x^{t+1}\left(\frac{x+1}{2}\right)^{s-t-1}(sx+4-s),\ x \in (0,\infty) \tag{4.38}$$

where $\xi_s(x)$ and $\psi_t(x)$ are as given by (3.13) and (3.7) respectively.

From (4.38) one has

$$g'_{(\xi_s,\psi_t)}(x) = \frac{x^t}{x+1}\left(\frac{x+1}{2}\right)^{s-t-1}\left[s(s+1)x^2 + s(t-s+6)x + (t+1)(4-s)\right] \geqslant 0,\ 0 \leqslant s \leqslant 4,\ t \geqslant -1. \tag{4.39}$$

In view of (4.39) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\xi_s,\psi_t)}(x) = r^{t+1}\left(\frac{r+1}{2}\right)^{s-t-1}(sr+4-s),\ 0 \leqslant s \leqslant 4,\ t \geqslant -1 \tag{4.40}$$

and

$$M = \sup_{x\in[r,R]} g_{(\xi_s,\psi_t)}(x) = R^{t+1}\left(\frac{R+1}{2}\right)^{s-t-1}(sR+4-s),\ 0 \leqslant s \leqslant 4,\ t \geqslant -1 \tag{4.41}$$

Now (4.40) and (4.41) together with (4.2) give the inequalities (4.37). □

Proposition 4.7. *We have the following inequalities among the measures $\zeta_s(Q||P)$ and $\Omega_t(P||Q)$:*

$$\begin{aligned}(4.42)\qquad &\left(\frac{r+1}{2r}\right)^{s-t-1}\left(\frac{(4-s)r+s}{r}\right)\Omega_t(P||Q)\\ &\leqslant \zeta_s(Q||P) \leqslant \left(\frac{R+1}{2R}\right)^{s-t-1}\left(\frac{(4-s)R+s}{R}\right)\Omega_t(P||Q),\\ &\qquad 0\leqslant s\leqslant 4,\ t\geqslant s,\ t(4-s)\geqslant 6s-s^2-4\end{aligned}$$

and

$$\begin{aligned}(4.43)\qquad &\left(\frac{R+1}{2R}\right)^{s-t-1}\left(\frac{(4-s)R+s}{R}\right)\Omega_t(P||Q)\\ &\leqslant \zeta_s(Q||P) \leqslant \left(\frac{r+1}{2r}\right)^{s-t-1}\left(\frac{(4-s)r+s}{r}\right)\Omega_t(P||Q),\\ &\qquad 0\leqslant s\leqslant 4,\ t\leqslant s,\ t(4-s)\leqslant 6s-s^2-4\end{aligned}$$

Proof. Let us consider

$$(4.44)\qquad g_{(\varsigma_s,\psi_t)}(x)=\frac{\varsigma_s''(x)}{\psi_t''(x)}=\left(\frac{x+1}{2x}\right)^{s-t-1}\left(\frac{(4-s)x+s}{x}\right),\ x\in(0,\infty)$$

where $\varsigma_s(x)$ and $\psi_t(x)$ are as given by (3.16) and (3.7) respectively.

From (4.44) one has

$$\begin{aligned}(4.45)\qquad g'_{(\varsigma_s,\psi_t)}(x)=&\frac{4}{(x+1)^3}\left(\frac{x+1}{2x}\right)^{s-t+1}\left[(4+4t-st-6s+s^2)x+s(t-s)\right]\\ &\begin{cases}\geqslant 0, & 0\leqslant s\leqslant 4,\ t\geqslant s,\ t(4-s)\geqslant 6s-s^2-4\\ \leqslant 0, & 0\leqslant s\leqslant 4,\ t\leqslant s,\ t(4-s)\leqslant 6s-s^2-4\end{cases}.\end{aligned}$$

In view of (4.45) we conclude the followings

$$\begin{aligned}(4.46)\qquad m=&\inf_{x\in[r,R]} g_{(\varsigma_s,\psi_t)}(x)\\ =&\begin{cases}\left(\frac{r+1}{2r}\right)^{s-t-1}\left(\frac{(4-s)r+s}{r}\right), & 0\leqslant s\leqslant 4,\ t\geqslant s,\\ & \quad t(4-s)\geqslant 6s-s^2-4\\ \left(\frac{R+1}{2R}\right)^{s-t-1}\left(\frac{(4-s)R+s}{R}\right), & 0\leqslant s\leqslant 4,\ t\leqslant s,\\ & t(4-s)\leqslant 6s-s^2-4\end{cases}\end{aligned}$$

