Chapter 15

Oscillatory Motion. Solutions of Selected Problems

15.1 Problem 15.18 *(In the text book)*

A block-spring system oscillates with an amplitude of 3.50 cm. If the spring constant is 250 N/m and the mass of the block is 0.500 kg, determine

(a) the mechanical energy of the system,

(b) the maximum speed of the block, and

(c) the maximum acceleration.

Solution

(a) The total energy of the system equals maximum kinetic energy or the maximum potential energy. Using the latter definition, we get:

$$\begin{aligned} E &= \frac{1}{2}kA^2 \\ &= \frac{1}{2} \times 250\ (N/m) \times (3.5 \times 10^{-2}\ (m))^2 \\ &= 0.153\ N/m^2 \\ &= 0.153\ J \end{aligned}$$

(b) The maximum speed of the block is given by:

$$v_{max} = A\omega = A \times \sqrt{\frac{k}{m}} = 3.5 \times 10^{-2}(m) \times \sqrt{\frac{250\ (N/m)}{0.500\ (kg)}} = 0.783\ m/s$$

(c) The maximum acceleration is:

$$a_{max} = A\omega^2 = A \times \frac{k}{m} = 3.5 \times 10^{2}\ (m) \times \frac{250; (N/m)}{0.500\ (kg)} = 17.5\ m/s^2$$

15.2 Problem 15.37 *(In the text book)*

Consider the physical pendulum of Figure (15.18).

(a) If its moment of inertia about an axis passing through its center of mass and parallel to the axis passing through its pivot point is I_{CM}, show that its period is

$$T = 2\pi\sqrt{\frac{I_{CM} + md^2}{mgd}}$$

where d is the distance between the pivot point and center of mass.

(b) Show that the period has a minimum value when d satisfies $md^2 = I_{CM}$.

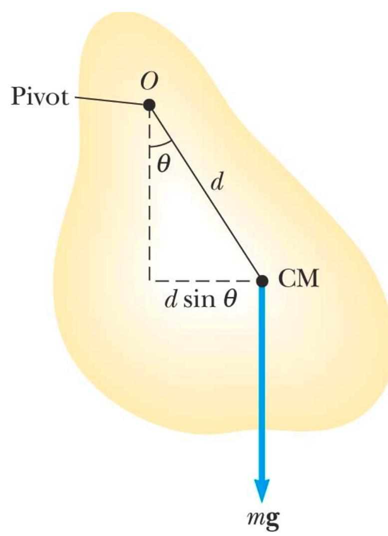


Figure 15.18:

Solution

(a) The period of a physical pendulum is given by:

$$T = 2\pi\sqrt{\frac{I}{mgd}}$$

where I is the moment of inertial about an axis through the point of suspension of the pendulum and d is the distance between the center of mass and the point of suspension. We are given the moment of inertial of the pendulum I_{CM} about an axis through the center of mass and parallel to the the axis through the pivot. Using the *parallel axis theorem*, we get:

$$I = I_{cm} + md^2$$

The period of the pendulum is then given by:

$$T = 2\pi\sqrt{\frac{I_{cm} + md^2}{mgd}}$$

(b) The period depends on d the distance between the pivot and the center of mass. So, to find the value of d that minimizes the period T we set the first derivative of of T with respect to d to zero,

$$\begin{aligned}
\frac{dT}{dd} &= 0 \\
0 &= \frac{d}{dd}\left\{2\pi\sqrt{\frac{I_{cm} + md^2}{mgd}}\right\} \\
&= 2\pi\frac{d}{dd}\left\{\sqrt{I_{cm} + md^2} \times (mgd)^{-1/2}\right\} \\
&= 2\pi\left\{\sqrt{I_{cm} + md^2} \times -\frac{1}{2} \times (mdg)^{-3/2} \times mg + (mgd)^{-1/2} \times \frac{1}{2} \times (I_{cm} + md^2)^{-1/2} \times 2md\right\} \\
&= \frac{-\pi(I_{cm} + md^2)mg}{\sqrt{(I_{cm} + md^2)(mgd)^3}} + \frac{2\pi(md)(mgd)}{\sqrt{(I_{cm} + md^2)(mgd)^3}} \\
&= \pi\frac{2(md)(mgd) - (I_{cm} + md^2)mg}{\sqrt{(I_{cm} + md^2)(mgd)^3}} \\
&= \pi\frac{2(md^2)(mg) - I_{cm}mg - (md^2)(mg)}{\sqrt{(I_{cm} + md^2)(mgd)^3}} \\
&= (md^2)(mg) - I_{cm}mg \\
&= (md^2) - I_{cm}
\end{aligned}$$

