Redshift-Distance Relationships

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1. Distances in Cosmology

This note considers two conceptually important definitions of cosmological distances, look-back distance and proper distance. Definitions of distance more directly related to observations, luminosity distance and angular diameter distance, are not treated in this note. Look-back distance is the time difference between when a galaxy emits light and when we receive the light, multiplied by the speed of light. Proper distance at time t is the distance between where the galaxy is at time t and where we are at time t.

2. Flat FLRW Universe

The metric for a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) universe gives (omitting angular coordinates) the spacetime interval

$$ds^2 = c^2 dt^2 - a\left(t\right)^2 dr^2. \tag{1}$$

Here, t is cosmological time since the Big Bang, r is a comoving spatial coordinate, and $a\left(t\right)$ is the scale factor of the universe. Just as in special relativity, the spacetime interval is: timelike if $ds^2 > 0$; lightlike if $ds^2 = 0$; and spacelikelike if $ds^2 < 0$.

To help understand comoving coordinates and scale factors, consider the surface of the Earth, with points on the surface located by their latitude and longitude coordinates. Suppose the Earth expands. Because latitude and longitude coordinates for points do not change as the Earth expands, they are examples of comoving coordinates. The distance between any two points, however, increases as the Earth expands. This can be represented by scale factor that increases as time goes on, i.e., the scale factor is a function of time.

Just as points on an expanding Earth have fixed values of latitude and longitude, galaxies in our expanding universe have fixed values of the comoving coordinate r. For convenience, assume that our Milky Way galaxy has the fixed value $r = 0$. The expansion of the universe is given by the change in the scale factor $a\left(t\right)$. The proper distance $d\left(t\right)$ at time t between the Milky Way and a galaxy that has comoving coordinate is $d\left(t\right) = a\left(t\right) r$.

Spatially flat spacetime universes are often called flat universes, but this is somewhat misleading, as (spatially) flat universes have non-zero spacetime curvature! For a flat universe that contains matter (including dark matter), dark energy, and no radiation, the scale factor $a\left(t\right)$ of the universe is (see exercise 15.23 from *General Relativity: An Introduction for Physicists* by Hobson, Efstathiou, and Lasenby)

$$a(t) = \left(\frac{1-\Omega_{\Lambda 0}}{\Omega_{\Lambda 0}}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{3}{2}\sqrt{\Omega_{\Lambda 0}} H_0 t\right).$$

Here, $\Omega_{\Lambda 0}$ is the (normalized) present density of dark energy, $1-\Omega_{\Lambda 0}$ is the (normalized) density of matter, and the scale factor $a(t)$ is normalized such that $a(t_0) = 1$, where t_0 is the present age of the universe. This is a very good approximation to the best fit to cosmological observations. (This expression for the scale factor is obtained by solving the Friedmann equation, which itself comes from the Einstein equation.)

Take the present dark energy density to be $\Omega_{\Lambda 0} = 0.692$, and the present value of the Hubble parameter to be

$$\begin{aligned} H_0 &= 67.80\,\mathrm{km}\,/\,\mathrm{s}\,/Mpc \\ &= 67.80 \times 3.240 \times 10^{-20}\,\mathrm{s}^{-1} \\ &= 2.197 \times 10^{-18}\,\mathrm{s}^{-1} \end{aligned}$$

(latest PLANCK satellite results). Then,

$$a(t) = A \sinh^{\frac{2}{3}}(Bt), \tag{2}$$

where

$$\begin{aligned} A &= \left(\frac{1-\Omega_{\Lambda 0}}{\Omega_{\Lambda 0}}\right)^{\frac{1}{3}} \\ &= \left(\frac{1-0.692}{0.692}\right)^{\frac{1}{3}} \\ &= 0.7635, \end{aligned}$$

$$\begin{aligned} B &= \frac{3}{2}\sqrt{\Omega_{\Lambda 0}} H_0 \\ &= \frac{3}{2}\sqrt{0.6922} \times 2.1972 \times 10^{-18}\,\mathrm{s}^{-1} \\ &= 2.742 \times 10^{-18}\,\mathrm{s}^{-1}. \end{aligned}$$