and

$$M = \sup_{x\in[r,R]} g_{(\varsigma_s,\psi_t)}(x) = \begin{cases} \left(\frac{R+1}{2R}\right)^{s-t-1}\left(\frac{(4-s)R+s}{R}\right), & 0 \leqslant s \leqslant 4,\ t \geqslant s, \\ & t(4-s) \geqslant 6s-s^2-4 \\ \left(\frac{r+1}{2r}\right)^{s-t-1}\left(\frac{(4-s)r+s}{r}\right), & 0 \leqslant s \leqslant 4,\ t \leqslant s, \\ & t(4-s) \leqslant 6s-s^2-4 \end{cases} \tag{4.47}$$

Now (4.46) and (4.47) together with (4.2) give the inequalities (4.42) and (4.43). □

Proposition 4.8. *We have the following inequalities among the measures $\zeta_s(P||Q)$ and $\Omega_t(Q||P)$:*

$$\left(\frac{r+1}{2}\right)^{s-t-1}(sr+4-s)\,\Omega_t(Q||P) \leqslant \zeta_s(P||Q) \leqslant \left(\frac{R+1}{2}\right)^{s-t-1}[sR+(4-s)]\,\Omega_t(Q||P), \quad 0 \leqslant s \leqslant 4,\ s \geqslant t,\ s(t-s+6) \geqslant 4(1+t) \tag{4.48}$$

and

$$\left(\frac{R+1}{2}\right)^{s-t-1}(sR+4-s)\,\Omega_t(Q||P) \leqslant \zeta_s(P||Q) \leqslant \left(\frac{r+1}{2}\right)^{s-t-1}(sr+4-s)\,\Omega_t(Q||P), \quad 0 \leqslant s \leqslant 4,\ s \leqslant t,\ s(t-s+6) \leqslant 4(1+t). \tag{4.49}$$

Proof. Let us consider

$$g_{(\xi_s,\upsilon_t)}(x) = \frac{\xi_s''(x)}{\upsilon_t''(x)} = \left(\frac{x+1}{2}\right)^{s-t-1}(sx+4-s),\ x \in (0,\infty) \tag{4.50}$$

where $\xi_s(x)$ and $\upsilon_t(x)$ are as given by (3.13) and (3.10) respectively.

From (4.50) one has

$$g'_{(\xi_s,\upsilon_t)}(x) = \frac{1}{2}\left(\frac{x+1}{2}\right)^{s-t-2}[s(s-t)x + (-4t-4+st+6s-s^2)] \begin{cases} \geqslant 0, & 0 \leqslant s \leqslant 4,\ s \geqslant t,\ s(t-s+6) \geqslant 4(1+t) \\ \leqslant 0, & 0 \leqslant s \leqslant 4,\ s \leqslant t,\ s(t-s+6) \leqslant 4(1+t) \end{cases}. \tag{4.51}$$

In view of (4.51) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\xi_s,\upsilon_t)}(x) = \begin{cases} \left(\frac{r+1}{2}\right)^{s-t-1}[sr+4-s], & 0\leqslant s\leqslant 4,\ s\geqslant t, \\ & s(t-s+6)\geqslant 4(1+t) \\ \left(\frac{R+1}{2}\right)^{s-t-1}[sR+4-s], & 0\leqslant s\leqslant 4,\ s\leqslant t, \\ & s(t-s+6)\leqslant 4(1+t) \end{cases} \tag{4.52}$$

and

$$M = \sup_{x\in[r,R]} g_{(\xi_s,\upsilon_t)}(x) = \begin{cases} \left(\frac{R+1}{2}\right)^{s-t-1}[sR+4-s], & 0\leqslant s\leqslant 4,\ s\geqslant t, \\ & s(t-s+6)\geqslant 4(1+t) \\ \left(\frac{r+1}{2}\right)^{s-t-1}[sr+4-s], & 0\leqslant s\leqslant 4,\ s\leqslant t, \\ & s(t-s+6)\leqslant 4(1+t) \end{cases} \tag{4.53}$$

Now (4.52) and (4.53) together with (4.2) give the inequalities (4.48) and (4.49). □