or

$$I_{CM} = md^2$$

15.3 Problem 15.43 *(In the text book)*

A 10.6-kg object oscillates at the end of a vertical spring which has a spring constant of $2.05 \times 10^4\ N/m$. The effect of air resistance is represented by the damping coefficient $b = 3.00\ N \cdot s/m$.

(a) Calculate the frequency of the damped oscillation.

(b) By what percentage does the amplitude of the oscillation decrease in each cycle?

(c) Find the time interval that elapses while the energy of the system drops to 5.00% of its initial value.

Solution

(a) The natural angular frequency of the system $\omega_\circ$ is:

$$\omega_\circ = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.05 \times 10^4\ (N/m)}{10.6\ (kg)}} = 43.98\ (N/kg{\cdot}m)^{\frac{1}{2}} = 43.98\ (kg{\cdot}m/kg{\cdot}m{\cdot}s^2)^{\frac{1}{2}} = 43.98\ s^{-1}$$

The angular frequency ω with damping is:

$$\begin{aligned}
\omega &= \sqrt{\omega_\circ^2 - \left(\frac{b}{2m}\right)^2} \\
&= \sqrt{(43.98\ s^{-1})^2 - \left(\frac{3.00\ (N \cdot s/m)}{2 \times 10.6\ (kg)}\right)^2} \\
&= 43.97\ s^{-1} \\
f &= \frac{\omega}{2\pi} \\
&= \frac{43.97}{2\pi} \\
&= 7.00\ Hz
\end{aligned}$$

(b) The position of the of the mass x is given by:

$$x = A_\circ e^{-bt/2m} \cos(\omega t + \phi)$$

Where $A_\circ$ is the amplitude without damping and $A_\circ e^{-bt/2m}$ is the amplitude with damping at a time t. After one period the amplitude becomes $A_\circ e^{-b(t+T)/2m}$ and the fractional change $\Delta A/A$ is then:

$$\begin{aligned}\frac{\Delta A}{A} &= \frac{A_\circ e^{-bt/2m} - A_\circ e^{-b(t+T)/2m}}{A_\circ e^{-bt/2m}} \\ &= \frac{A_\circ e^{-bt/2m}\left[1 - e^{-bT/2m}\right]}{A_\circ e^{-bt/2m}} \\ &= 1 - e^{-bT/2m}\end{aligned}$$

Since $T = 1/f$, we get:

$$\begin{aligned}\frac{\Delta A}{A} &= 1 - e^{-bT/2m} \\ &= 1 - e^{-b/2mf\omega} \\ &= 1 - e^{-3.00/(2\times 10.06\times 7)} = 1 - e^{-2.02\times 10^2} = 1 - 0.98 \\ &= 0.02 = 2.00\%\end{aligned}$$

(c) The total energy of the system is:

$$\begin{aligned}E &= \frac{1}{2}kA^2 \\ &= \frac{1}{2}kA_\circ^2 e^{-2bt/2m} \\ &= E_\circ e^{-bt/m}\end{aligned}$$

where $E_\circ = \frac{1}{2}kA_\circ^2$ is the total energy without damping. The time after which the total energy drops by 5% is then:

$$\begin{aligned}E &= 0.05E_\circ \\ &= E_\circ e^{-bt/m} \\ 0.05 &= e^{-bt/m} \\ 20.0 &= e^{bt/m} \\ t &= \frac{\ln(20)\times m}{b} \\ &= \frac{\ln(20)\times 10.6\ (kg)}{3.00\ (N\cdot s/m)} \\ &= 10.6\ s\end{aligned}$$

