Chapter 5

Matter Waves. Solutions of Selected Problems

5.1 Problem 5.11 (In the text book)

For an electron to be confined to a nucleus, its de Broglie wavelength would have to be less than $10^{-14}$ m.

(a) What would be the kinetic energy of an electron confined to this region?

(b) On the basis of this result, would you expect to find an electron in a nucleus? Explain.

Solution

(a) The momentum of the electron is defined by its de Broglie wavelength as:

\[
p = \frac{h}{\lambda}
\]

\[
pe = \frac{hc}{\lambda}
\]

\[
= \frac{1.240 \times 10^3 (eV \cdot nm)}{10^{-5} (nm)}
\]

\[
= 1.240 \times 10^8 eV
\]

The energy of the electron is:
\[ E = \sqrt{(pc)^2 + m_e^2c^4} \]
\[ = \sqrt{(1.240 \times 10^8)^2 + (511 \times 10^3)^2} \]
\[ = 1.2400 \times 10^8 \text{ eV} \]
\[ = 124.0 \text{ MeV} \]

The kinetic energy \( K \) of the electron is:

\[ K = E - m_ec^2 \]
\[ = 124.0 - 0.511 \]
\[ = 123.499 \text{ MeV} \]

(b) The kinetic energy of the electron is too large for the electron to be confined in such small space.
5.2 Problem 5.17 (In the text book)

The dispersion relation for free relativistic electron waves is

$$\omega(k) = \sqrt{c^2 k^2 + \left(\frac{m_e c^2}{\hbar}\right)^2}$$

Obtain expressions for the phase velocity $v_p$ and group velocity $v_g$ of these waves and show that their product is a constant, independent of $k$. From your result, what can you conclude about $v_g$ if $v_p > c$?

Solution

The total energy of the electron is:

$$E = \sqrt{p^2 c^2 + m_e^2 c^2}$$

Since $E = h\omega$ and $p = \hbar k$, we get:

$$h\omega = \sqrt{(\hbar kc)^2 + (m_e^2 c^2)^2}$$

$$\omega(k) = \sqrt{(kc)^2 + \left(\frac{m_e c^2}{\hbar}\right)^2}$$

The phase velocity $v_p$ is given by:

$$v_p = \frac{\omega}{k}$$

$$= \frac{\sqrt{(kc)^2 + \left(\frac{m_e c^2}{\hbar}\right)^2}}{k}$$

$$= \sqrt{c^2 + \left(\frac{m_e c^2}{\hbar k}\right)^2}$$

and the group velocity $v_g$ is given by:

$$v_g = \frac{d\omega}{dk}\bigg|_{k_0}$$

$$= \frac{1}{2} \frac{2kc^2}{2 \sqrt{k^2 c^2 + \left(\frac{m_e c^2}{\hbar}\right)^2}}$$

$$= \frac{kc^2}{\sqrt{k^2 c^2 + \left(\frac{m_e c^2}{\hbar}\right)^2}}$$
The product $v_p v_g$ is:

$$v_p v_g = \frac{\sqrt{(k c)^2 + \left(\frac{m e c^2}{\hbar}\right)^2}}{k} \times \frac{k c^2}{\sqrt{k^2 c^2 + \left(\frac{m e c^2}{\hbar}\right)^2}}$$

$$= c^2$$

Therefore, if $v_p > c$, then $v_g < c$
5.3 Problem 5.22 (In the text book)

A beam of electrons is incident on a slit of variable width. If it is possible to resolve a 1% difference in momentum, what slit width would be necessary to resolve the interference pattern of the electrons if their kinetic energy is

1. 0.010 MeV,
2. 1.0 MeV, and
3. 100 MeV?

Solution

Using the uncertainty principle, with \( \Delta x = a \) and \( \Delta p = 0.01p \), where \( a \) is the slit width, we get:

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \\
a \times 0.01p = \frac{\hbar}{2} \\
\Rightarrow a = \frac{\hbar}{2 \times 0.01p} = \frac{\hbar}{0.02pc}
\]

since we are given the kinetic energy of the electron, we can find \( pc \) from:

\[
E^2 = p^2c^2 + m_e^2c^4 \\
pc = \sqrt{E^2 - m_e^2c^4} \\
= \sqrt{(K + m_e^2c^2) - m_e^4c^4} \\
= \sqrt{K^2 + 2Km_e^2c^2 + m_e^4c^4 - m_e^4c^4} \\
= \sqrt{K^2 + 2Km_e^2c^2}
\]

Now using Equation (5.2) in Equation (5.1) and take \( m_e^2c^2 = 0.511 \text{ MeV} \) we get:
\[
a = \frac{\hbar c}{0.02 pc} \\
= \frac{1.973 \times 10^{-4} (MeV \cdot nm)}{0.02 \sqrt{K^2 (MeV)^2 + 2K (MeV) m_e c^2 (MeV)}} \\
= \frac{9.865 \times 10^{-3}}{\sqrt{K^2 + 2 \times 0.511 K}} \\
= 9.865 \times 10^{-3} \sqrt{K^2 + 1.022K} \tag{5.3}
\]

