Toward consistent branch and stem dynamic models.
Working progress draft.

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1 Introduction

The objective is to build models that explain the growth of branch and stem diameters in an integrated way, being able to generate the observed patterns for different silvicultural treatments.

This is just a periodic record of progress to date, largely for my own use. As such, it does not pretend to be definitive nor very readable. I realize that more explanation may be needed for a full understanding.

2 Some previous work

There is a great deal of information on the subject in the thesis of F. Colin [4]. In particular, data and regression models for the relationship between branch diameter and distance from the tip. As indicated there, those are descriptive empirical models, and do not address the developmental dynamics. We try to interpret those observations introducing the dynamic aspects here. Also to link with stem diameter growth.

Task 5 of the EEC forestry project obtained further branch diameter regressions with additional data [5]. The Douglas fir results from the German team are shown in Appendix 1. The graphs show the basal areas for the whorl mean diameter branch. Presumably this variable would be closely proportional to the total sectional area of branches per unit of stem length, which may be a better variable for a first mechanistic model. The equation for each tree can be written, with the notation to be used later, as

$$y(x,H) = (\pi/4)(ax/H + 2)^3 e^{-2bx/H} \ ,$$

where x is the distance from the top, H the total tree height, and a and b the regression parameters. Only the values for the green crown are shown.

3 Growth distributions

I use, at least initially, continuous models (on continuous time, and describing branch sectional areas as continuous functions along the stem). Discrete models can be used more directly for simulation, require less mathematics, and may describe more realistically the presence of individual branches. But continuous models are more easily manipulated to find relationships between different aspects of the growth processes. They can later be discretized for their application.

In this section I will deal with the distribution of branch and stem increments, sectional area and diameters along the stem and over time. Actually, it is more convenient to work with total height instead of directly with time. There is a one-to-one relationship through the height-age curve that can be used for conversion later. This curve is usually almost linear over most of the life of the tree, so we can also think of height and time as almost equivalent.

Notation:

H: total tree height
h, u: height above ground
$x = H - h, v$: distance from tip
$y(x, H) = y'(h, H)$: branch sectional area, per unit of stem length
$z(x, H) = z'(h, H)$: stem sectional area
$f(x, H)$: branch sectional area increment (relative to $\mathrm{d}H$)
(All values under bark).

3.1 Branches

By definition,

$$\frac{\mathrm{d}y'(h,H)}{\mathrm{d}H} = f(H-h,H)\ ; \qquad y'(h,h) = y_0\ .$$

There may be a non-zero sectional area y_0 at the time of formation of the branches, essentially the size of the pith. Then,

$$y'(h,H) = y_0 + \int_h^H f(u-h,u)\,\mathrm{d}u = y_0 + \int_0^{H-h} f(v,v+h)\,\mathrm{d}v$$

$$\begin{aligned} y(x,H) = y'(H-x,H) &= y_0 + \int_{H-x}^{H} f(u-H+x,u)\,\mathrm{d}u \\ &= y_0 + \int_0^x f(v,v+H-x)\,\mathrm{d}v\ . \end{aligned}$$

The increment $f(x, H)$ should become zero beyond some distance x from the tip (in general, varying with H). That is, branches do not grow in diameter below this “base of the active crown” (BAC). Non-growing branches may still remain alive for some time, so the green crown base would be somewhat lower. The relationship between these two crown bases will have to be examined later.

Under some circumstances, for example if the stand density remains constant, growth could be assumed to be approximately time-invariant (stationary). That is, independent of H: $f(x,H) \equiv f(x)$, $y(x,H) \equiv y(x)$. The crown just moves up, without changing size or shape. Then, in this simple case,

$$y(x) = y_0 + \int_0^x f(v)\,\mathrm{d}v\ .$$

Note that under invariance the branch diameters below the BAC remain constant (dying or dead branches). The decrease in branch diameters near or below the crown base that is often observed must be due to an increase in $f(x,H)$ with H associated to decreasing stand density.

3.2 Stem

We will assume that the "pipe theory", developed by Jaccard and others around the end of last century [2] and re-discovered by Shiniozaki *et al* in the 60's, holds. At least in terms of increments. That is, the increment in stem sectional area at some point equals the increment in sectional area of all branches above that point. Actually, a simple proportionality could easily be accomodated, to acount for different conducting efficiencies, etc. Then,

$$\frac{\mathrm{d}z'(h,H)}{\mathrm{d}H} = \int_h^H \frac{\mathrm{d}y'(u,H)}{\mathrm{d}H}\,\mathrm{d}u = \int_h^H f(H-u,H)\,\mathrm{d}u = \int_0^{H-h} f(v,H)\,\mathrm{d}v\ ,$$

with $z'(h,h) = z_0$ being the sectional area at the tip (pith).