15.4 Problem 15.56 *(In the text book)*

A solid sphere (radius = R) rolls without slipping in a cylindrical trough (radius = $5R$) as shown in Figure (15.56). Show that, for small displacements from equilibrium perpendicular to the length of the trough, the sphere executes simple harmonic motion with a period

$$T = 2\pi\sqrt{\frac{28R}{5g}}$$

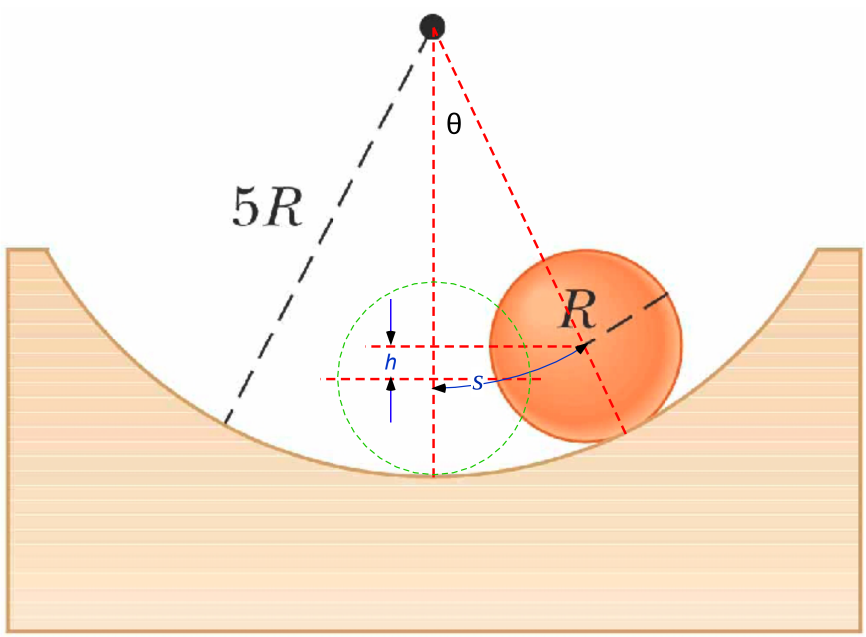


Figure 15.56:

Solution

The kinetic energy K of the ball is coming from the translational and angular motions. Let v be the linear velocity of the center of mass of the ball and the Ω is the angular velocity of the rolling motion of the ball, we then have:

$$K = \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2$$

where $I = (2/5)mR^2$ is the moment of inertia of the ball around its center of mass. The center of the ball moves along a circle of radius $4R$, and its displacement from the equilibrium position is $s = 4R\theta$. The linear velocity v of the the center of the ball is then:

$$v = \frac{ds}{dt} = 4R\left(\frac{d\theta}{dt}\right)$$

Since the ball is rolling without slipping, we get:

$$v = \frac{ds}{dt} = R\Omega = 4R\left(\frac{d\theta}{dt}\right)$$

and

$$\Omega = \frac{v}{r} = 4\left(\frac{d\theta}{dt}\right)$$

The kinetic energy then becomes:

$$\begin{aligned} K &= \frac{1}{2}m\left(4R\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(4\frac{d\theta}{dt}\right)^2 \\ &= \frac{112mR^2}{10}\left(\frac{d\theta}{dt}\right)^2 \end{aligned}$$

When the ball is displaced by an angle θ, its center is higher than its equilibrium position by a distance h,

$$h = 4R(1 - \cos\theta)$$

Since $\sin^2\left(\frac{1}{2}\theta\right)$ can be written as:

$$\sin^2\left(\frac{1}{2}\theta\right) = \frac{1}{2}(1 - \cos\theta)$$

for a small $\sin\theta \approx \theta$, so:

$$1 - \cos\theta \approx \frac{1}{2}\theta^2$$

The change in the potential energy of the displaced ball (for small angles) $U = mgh = 4mgR(1 - \cos\theta) = 2mgR\theta^2$. The total energy E of the ball is then:

$$E = K + E = \frac{112mR^2}{10}\left(\frac{d\theta}{dt}\right)^2 + 2mgR\theta^2$$