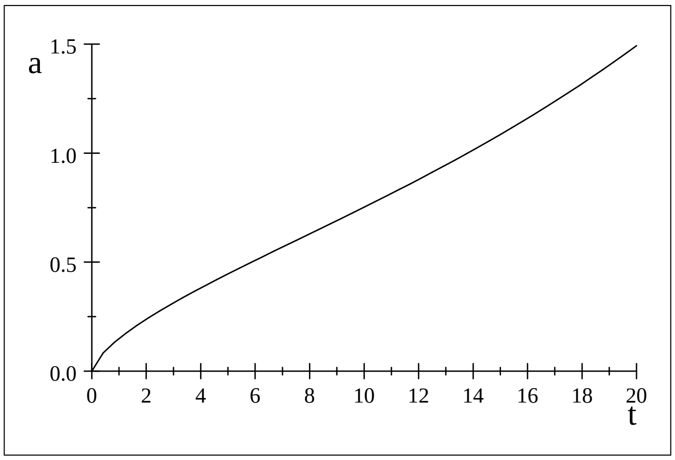
Inverting (2) gives (exercise: show that this is equivalent to equation (29.129) in Carroll and Ostlie, where their $R(t)$ is my $a(t)$)

$$t = \frac{1}{B}\sinh^{-1}\left[\left(\frac{a}{A}\right)^{\frac{3}{2}}\right], \tag{3}$$

which can be used to find t_0, the present age of the universe from the PLANCK parameters. To do this, set $a = 1$ in (3), which gives

$$
\begin{aligned}
t_0 &= \frac{1}{2.742 \times 10^{-18}} \sinh^{-1}\left(0.7635^{-\frac{3}{2}}\right) \\
&= 4.354 \times 10^{17}\,\mathrm{s} \\
&= 13.80 \times 10^{9}\,\mathrm{y}\,.
\end{aligned}
$$

Plot of scale factor $a(t)$ versus t measured in billions of years:



3. Cosmological Redshift

As light travels from a galaxy to us, the universe expands, and the wavelength of the light stretches in proportion to the amount of expansion of the universe. This means that if λ_e is the wavelenth of the emitted light and λ_o is the wavelength of the light that we observe, then

$$
\frac{\lambda_o}{\lambda_e} = \frac{a(t_0)}{a(t_e)},
$$

where t_e is the time at which the galaxy emitted the light. Redshift is quantified by the relative change in wavelength z, which is expressed as

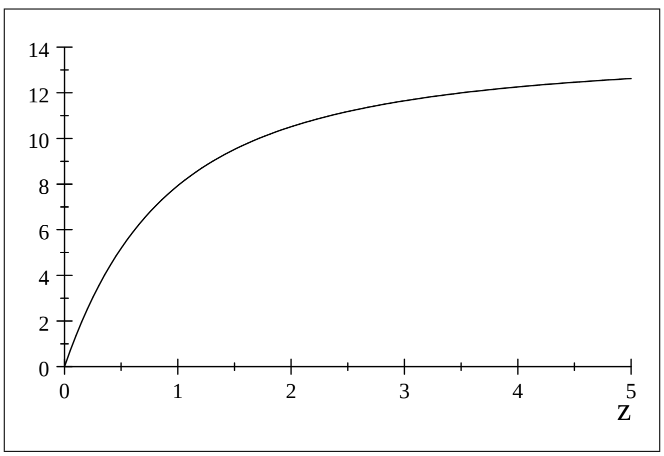
$$
\begin{aligned}
z &= \frac{\lambda_o - \lambda_e}{\lambda_e} \\
&= \frac{\lambda_o}{\lambda_e} - 1 \\
&= \frac{a(t_0)}{a(t_e)} - 1. \qquad (4)
\end{aligned}
$$

4. Look-Back Time and Distance

Given z, the redshift of a galaxy that we observe now, we can find t_e,

$$\begin{aligned}
1+z &= \frac{a(t_0)}{a(t_e)} \\
a(t_e) &= \frac{a(t_0)}{1+z} \\
A\sinh^{\frac{2}{3}}(Bt_e) &= \frac{a(t_0)}{1+z} \\
\sinh(Bt_e) &= \left(\frac{a(t_0)}{A(1+z)}\right)^{\frac{3}{2}} \\
t_e &= \frac{1}{B}\sinh^{-1}\left(\left[\frac{a(t_0)}{A(1+z)}\right]^{\frac{3}{2}}\right).
\end{aligned}$$

The time of travel for the light is the difference $t_0 - t_e$ between the time at which the light is observed and the time at which the light was emitted, and is called the look-back time. Plot of look-back distance (in billions of light-years) versus redshift z:



For example, for an observed $z = 0.4$, light was emitted at cosmological time

$$\begin{aligned}
t_e &= \frac{1}{2.742\times10^{-18}\,\mathrm{s}^{-1}}\sinh^{-1}\left(\left[\frac{1}{0.7635\,(1+0.4)}\right]^{\frac{3}{2}}\right) \\
&= 2.963\times10^{17}\,\mathrm{s} \\
&= 9.389\times10^{9}\,\mathrm{y},
\end{aligned}$$

the look-back time is

$$\begin{aligned}
t_0 - t_e &= (13.8 - 9.4)\times10^{9}\,\mathrm{y} \\
&= 4.4\times10^{9}\,\mathrm{y},
\end{aligned}$$

and the look-back distance is 4.4 billion light-years.