Integrating, and changing orders of integration, it is found that

$$z'(h,H) = z_0 + \int_h^H \frac{\mathrm{d}z'(h,w)}{\mathrm{d}H}\,\mathrm{d}w = z_0 + \int_h^H \int_h^w f(w-u,w)\,\mathrm{d}u\,\mathrm{d}w$$

$$= z_0 + \int_h^H \int_0^{w-h} f(v,w)\,\mathrm{d}v\,\mathrm{d}w = z_0 + \int_0^{H-h}\int_{v+h}^H f(v,w)\,\mathrm{d}w\,\mathrm{d}v$$

and

$$z(x,H) = z_0 + \int_{H-x}^H \int_0^{w-x+H} f(v,w)\,\mathrm{d}v\,\mathrm{d}w$$

$$= z_0 + \int_0^x \int_{v+H-x}^H f(v,w)\,\mathrm{d}w\,\mathrm{d}v\ .$$

Under invariance:

$$z(x) = z_0 + \int_0^x (x-v) f(v)\,\mathrm{d}v\ .$$

3.3 Stem, in terms of y, y'

$$z'(h,H) = z_0 + \int_h^H \int_h^w f(w-u,w)\,\mathrm{d}u\,\mathrm{d}w = z_0 + \int_h^H \int_u^H f(w-u,w)\,\mathrm{d}w\,\mathrm{d}u$$
$$= z_0 - y_0(H-h) + \int_h^H y'(u,H)\,\mathrm{d}u$$

$$z(x,H) = z_0 - y_0 x + \int_{H-x}^H y(H-u,H)\,\mathrm{d}u = z_0 - y_0 x + \int_0^x y(v,H)\,\mathrm{d}v\ .$$

This shows that the pipe theory is implied also for the accumulated branch and stem sectional areas, not just for the increments.

3.4 Summary

$$y'(h,H) = y_0 + \int_h^H f(u-h,u)\,\mathrm{d}u = y_0 + \int_0^{H-h} f(v,v+h)\,\mathrm{d}v \tag{1}$$

$$y(x,H) = y_0 + \int_{H-x}^H f(u-H+x,u)\,\mathrm{d}u = y_0 + \int_0^x f(v,v+H-x)\,\mathrm{d}v \tag{2}$$

$$z'(h,H) = z_0 + \int_h^H \int_0^{u-h} f(v,u)\,\mathrm{d}v\,\mathrm{d}u = z_0 + \int_0^{H-h} \int_{v+h}^H f(v,u)\,\mathrm{d}u\,\mathrm{d}v \tag{3}$$

$$z'(h,H) = z_0 - y_0(H-h) + \int_h^H y'(u,H)\,\mathrm{d}u \tag{4}$$

$$z(x,H) = z_0 + \int_{H-x}^H \int_0^{u-H+x} f(v,u)\,\mathrm{d}v\,\mathrm{d}u = z_0 + \int_0^x \int_{v+H-x}^H f(v,u)\,\mathrm{d}u\,\mathrm{d}v \tag{5}$$

$$z(x,H) = z_0 - y_0 x + \int_0^x y(v,H)\,\mathrm{d}v = z_0 + \int_0^x [y(v,H) - y_0]\,\mathrm{d}v \tag{6}$$

Under H-invariance (time-invariance):

$$y(x) = y_0 + \int_0^x f(v)\,\mathrm{d}v \tag{7}$$

$$z(x) = z_0 + \int_0^x (x-v) f(v)\,\mathrm{d}v = z_0 + \int_0^x [y(v) - y_0]\,\mathrm{d}v \tag{8}$$