(a) Using Equation (5.3) and \( K = 0.01 \) MeV we get:

\[
a = \frac{9.865 \times 10^{-3}}{\sqrt{(0.01)^2 + 1.022 \times 0.01}} \\
= \frac{9.865 \times 10^{-3}}{9.711 \times 10^{-2} nm} \\
= 9.711 \times 10^{-2} \text{ nm}
\]

(b) Using Equation (5.3) and \( K = 1.00 \) MeV we get:

\[
a = \frac{9.865 \times 10^{-3}}{\sqrt{(1.00)^2 + 1.022 \times 1.00}} \\
= \frac{9.865 \times 10^{-3}}{6.938 \times 10^{-3} nm} \\
= 6.938 \times 10^{-3} \text{ nm}
\]

(c) Using Equation (5.3) and \( K = 100 \) MeV we get:

\[
a = \frac{9.865 \times 10^{-3}}{\sqrt{(100)^2 + 1.022 \times 100}} \\
= \frac{9.865 \times 10^{-3}}{9.815 \times 10^{-5} nm} \\
= 9.815 \times 10^{-5} \text{ nm}
\]
5.4 Problem 5.29 (In the text book)

A two-slit electron diffraction experiment is done with slits of unequal widths. When only slit 1 is open, the number of electrons reaching the screen per second is 25 times the number of electrons reaching the screen per second when only slit 2 is open. When both slits are open, an interference pattern results in which the destructive interference is not complete. Find the ratio of the probability of an electron arriving at an interference maximum to the probability of an electron arriving at an adjacent interference minimum. (Hint: Use the superposition principle).

Solution

With slit 1 open and slit 2 closed, we have:

\[ P_1 = |\Psi_1|^2 \]

With slit 2 open and slit 1 closed, we have:

\[ P_2 = |\Psi_2|^2 \]

where \( P_1 \) and \( P_2 \) are the probabilities that the electrons reach the screen though slit 1 and slit 2 respectively. \( \Psi_1 \) and \( \Psi_2 \) are the wave functions of electrons going through slit 1 and slit 2 respectively.

When both slits are open we get:

\[
P = |\Psi_1 + \Psi_2|^2 = |\Psi_1|^2 + |\Psi_2|^2 + 2|\Psi_1||\Psi_2| \cos \phi
\]

where \( \phi \) is the phase angle between the waves arriving at the screen from the two slits. So \( P = P_{max} \) when \( \cos \phi = +1 \) and \( P = P_{min} \) when \( \cos \phi = -1 \). Where \( P_{max} \) is the probability that there will be a maximum intensity on the screen and \( P_{min} \) is the probability that there will minimum intensity on the screen. Note that all probabilities are functions of position on the screen, we then have:

\[
P_{max} = (|\Psi_1 + \Psi_2|)^2
\]

\[
P_{min} = (|\Psi_1 - \Psi_2|)^2
\]

Since,
we then get:

\[
\frac{P_1}{P_2} = \frac{|\Psi_1|^2}{|\Psi_2|^2} = 25
\]
\[
\frac{|\Psi_1|}{|\Psi_2|} = 5
\]

\[
\frac{P_{max}}{P_{min}} = \frac{(|\Psi_1 + \Psi_2|)^2}{(|\Psi_1 - \Psi_2|)^2} = \frac{(5|\Psi_2 + \Psi_2|)^2}{(5|\Psi_2 - \Psi_2|)^2} = \frac{(6\Psi_2)^2}{(4\Psi_2)^2} = \frac{36}{16} = 2.25
\]
5.5 Problem 5.35 (In the text book)

A matter wave packet.

(a) Find and sketch the real part of the matter wave pulse shape \( f(x) \) for a Gaussian amplitude distribution \( a(k) \), where

\[
a(k) = Ae^{-\alpha^2(k-k_0)^2}
\]

Note that \( a(k) \) is peaked at \( k_0 \) and has a width that decreases with increasing \( \alpha \). (Hint: In order to put

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k)e^{ikx}dk
\]

into the standard form

\[
\int_{-\infty}^{\infty} e^{-\alpha z^2}dz
\]

complete the square in \( k \).)

(b) By comparing the result for the real part of \( f(x) \) to the standard form of a Gaussian function with width \( \Delta x \), \( f(x) \propto Ae^{-(x/2\Delta x)^2} \) show that the width of the matter wave pulse is \( \Delta x = \alpha \).

(c) Find the width \( \Delta k \) of \( a(k) \) by writing \( a(k) \) in standard Gaussian form and show that \( \Delta x \Delta k = \frac{1}{2} \), independent of \( \alpha \).