Proposition 4.9. *We have the following inequalities among the measures $\zeta_s(Q||P)$ and $\Omega_t(Q||P)$:*

$$\frac{1}{R^{s+1}}\left(\frac{R+1}{2}\right)^{s-t-1}[(4-s)R+s]\,\Omega_t(Q||P) \leqslant \zeta_s(Q||P) \leqslant \frac{1}{r^{s+1}}\left(\frac{r+1}{2}\right)^{s-t-1}[(4-s)R+s]\,\Omega_t(Q||P),\ 0\leqslant s\leqslant 4,\ t\geqslant -1. \tag{4.54}$$

Proof. Let us consider

$$g_{(\varsigma_s,\upsilon_t)}(x) = \frac{\varsigma_s''(x)}{\upsilon_t''(x)} = \left(\frac{x+1}{2}\right)^{s-t-1}\left(\frac{(4-s)x+s}{x^{s+1}}\right),\ x\in(0,\infty) \tag{4.55}$$

where $\varsigma_s(x)$ and $\upsilon_t(x)$ are as given by (3.16) and (3.10) respectively.

From (4.55) one has

$$g'_{(\varsigma_s,\upsilon_t)}(x) = \frac{1}{2x^{t+4}}\left(\frac{x+1}{2}\right)^{s-t-2}\left[(s-4)(t+1)x^2 + s(s-t-6)x - s(s+1)\right] \leqslant 0,\ 0\leqslant s\leqslant 4,\ t\geqslant -1. \tag{4.56}$$

In view of (4.56) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\xi_s,\psi_t)}(x) = \left(\frac{R+1}{2}\right)^{s-t-1}\left(\frac{(4-s)R+s}{R^{s+1}}\right),\ 0\leqslant s\leqslant 4,\ t\geqslant -1 \tag{4.57}$$

and

$$M = \sup_{x\in[r,R]} g_{(\xi_s,\psi_t)}(x) = \left(\frac{r+1}{2}\right)^{s-t-1}\left(\frac{(4-s)r+s}{r^{s+1}}\right),\ 0\leqslant s\leqslant 4,\ t\geqslant -1 \tag{4.58}$$

Now (4.57) and (4.58) together with (4.2) give the inequalities (4.54). □

Proposition 4.10. *We have the following inequalities among the measures $\zeta_s(Q||P)$ and $\zeta_t(P||Q)$:*

$$\frac{1}{R^{s+1}}\left(\frac{R+1}{2}\right)^{s-t}\left(\frac{(4-s)R+s}{tR+4-t}\right)\zeta_t(P||Q) \leqslant \zeta_s(Q||P) \leqslant \frac{1}{r^{s+1}}\left(\frac{r+1}{2}\right)^{s-t}\left(\frac{(4-s)r+s}{tr+4-t}\right)\zeta_t(P||Q),\ 2\leqslant s\leqslant 4,\ 2\leqslant t\leqslant 4. \tag{4.59}$$

Proof. Let us consider

$$g_{(\varsigma_s,\xi_t)}(x) = \frac{\varsigma_s''(x)}{\xi_t''(x)} = \frac{1}{x^{s+1}}\left(\frac{x+1}{2}\right)^{s-t}\left(\frac{(4-s)x+s}{tx+4-t}\right),\ x\in(0,\infty) \tag{4.60}$$

where $\varsigma_s(x)$ and $\xi_t(x)$ are as given by (3.16) and (3.13) respectively.

From (4.60) one has

$$\begin{aligned} g'_{(\varsigma_s,\xi_s)}(x) &= \left(\frac{x+1}{2}\right)^{s-t-1}\frac{1}{2x^{s+2}(x+1)^3(tx+4-t)^2}\times \\ &\quad \big[t(t+1)(s-4)x^3 + \big(t(s+4)(s-5)+2t^2(2-s)\big)x^2 \\ &\quad + \big(s(t-5)(t+4)+2s^2(2-t)\big)x + s(s+1)(t-4)\big] \\ &\leqslant 0,\ 2\leqslant s\leqslant 4,\ 2\leqslant t\leqslant 4. \end{aligned} \tag{4.61}$$

In view of (4.61) we conclude the followings

$$m = \inf_{x\in[r,R]} g_{(\varsigma_s,\xi_s)}(x) = \frac{1}{R^{s+1}}\left(\frac{R+1}{2}\right)^{s-t}\left(\frac{(4-s)R+s}{tR+4-t}\right),\ 2\leqslant s\leqslant 4,\ 2\leqslant t\leqslant 4 \tag{4.62}$$

and

$$M = \sup_{x\in[r,R]} g_{(\varsigma_s,\xi_s)}(x) = \frac{1}{r^{s+1}}\left(\frac{r+1}{2}\right)^{s-t}\left(\frac{(4-s)r+s}{tr+4-t}\right),\ 2\leqslant s\leqslant 4,\ 2\leqslant t\leqslant 4 \tag{4.63}$$

Now (4.62) and (4.63) together with (4.2) give the inequalities (4.59). □

5. Inequalities Among Classical Measures

We shall present the inequalities showing relationship among the classical measures. These inequalities are developed from the propositions established in Section 4. In some cases we have combined these in order to have a single inequality. These inequalities we have divided in subsections according to measures given in Section 1.1.