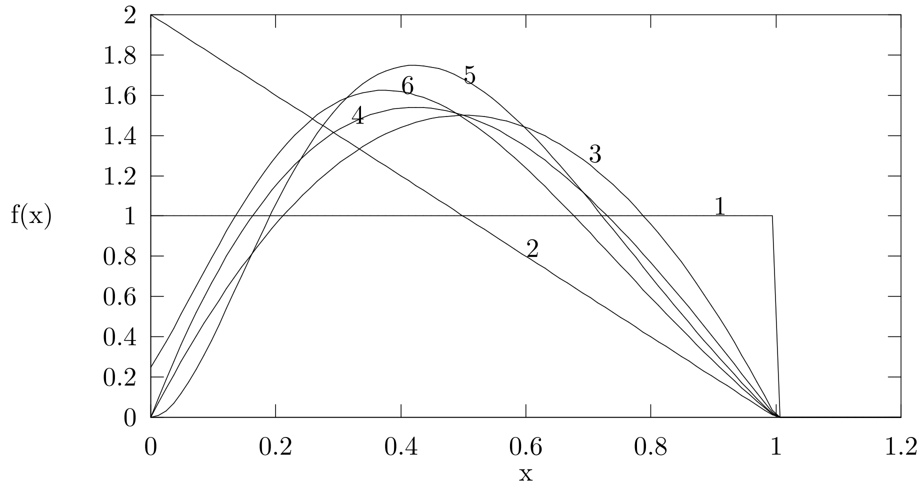

Figure 1: Examples of branch sectional distributions. Average $c = 0.078$ used for example 6

3.5 Examples

Some examples of growth functions and resulting branch and stem sizes. Time invariance is assumed here. To facilitate comparisons the functions are normalized so that the length of the active crown is one ($H - \mathrm{BAC} = 1$), and the total branch sectional increment is one ($y(1) = 1 + y_0$). Then $f(x)$ can be interpreted as a relative increment distribution. Results could be re-scaled as necessary later.

The resulting curves are displayed in Figures 3.5 to 2. Fig. 2 is more directly comparable with the published relationships; the "diameters" are the square roots of the sections.

1. The simplest branch sectional increment function: constant at 1 above the BAC.

   $$f(x, H) = f(x) = 1 \text{ for } x \leq 1 \;\; 0 \text{ otherwise.}$$

   Then,

   $$y(x) = y_0 + \int_0^x f(v)\,\mathrm{d}v = \begin{cases} y_0 + x & \text{if} \quad x \leq 1 \\ y_0 + 1 & \text{if} \quad x \geq 1 \end{cases}$$

   $$z(x) = z_0 + \int_0^x [y(v) - y_0]\,\mathrm{d}v = \begin{cases} z_0 + x^2/2 & \text{if} \quad x \leq 1 \\ z_0 + x - 1/2 & \text{if} \quad x \geq 1 \end{cases}$$

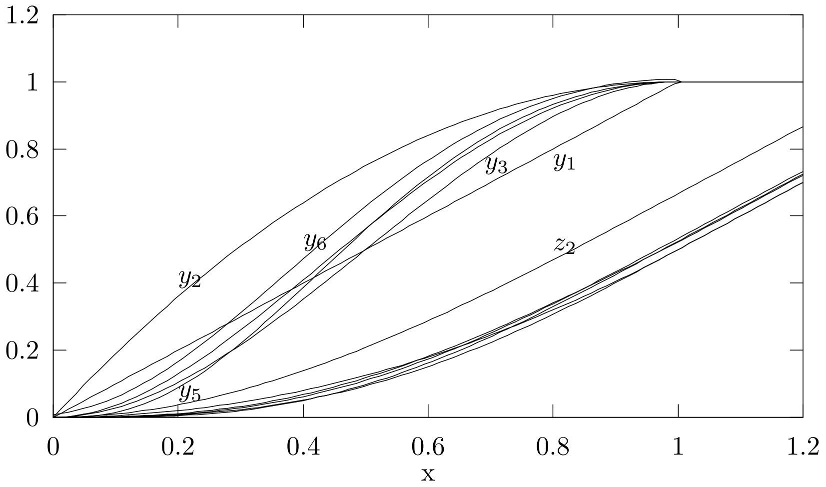

Figure 2: Branch and stem relative sections for the examples

As always in the stationary case, z is linear below the crown, giving the usually-assumed paraboloid stem shape. Here the form of the stem within the crown is conical, another common assumption, but this is not so with other branch increment distributions (see the examples below).