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Solution

(a) The spectral distribution function \( a(k) \) is given by:

\[
a(k) = Ae^{-\alpha^2(k-k_0)^2}
\]

\[
= Ae^{-\alpha^2(2k-k_0+k_0)^2}
\]

\[
= Ae^{-\alpha^2k_0^2}e^{-\alpha^2(2k-k_0)^2}
\]

The matter wave pulse shape \( f(x) \) becomes:
\[
\begin{align*}
  f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) e^{ikx} \, dk \\
  &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A e^{-\alpha^2 k^2} e^{-\alpha^2 (k^2 - 2k k_o) e^{ikx}} \, dk \\
  &= \frac{1}{\sqrt{2\pi}} A e^{-\alpha^2 k_o^2} \int_{-\infty}^{\infty} e^{-\alpha^2 (k^2 - 2k k_o) + i k x} \, dk \\
  &= \frac{1}{\sqrt{2\pi}} A e^{-\alpha^2 k_o^2} \int_{-\infty}^{\infty} e^{-\alpha^2 (k^2 - 2k k_o - i k x / \alpha^2)} \, dk \\
  &= \frac{1}{\sqrt{2\pi}} A e^{-\alpha^2 k_o^2} \int_{-\infty}^{\infty} e^\beta \, dk
\end{align*}
\]

where \( \beta = -\alpha^2 (k^2 - 2k k_o - \frac{i k x}{\alpha^2}) \).

\[
\begin{align*}
  \beta &= -\alpha^2 \left( k^2 - 2k k_o - \frac{i k x}{\alpha^2} \right) \\
  &= -\alpha^2 \left[ k^2 - 2k \left( k_o + \frac{i x}{2\alpha^2} \right) \right] \\
  &= -\alpha^2 \left[ k^2 - 2k \left( k_o + \frac{i x}{2\alpha^2} \right) + \left( k_o + \frac{i x}{2\alpha^2} \right)^2 - \left( k_o + \frac{i x}{2\alpha^2} \right)^2 \right] \\
  &= -\alpha^2 \left[ k + \left( k_o + \frac{i x}{2\alpha^2} \right)^2 \right] + \alpha^2 \left( k_o + \frac{i x}{2\alpha^2} \right)^2
\end{align*}
\]

\( f(x) \), then becomes:

\[
\begin{align*}
  f(x) &= \frac{1}{\sqrt{2\pi}} A e^{-\alpha^2 k_o^2} \int_{-\infty}^{\infty} e^{-\alpha^2 \left[ k + \left( k_o + \frac{i x}{2\alpha^2} \right)^2 \right]} \, dk \\
  &= \frac{1}{\sqrt{2\pi}} A e^{-\alpha^2 k_o^2} \int_{-\infty}^{\infty} e^{-\alpha^2 z^2} \, dz
\end{align*}
\]

where \( z = \left[ k + \left( k_o + \frac{i x}{2\alpha^2} \right) \right] \) and \( dz = dk \). Since,

\[
\int_{-\infty}^{\infty} e^{-\alpha^2 z^2} = \frac{\sqrt{\pi}}{\alpha}
\]
then \( f(x) \) becomes:

\[
f(x) = \frac{1}{\sqrt{2\pi}} A e^{-\alpha^2 k_o^2} e^{\alpha^2 \left( k_o + \frac{ix}{2\alpha^2} \right)^2} \frac{\sqrt{\pi}}{\alpha} \\
= \frac{A}{\alpha\sqrt{2}} e^{-\alpha^2 k_o^2} e^{\alpha^2 \left( k_o^2 + \frac{(ik_o x/\alpha^2) - (x^2/4\alpha^4)}{4\alpha^2} \right)} \\
= \frac{A}{\alpha\sqrt{2}} e^{-x^2/4\alpha^2} e^{ik_o x} e^{-x^2/4\alpha^2} \\
= \frac{A}{\alpha\sqrt{2}} e^{-x^2/4\alpha^2} (\cos k_o x + i \sin k_o x)
\]

The real part of \( f(x) \), \( \text{Re}\, f(x) \), is:

\[
\text{Re}\, f(x) = \frac{A}{\alpha\sqrt{2}} e^{-x^2/4\alpha^2} \cos k_o x
\]

and is composed of a Gaussian envelope multiplied by a harmonic wave with a wave number \( k_o \). A plot of \( \text{Re}\, f(x) \) is shown in Figure (5.1).

![Figure 5.1:](image)

(b) Comparing

\[
\frac{A}{\alpha\sqrt{2}} e^{-x^2/4\alpha^2}
\]
to

\[ Ae^{-\frac{(x/2\Delta x)^2}{\alpha^2}} \]

implies that \( \Delta x = \alpha \).

(c) Given that \( a(k) \) is:

\[ a(k) = Ae^{-\alpha^2(k-k_0)^2} \]

putting \( \Delta k = 1/2\alpha \), than \( a(k) \) can be written as:

\[ a(k) = Ae^{-(k-k_0)^2/(2\Delta k)^2} \]

the last equation makes \( a(k) \) takes the standard Gaussian form, so we then have:

\[ \Delta x \Delta k = \frac{1}{2\alpha} \times \alpha = \frac{1}{2} \]