5.1. Bounds on Chi-square Divergence.

We have following bounds on *Chi-square divergence* and its *adjoint* given in (1.1) and (1.2):

- For $t = \frac{1}{2}, s = 2$ in (4.9) or in (4.15) one gets

$$8r\sqrt{r}\ h(P||Q) \leqslant \chi^2(P||Q) \leqslant 8R\sqrt{R}\ h(P||Q). \tag{5.1}$$

- For $t = 2,\ s = -1$ in (4.4) or $t = 2,\ s = 0$ in (4.24) or (4.49) or in (4.54) or $t = -1,\ s = 2$ in (4.31) or in (4.37) one gets

$$\frac{(r+1)^3}{4}\Delta(P||Q) \leqslant \chi^2(P||Q) \leqslant \frac{(R+1)^3}{4}\Delta(P||Q). \tag{5.2}$$

- For $t = \frac{1}{2},\ s = 2$ in (4.4) or in (4.24) one gets

$$\frac{8}{R\sqrt{R}}h(P||Q) \leqslant \chi^2(Q||P) \leqslant \frac{8}{r\sqrt{r}}h(P||Q). \tag{5.3}$$

- For $t = -1,\ s = -1$ in (4.3) or in (4.9) $t = -1,\ s = 0$ in (4.15) or $t = 2,\ s = -1$ in (4.31) or $t = 2,\ s = 0$ in (4.37) or in (4.43) or $t = -1,\ s = 2$ in (4.43) or in (4.54) one gets

$$\frac{1}{4}\left(\frac{R+1}{R}\right)^3\Delta(P||Q) \leqslant \chi^2(Q||P) \leqslant \frac{1}{4}\left(\frac{r+1}{r}\right)^3\Delta(P||Q). \tag{5.4}$$

- For $t = 2,\ s = 2$ in (4.4) or in (4.24) or in (4.31) or in (3.37) or in (4.54) or in (4.59) or $t = -1,\ s = 2$ in (4.9) or in (4.15) one gets

$$\frac{1}{R^3}\chi^2(P||Q) \leqslant \chi^2(Q||P) \leqslant \frac{1}{r^3}\chi^2(P||Q). \tag{5.5}$$

5.2. Bounds on Relative Information. We have following bounds on *relative information* and its *adjoint* given in (1.3) and (1.4):

- For $t = 1,\ s = 2$ in (4.9) or in (4.15) one gets

$$\frac{1}{2R}\chi^2(P||Q) \leqslant K(P||Q) \leqslant \frac{1}{2r}\chi^2(P||Q). \tag{5.6}$$

- For $t = 1,\ s = 2$ in (4.4) or in (4.24) one gets

$$\frac{r^2}{2}\chi^2(Q||P) \leqslant K(P||Q) \leqslant \frac{R^2}{2}\chi^2(Q||P). \tag{5.7}$$

- For $t = 0,\ s = 2$ in (4.9) or in (4.15) one gets

$$\frac{1}{2R^2}\chi^2(P||Q) \leqslant K(Q||P) \leqslant \frac{1}{2r^2}\chi^2(P||Q). \tag{5.8}$$

- For $t = 0,\ s = 2$ in (4.4) or in (4.24) one gets

$$\frac{r}{2}\chi^2(Q||P) \leqslant K(Q||P) \leqslant \frac{R}{2}\chi^2(Q||P). \tag{5.9}$$

5.3. Bounds on Relative JS-Divergence. We have following bounds on *relative JS-divergence* and its *adjoint* given in (1.5) and (1.6):

- For $t = 0,\ s = -1$ in (4.31) or $t = 0,\ s = 0$ in (4.37) or in (4.43) one gets

$$\frac{R+1}{8R}\Delta(P||Q) \leqslant F(P||Q) \leqslant \frac{r+1}{8r}\Delta(P||Q). \tag{5.10}$$