2. Increment linearly decreasing.

$$f(x) = 2\max\{1-x, 0\} = 2(1-x)_+$$

Then,

$$y(x) = \begin{cases} y_0 + x(2-x) & \text{if} \quad x \le 1 \\ y_0 + 1 & \text{if} \quad x \ge 1 \end{cases}$$

$$z(x) = z_0 + \int_0^x [y(v) - y_0]\,\mathrm{d}v = \begin{cases} z_0 + \frac{1}{3}x^2(3-x) & \text{if} \quad x \le 1 \\ z_0 + x - 1/3 & \text{if} \quad x \ge 1 \end{cases}$$

3. Parabolic increment distribution.

$$f(x) = [6x(1-x)]_+$$

$$y(x) = \begin{cases} y_0 + x^2(3-2x) & \text{if} \quad x \le 1 \\ y_0 + 1 & \text{if} \quad x \ge 1 \end{cases}$$

$$z(x) = z_0 + \int_0^x [y(v) - y_0]\,\mathrm{d}v = \begin{cases} z_0 + \frac{1}{2}x^3(2-x) & \text{if} \quad x \le 1 \\ z_0 + x - 1/2 & \text{if} \quad x \ge 1 \end{cases}$$

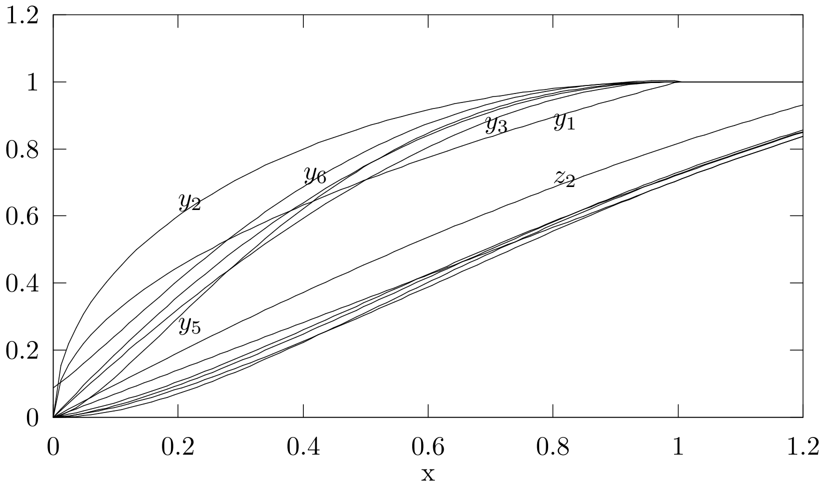

Figure 3: Branch and stem relative diameters for the examples

4. The increment implied by the segmented polynomial of Colin and Houllier [3], in the stationary case. Above the BAC it is a quadratic for branch diameter (assume constant number of branches per unit stem length):

$$y(x) = \begin{cases} [x(2-x)]^2 & \text{if} \quad x \leq 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

Then,

$$f(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = [4x(1-x)(2-x)]_+$$

$$z(x) = \begin{cases} z_0 + \frac{1}{15}x^3(20 - 15x + 3x^2) & \text{if} \quad x \leq 1 \\ z_0 + x - 7/15 & \text{if} \quad x \geq 1 \end{cases}$$

5. The increment implied by the model used by Task 5 of the EEC project, in the stationary case [5]. Simplified by making the *io* parameter equal to zero.

$$y(x) = \begin{cases} (xe^{1-x})^3 & \text{if} \quad x \leq 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

Then,

$$f(x) = [3x^2(1-x)e^{3(1-x)}]_+$$

$$z(x) = \begin{cases} z_0 + 2e^3/27 - \frac{1}{27}e^{3(1-x)}(9x^3 + 9x^2 + 6x + 2) & \text{if} \quad x \leq 1 \\ z_0 + x - (53 - 2e^3)/27 & \text{if} \quad x \geq 1 \end{cases}$$

6. The increment implied by the model used by Task 5 of the EEC project, in the stationary case, with their $io = 2$ (Appendix 1). The normalization makes the whole thing a bit messy.

   $$y(x) = \begin{cases} y_0 + \frac{(dx+c)^3 e^{3d(1-x)} - c^3 e^{3d}}{1-e^{3d}} & \text{if} \quad x \leq 1 \\ 1 & \text{if} \quad x \geq 1 \end{cases}$$

   where $c = \frac{4b}{3a}$, $d = 1 - c$, and $y_0 = c^3 e^{3d}/(1 - c^3 e^{3d})$.

   For the Germany Douglas fir data the estimated values of c range from 0.061 to 0.093, with a mean of 0.078. With $c = 0$ we have the previous example.