5. Proper Distance

The time t_e can be used to find the comoving coordinate r_e of the object that emitted the light. Since the light follows a lightlike path, the spacetime interval all along its path is $ds^2 = 0$, so (1) gives

$$\begin{aligned} 0 &= c^2 dt^2 - a(t)^2 dr^2 \\ -a(t)\, dr &= c dt. \end{aligned}$$

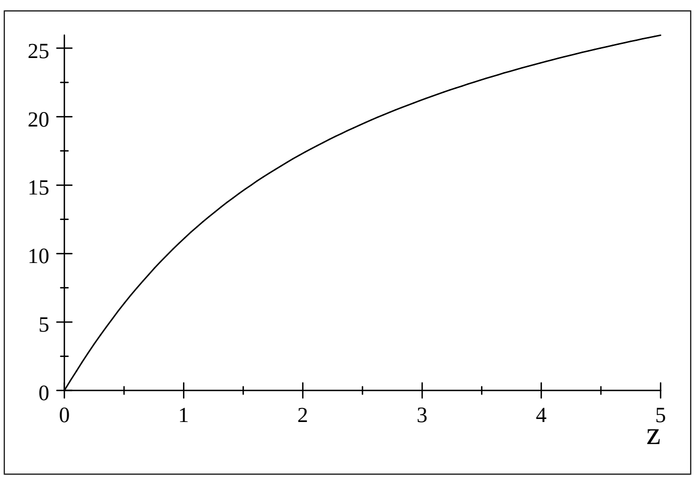
The negative sign arises because the light starts at positive r_e and is observed by us at $r = 0$. Integrating this gives

$$\begin{aligned} -\int_{r_e}^{0} dr &= c \int_{t_e}^{t_0} \frac{dt}{a(t)} \\ r_e &= c \int_{t_e}^{t_0} \frac{dt}{a(t)}. \end{aligned}$$

The proper distance between where we are now ($t = t_0$) and where the galaxy is now (i.e., not where the galaxy was when the light was emitted) $d(t_0)$ is given by

$$\begin{aligned} d(t_0) &= a(t_0)\, r_e \\ &= r_e, \end{aligned}$$

since $a(t_0) = 1$. Plot of present proper distance (in billions of light-years) versus redshift z:



For light observed at $z = 0.4$, the present proper distance to the galaxy is

$$\begin{aligned} d\left(t_0\right) &= r_e \\ &= c\int_{t_e}^{t_0} \frac{dt}{a\left(t\right)} \\ &= \frac{c}{0.7635}\int_{2.963\times10^{17}\,\mathrm{s}}^{4.354\times10^{17}\,\mathrm{s}} \frac{dt}{\sinh^{\frac{2}{3}}\left(2.742\times10^{-18}\,\mathrm{s}^{-1}\,t\right)} \\ &= c\times1.644\times10^{17}\,\mathrm{s} \\ &= c\times5.212\times10^{9}\,\mathrm{y} \\ &= 5.212\times10^{9}\mathrm{light-years}. \end{aligned} \tag{5}$$

6. Cosmological Horizon

Because both the age of the universe and the speed of light are finite, there is a cosmological horizon, beyond which we cannot see. The farthest we can look back is the present age of the universe, 13.8 billion years, which is the look-back distance of the cosmic horizon. Where are the ojects now that were then on the cosmic horizon, i.e., what is the current proper distance of something on the cosmic horizon? This is found by setting $t_e = 0$ in (5)

$$\begin{aligned} d_{\mathrm{horizon}} &= c\int_{0}^{t_0} \frac{dt}{a\left(t\right)} \\ &= \frac{c}{0.7635}\int_{0\,\mathrm{s}}^{4.354\times10^{17}\,\mathrm{s}} \frac{dt}{\sinh^{\frac{2}{3}}\left(2.742\times10^{-18}\,\mathrm{s}^{-1}\,t\right)} \\ &= c\times1.487\times10^{18}\,\mathrm{s} \\ &= c\times47.15\times10^{9}\,\mathrm{y} \\ &= 47.15\times10^{9}\mathrm{light-years}. \end{aligned}$$

Using $t_e = 0$ in (2) and (4) shows that the cosmic horizon is a surface of infinite redshift.

The calculated value for d_{horizon} is somewhat larger than the accepted value of 46 billion light-years because t_e should actually be the time from which we receive the Comic Microwave Background (CMB) radiation.