- For $t = 2,\ s = 0$ in (4.4) or $t = 0,\ s = 2$ in (4.31) or in (4.37) one gets

$$\frac{1}{2R(R+1)^2}\chi^2(P||Q) \leqslant F(P||Q) \leqslant \frac{1}{2r(r+1)^2}\chi^2(P||Q). \tag{5.11}$$

- For $t = -1,\ s = 0$ in (4.3) or $t = 0,\ s = 2$ in (4.43) one gets

$$\frac{1}{2}\left(\frac{r}{r+1}\right)^2\chi^2(Q||P) \leqslant F(P||Q) \leqslant \frac{1}{2}\left(\frac{R}{R+1}\right)^2\chi^2(Q||P). \tag{5.12}$$

- For $t = 1,\ s = 0$ in (4.4), one gets

$$\frac{1}{(R+1)^2}K(P||Q) \leqslant F(P||Q) \leqslant \frac{1}{(r+1)^2}K(P||Q). \tag{5.13}$$

- For $t = -1,\ s = 0$ in (4.31) or $t = 0,\ s = 0$ in (4.49) or in (4.54) one gets

$$\frac{r+1}{8}\Delta(P||Q) \leqslant F(Q||P) \leqslant \frac{R+1}{8}\Delta(P||Q). \tag{5.14}$$

• For $t=-1,\ s=0$ in (4.9) or $t=2,\ s=0$ in (4.31) or $t=0,\ s=2$ in (4.54) one gets

$$\frac{r}{2}\left(\frac{r}{r+1}\right)^2\chi^2(Q||P)\leqslant F(Q||P)\leqslant\frac{R}{2}\left(\frac{R}{R+1}\right)^2\chi^2(Q||P). \tag{5.15}$$

• For $t=0,\ s=0$ in (4.9) one gets

$$\left(\frac{r}{r+1}\right)^2K(Q||P)\leqslant F(Q||P)\leqslant\left(\frac{R}{R+1}\right)^2K(Q||P). \tag{5.16}$$

• For $t=0,\ s=0$ in (4.31) one gets

$$r\ F(P||Q)\leqslant F(Q||P)\leqslant R\ F(P||Q). \tag{5.17}$$

5.4. Bounds on Relative AG-Divergence. We have following bounds on *relative AG-divergence* and its *adjoint* given in (1.7) and (1.8):

• For $t=\frac{1}{2},\ s=1$ in (4.4) one gets

$$\frac{2}{\sqrt{R}\,(R+1)}h(P||Q)\leqslant G(P||Q)\leqslant\frac{2}{\sqrt{r}\,(r+1)}h(P||Q). \tag{5.18}$$

• For $t=1,\ s=-1$ in (4.31) or in $t=1,\ s=0$ in (4.37) or (4.42) one gets

$$\left(\frac{R+1}{4R}\right)^2\Delta(P||Q)\leqslant G(P||Q)\leqslant\left(\frac{r+1}{4r}\right)^2\Delta(P||Q). \tag{5.19}$$

• For $t=2,\ s=1$ in (4.4) or $t=1,\ s=2$ in (4.31) or in (4.37) one gets

$$\frac{1}{4R^2(R+1)}\chi^2(P||Q)\leqslant G(P||Q)\leqslant\frac{1}{4r^2(r+1)}\chi^2(P||Q). \tag{5.20}$$

• For $t=-1,\ s=1$ in (4. 3) or $t=1,\ s=2$ in (4.43) one gets

$$\frac{r}{4(r+1)}\chi^2(Q||P)\leqslant G(P||Q)\leqslant\frac{R}{4(R+1)}\chi^2(Q||P). \tag{5.21}$$

• For $t=1,\ s=1$ in (4.4) one gets

$$\frac{1}{2R(R+1)}K(P||Q)\leqslant G(P||Q)\leqslant\frac{1}{2r(r+1)}K(P||Q). \tag{5.22}$$

• For $t=0,\ s=1$ in (4.4) one gets

$$\frac{1}{2(R+1)}K(Q||P)\leqslant G(P||Q)\leqslant\frac{1}{2(r+1)}K(Q||P). \tag{5.23}$$

• For $t = 1,\ s = 0$ in (4.31) one gets

$$\frac{R+1}{2R^2}F(Q||P) \leqslant G(P||Q) \leqslant \frac{r+1}{2r^2}F(Q||P). \tag{5.24}$$