Since the total energy is constant in time, we get:

$$\begin{aligned} \frac{dE}{dt} &= 0 \\ &= \frac{112mR^2}{5}\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{ddt^2}\right) + 4mgR\theta\left(\frac{d\theta}{dt}\right) \end{aligned}$$

dividing the last equation by $4mR\left(\frac{d\theta}{dt}\right)$ we get:

$$\frac{28R}{5}\left(\frac{d^2\theta}{dt^2}\right)+g\theta=0$$

or

$$\frac{d^2\theta}{dt^2}=-\left(\frac{5g}{28R}\right)\theta$$

The last equation is a that of a simple harmonic motion with angular frequency ω given by:

$$\omega=\sqrt{\frac{5g}{28R}}$$

and the period of the simple harmonic motion is:

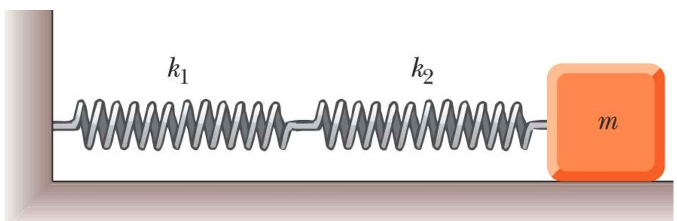
$$T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{28R}{5g}}$$

15.5 Problem 15.71 *(In the text book)*

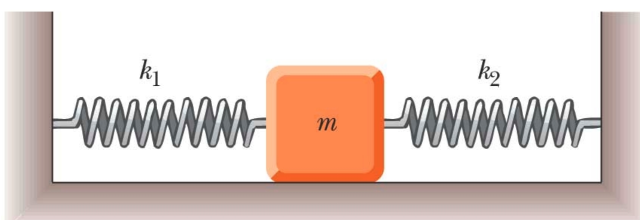
A block of mass m is connected to two springs of force constants k_1 and k_2 as shown in Figure (15.71a) and Figure (15.71b) Figures. In each case, the block moves on a frictionless table after it is displaced from equilibrium and released. Show that in the two cases the block exhibits simple harmonic motion with periods

$$\text{(a)} \qquad T = 2\pi\sqrt{\frac{m(k_1+k_2)}{k_1k_2}}$$

$$\text{(b)} \qquad T = 2\pi\sqrt{\frac{m}{k_1+k_2}}$$



(a)



(b)

Figure 15.71:

Solution

(a) When the block is displaced a distance x from the equilibrium position, spring 1 is stretched a distance x_1 and spring 2 is stretched a distance x_2.There is one force acting through out the system shown in Figure (15.71a), so

$$k_1x_1 = k_2x_2$$

in addition, the blocks displacement is the sum of the the distances that the two springs stretched through, i.e.

$$x = x_1 + x_2$$

We then find:

$$\begin{aligned} x &= x_1 + \frac{k_2}{k_2} x_1 = \left[1 + \frac{k_1}{k_2}\right] x_1 = \left[\frac{k_1 + k_2}{k_2}\right] x_1 \\ x_1 &= \left[\frac{k_2}{k_1 + k_2}\right] x \end{aligned}$$

The force F acting on either spring is then:

$$F = k_1 x_1 = \left[\frac{k_1 k_2}{k_1 + k_2}\right] x = k_{eff} x$$

where:

$$k_{eff} = \frac{k_1 k_2}{k_1 + k_2}$$

The period T of the motion is then:

$$T = 2\pi \sqrt{\frac{m}{k_{eff}}} = 2\pi \sqrt{\frac{m(k_1 + k_2)}{k_1 k_2}}$$

(b) In this case (see Figure (15.71b)), when the block is displaced a distance x, then, one spring is stretched a distance x and the other is compressed by the same distance x. Therefore, the block is acted upon by two forces acting in same direction which is opposite to the direction of the motion of the block, so:

$$F = -(k_1 + k_2)x = k_{eff} x$$

where $k_{eff} = k_1 + k_2$, the period T of the motion in this case is then:

$$T = 2\pi \sqrt{\frac{m}{k_{ff}}} = 2\pi \sqrt{\frac{m}{k_1 + k_2}}$$