• For $t = \frac{1}{2},\ s = 1$ in (4.9) one gets

$$\frac{2r\sqrt{r}}{r+1}h(P||Q) \leqslant G(Q||P) \leqslant \frac{2R\sqrt{R}}{R+1}h(P||Q). \tag{5.25}$$

• For $t = -1,\ s = 1$ in (4.31) or $t = 1,\ s = 0$ in (4.49) or in (4.54) one gets

$$\left(\frac{r+1}{4}\right)^2 \Delta(P||Q) \leqslant G(Q||P) \leqslant \left(\frac{R+1}{4}\right)^2 \Delta(P||Q). \tag{5.26}$$

• For $t = 1,\ s = 2$ in (4.54) or $t = -1,\ s = 1$ in (4.9) or $t = 2,\ s = 1$ in (4.31) one gets

$$\frac{r^3}{4(r+1)}\chi^2(Q||P) \leqslant G(Q||P) \leqslant \frac{R^3}{4(R+1)}\chi^2(Q||P). \tag{5.27}$$

• For $t = 1,\ s = 1$ in (4.9) one gets

$$\frac{r}{2(r+1)}K(P||Q) \leqslant G(Q||P) \leqslant \frac{R}{2(R+1)}K(P||Q). \tag{5.28}$$

• For $t = 0,\ s = 1$ in (4.9) one gets

$$\frac{r^2}{2(r+1)}K(Q||P) \leqslant G(Q||P) \leqslant \frac{R^2}{2(R+1)}K(Q||P). \tag{5.29}$$

• For $t = 0,\ s = 1$ in (4.31) one gets

$$\frac{r(r+1)}{2}F(P||Q) \leqslant G(Q||P) \leqslant \frac{R(R+1)}{2}F(P||Q). \tag{5.30}$$

• For $t = 1,\ s = 1$ in (4.31) one gets

$$r^2G(P||Q) \leqslant G(Q||P) \leqslant R^2G(P||Q). \tag{5.31}$$

5.5. Bounds on Relative J-Divergence. We have following bounds on *relative J-divergence* and its *adjoint* given in (1.9) and (1.10):

- For $t = -1,\ s = 1$ in (4.37) one gets

$$\frac{(r+1)(r+3)}{8}\Delta(P||Q) \leqslant D(P||Q) \leqslant \frac{(R+1)(R+3)}{8}\Delta(P||Q). \tag{5.32}$$

- For $t = 2,\ s = 1$ in (4.49) one gets

$$\frac{R+3}{2(R+1)^2}\chi^2(P||Q) \leqslant D(P||Q) \leqslant \frac{r+3}{2(r+1)^2}\chi^2(P||Q). \tag{5.33}$$

- For $t = -1,\ s = 1$ in (4.15) or $t = 2,\ s = 1$ in (4.37) one gets

$$\frac{r^3(r+3)}{2(r+1)^2}\chi^2(Q||P) \leqslant D(P||Q) \leqslant \frac{R^3(R+3)}{2(R+1)^2}\chi^2(Q||P). \tag{5.34}$$

- For $t = 0,\ s = 1$ in (4.15) one gets

$$\frac{r^2(r+3)}{(r+1)^2}K(Q||P) \leqslant D(P||Q) \leqslant \frac{R^2(R+3)}{(R+1)^2}K(Q||P). \tag{5.35}$$

- For $t = 0,\ s = 1$ in (4.37) one gets

$$r(r+3)F(P||Q) \leqslant D(P||Q) \leqslant R(R+3)F(P||Q). \tag{5.36}$$

- For $t = 1,\ s = 1$ in (4.37) one gets

$$\frac{2r^2(r+3)}{r+1}G(P||Q) \leqslant D(P||Q) \leqslant \frac{2R^2(R+3)}{R+1}G(P||Q). \tag{5.37}$$

- For $t = 1,\ s = 1$ in (4.49) one gets

$$\frac{2(R+3)}{R+1}G(Q||P) \leqslant D(P||Q) \leqslant \frac{2(r+3)}{r+1}G(Q||P). \tag{5.38}$$

- For $t = -1,\ s = 1$ in (4.43) or in (4.54) one gets

$$\frac{(R+1)(3R+1)}{8R^2}\Delta(P||Q) \leqslant D(Q||P) \leqslant \frac{(r+1)(3r+1)}{8r^2}\Delta(P||Q). \tag{5.39}$$

- For $t = 2,\ s = 1$in (4.24) or in (4.54) one gets

$$\frac{3R+1}{2R^2(R+1)^2}\chi^2(P||Q) \leqslant D(Q||P) \leqslant \frac{3r+1}{2r^2(r+1)^2}\chi^2(P||Q). \tag{5.40}$$

- For $t = 2,\ s = 1$ in (4.42) one gets

$$\frac{r(3r+1)}{2(r+1)^2}\chi^2(Q||P) \leqslant D(Q||P) \leqslant \frac{R(3R+1)}{2(R+1)^2}\chi^2(Q||P). \tag{5.41}$$

• For $t = 1,\ s = 1$ in (4.24) one gets

$$\frac{3R+1}{R(R+1)^2}K(P||Q) \leqslant D(Q||P) \leqslant \frac{3r+1}{r(r+1)^2}K(P||Q). \tag{5.42}$$

• For $t = 0,\ s = 1$ in (4.43) one gets

$$\frac{3R+1}{R}F(P||Q) \leqslant D(Q||P) \leqslant \frac{3r+1}{r}F(P||Q). \tag{5.43}$$

• For $t = 0,\ s = 1$ in (4.54) one gets

$$\frac{3R+1}{R^2}F(Q||P) \leqslant D(Q||P) \leqslant \frac{3r+1}{r^2}F(Q||P). \tag{5.44}$$

• For $t = 1,\ s = 1$ in (4.43) one gets

$$\frac{2(3r+1)}{r+1}G(P||Q) \leqslant D(Q||P) \leqslant \frac{2(3R+1)}{R+1}G(P||Q). \tag{5.45}$$

• For $t = 1,\ s = 1$ in (4.54) one gets

$$\frac{2(3R+1)}{R^2(R+1)}G(Q||P) \leqslant D(Q||P) \leqslant \frac{2(3r+1)}{r^2(r+1)}G(Q||P). \tag{5.46}$$

The inequalities given above are summarized in the following table:

h												
Δ	NA											
χ_1	(5.1)	(5.2)										
χ_2	(5.3)	(5.4)	(5.5)									
K_1	NC	NA	(5.6)	(5.7)								
K_2	NC	NA	(5.8)	(5.9)	NC							
F_1	NA	(5.10)	(5.11)	(5.12)	(5.13)	NA						
F_2	NA	(5.14)	NA	(5.15)	NA	(5.16)	(5.17)					
G_1	(5.18)	(5.19)	(5.20)	(5.21)	(5.22)	(5.23)	NC	(5.24)				
G_2	(5.25)	(5.26)	NA	(5.27)	(5.28)	(5.29)	(5.30)	NC	(5.31)			
D_1	NA	(5.32)	(5.33)	(5.34)	NA	(5.35)	(5.36)	NA	(5.37)	(5.38)		
D_2	NA	(5.39)	(5.40)	(5.41)	(5.42)	NA	(5.43)	(5.44)	(5.45)	(5.46)	NA	
	h	Δ	χ_1	χ_2	K_1	K_2	F_1	F_2	G_1	G_2	D_1	D_2

Legend:
$NC-$ Not considered here;
$NA-$ Not available here.

We can obtain bounds referring to "NC" and "NA". Some of these bounds can be seen in Taneja [23].

Now se shall express some of the inequalities given above in the unified form, that is, some of the above inequalities can be written as

$$r \leqslant \eta_u(P||Q) \leqslant R,\ u = 1, 2, ..., 21, \tag{5.47}$$

where

$$\eta_1(P||Q) = \frac{\sqrt[3]{\chi^2(P||Q)^2}}{\sqrt[3]{h(P||Q)^2}},$$

$$\eta_2(P||Q) = \frac{\sqrt[3]{4\chi^2(P||Q)} - \sqrt[3]{\Delta(P||Q)}}{\sqrt[3]{\Delta(P||Q)}},$$

$$\eta_3(P||Q) = \frac{\sqrt[3]{h(P||Q)^2}}{\sqrt[3]{\chi^2(Q||P)^2}},$$

$$\eta_4(P||Q) = \frac{\sqrt[3]{\Delta(P||Q)}}{\sqrt[3]{4\chi^2(Q||P)} - \sqrt[3]{\Delta(P||Q)}},$$

$$\eta_5(P||Q) = \frac{\sqrt[3]{\chi^2(P||Q)}}{\sqrt[3]{\chi^2(Q||P)}},$$

$$\eta_6(P||Q) = \frac{\chi^2(P||Q)}{2K(P||Q)},$$

$$\eta_7(P||Q) = \frac{\sqrt{2K(P||Q)}}{\sqrt{\chi^2(Q||P)}},$$

$$\eta_8(P||Q) = \frac{\sqrt{\chi^2(P||Q)}}{\sqrt{2K(Q||P)}},$$

$$\eta_9(P||Q) = \frac{2K(Q||P)}{\chi^2(Q||P)},$$

$$\eta_{10}(P||Q) = \frac{\Delta(P||Q)}{8F(P||Q) - \Delta(P||Q)},$$

$$\eta_{11}(P||Q) = \frac{\sqrt{2F(P||Q)}}{\sqrt{\chi^2(Q||P)} - \sqrt{2F(P||Q)}},$$

$$\eta_{12}(P||Q) = \frac{\sqrt{K(P||Q)} - \sqrt{F(P||Q)}}{\sqrt{F(P||Q)}},$$

$$\eta_{13}(P||Q) = \frac{8F(Q||P) - \Delta(P||Q)}{\Delta(P||Q)},$$

$$\eta_{14}(P||Q) = \frac{\sqrt{F(Q||P)}}{\sqrt{K(Q||P)} - \sqrt{F(Q||P)}},$$

$$\eta_{15}(P||Q) = \frac{F(Q||P)}{F(P||Q)},$$

$$\eta_{16}(P||Q) = \frac{\sqrt{\Delta(P||Q)}}{4\sqrt{G(P||Q)} - \sqrt{\Delta(P||Q)}},$$
$$\eta_{17}(P||Q) = \frac{4G(P||Q)}{\chi^2(Q||P) - 4G(P||Q)},$$
$$\eta_{18}(P||Q) = \frac{K(Q||P) - 2G(P||Q)}{2G(P||Q)},$$
$$\eta_{19}(P||Q) = \frac{4\sqrt{G(Q||P)} - \sqrt{\Delta(P||Q)}}{\sqrt{\Delta(P||Q)}},$$
$$\eta_{20}(P||Q) = \frac{2G(P||Q)}{K(P||Q) - 2G(P||Q)}$$

and

$$\eta_{21}(P||Q) = \frac{\sqrt{G(Q||P)}}{\sqrt{G(P||Q)}}.$$

Here $u = 1$ stands for (5.1), $u = 2$ for (5.2), $u = 3$ for (5.3), $u = 4$ for (5.4), $u = 5$ for (5.5), $u = 6$ for (5.6), $u = 7$ for (5.7), $u = 8$ for (5.8), $u = 9$ for (5.9), $u = 10$ for (5.10), $u = 11$ for (5.12), $u = 12$ for (5.13), $u = 13$ for (5.14), $u = 14$ for (5.16), $u = 15$ for (5.17), $u = 16$ for (5.19), $u = 17$ for (5.21), $u = 18$ for (5.23), $u = 19$ for (5.26), $u = 20$ for (5.28) and finally $u = 21$ stands for (5.31).

Remark 5.1. (*i*) *From the table we observe that there are some inequalities among the mesures not considered here, such as: among $K(P||Q)$ and $K(Q||P)$; $F(P||Q)$ and $G(P||Q)$; $F(Q||P)$ and $G(Q||P)$. These are given by*

$$r \leqslant \frac{K(P||Q)}{K(Q||P)} \leqslant R, \tag{5.48}$$

$$r \leqslant \frac{F(P||Q)}{2G(P||Q) - F(P||Q)} \leqslant R \tag{5.49}$$

and

$$r \leqslant \frac{2G(Q||P) - F(Q||P)}{F(Q||P)} \leqslant R. \tag{5.50}$$

The inequalities (5.48) can be seen in Taneja and Kumar [26] *and the inequalities (5.49) and (5.50) are given in Taneja* [25].

(*ii*) *In view of inequalities appearing in (5.47), (5.49) and (5.50), we have the following two sequence of inequalities:*

$$\frac{1}{16}\Delta(P||Q) \leqslant \frac{1}{2}F(P||Q) \leqslant G(P||Q) \leqslant \left(\frac{1}{2}K(Q||P) \text{ or } \frac{1}{4}\chi^2(Q||P)\right) \tag{5.51}$$

and

$$\frac{1}{16}\Delta(P||Q) \leqslant \frac{1}{2}F(Q||P) \leqslant G(Q||P) \leqslant \left(\frac{1}{2}K(P||Q) \text{ or } \frac{1}{4}\chi^2(P||Q)\right). \tag{5.52}$$

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