   It is found,

   $$f(x) = [\frac{3d^2(dx + c)^2(1 - x)e^{3d(1-x)}}{1 - c^3 e^{3d}}]_+$$

   $$z(x) = \begin{cases} z_0 + [(e^{3z}(9z^3 - 36z^2 + 51z - 26) & \\ -e^{3d}(27c^3 dx + 9d^3 - 36d^2 + 51d - 26)] & \\ /[27(1 - c^3 e^{3d})] & \text{if} \quad x \leq 1 \\ z_0 + x - \frac{e^{3d}(-27c^4 + 9d^3 - 36d^2 + 51d - 26) + 53}{27(1 - c^3 e^{3d})} & \text{if} \quad x \geq 1 \end{cases}$$

3.6 Comments

Considering the high variability in the data, perhaps a simple model such as the parabolic of Example 3 may be a reasonable approximation, at least for the moment.

It is not entirely clear why the increment distribution should have the observed shape. It would be desirable to look further into mechanistic models for this.

Models where branch growth rate is a function of current branch size, and not just depth into the canopy, might become necessary. That would involve (partial?) differential equations, but it should not be too difficult to handle.

4 Effect of stand density

The examples looked at the stationary case. More generally, the crown growth and dimensions will vary over time (and H), following changes in stand density.

We may start from a normalized branch growth distribution $\bar{f}(x)$, and scale it as

$$f(x, H) = \alpha \bar{f}(x/\beta) \ , \tag{9}$$

where the scale factors α and β vary with H. The factor β is roughly proportional to crown depth, and α to the average branch sectional increment.

4.1 Closed stands

The total branch sectional area growth, and therefore the stem basal area increment, is $\alpha\beta$. Let us assume for now that the basal area increment per hectare in a closed stand is constant, c. Then, with N being the number of stems per hectare, we must have (in "average")

$$\begin{gathered} c/N = \\ \int_0^\infty f(x, H)\,\mathrm{d}x = \int_0^\infty \alpha \bar{f}(x/\beta)\,\mathrm{d}x = \alpha\beta \int_0^\infty \bar{f}(u)\,\mathrm{d}u \\ = \alpha\beta \ . \end{gathered}$$

Actually, we assume for now that all trees are identical, and will worry about size variation later. What about the relative values of α and β?

Many canopy models have assumed similarity in crown shapes, with crown diameter and length proportional to tree spacing. That is, $\alpha \propto N^{-1/2}, \beta \propto N^{-1/2}$. Actually, studies with radiata pine in New Zealand have shown that this is not so. Beekhuis [1] found that canopy depth is related to both average spacing and top height. Models of the form *depth* $= aN^b H^c$ have been used (García published in [8], and unpublished work by Lawrence and Dunningham). The effect of H is much less important than that of spacing. Ignoring H, crown depth was roughly proportional to the square root of average spacing. Then, we may take

$$\alpha = aN^{-3/4} \ , \qquad \beta = bN^{-1/4} \ . \tag{10}$$

Note that, with branch extension proportional to the spacing $N^{-1/2}$, this would imply branch diameters approximately proportional to the branch length to the power of 3/4. This seems reasonable, meaning a branch form somewhere between parabolic (1/2) and conic (1).

Having the history of stand density over time (and thus N as a function of H), equations (9) and (10), together with the relationships from the previous section, allow us to simulate the branch and stem development of the "average" tree in any closed stand.

4.2 Open stands

The above is probably sufficient to model trees over most of their life under typical European silviculture employing light to moderate thinnings. The modelling of less than fully closed stands may be necessary for the early growth, and for regimes using heavy thinning and/or pruning.

Here we may think of α and β describing a "potential" or "equilibrium" crown. The potential crown length (and branch size) increases instantaneously with a thinning. Then the crown base remains stationary while the potential is reached through height and branch growth. Pruning also produces a similar discrepancy between actual and potential crown. The same happens in the early growth, while crown length is limited by the tree total height.

Let us try finding the instantaneous change in growth caused by a thinning. The amount of assimilating materials (foliage, active roots) is reduced proportionally to the basal area or volume removed. Expressed as a proportion of the equilibrium amount, this is called *relative closure* in [6, 7]. If the stand is closed before the thinning, and with our uniform tree sizes, the relative closure after the thinning is $R = N/N_0$, where N_0 and N are the numbers of trees before and after thinning, respectively. The increment per hectare, however, does not drop in the same proportion. In [7] the (total volume) increment per hectare, relative to that in a closed stand, is called *occupancy*. Let us take it in terms of basal area, and denote it by Ω.

Before the thinning the increment is $c = ab$, and the crown parameters are $\alpha_0 = aN_0^{-3/4}$ and $\beta_0 = bN_0^{-1/4}$. Inmediately after the thinning the base of the active crown (BAC) remains at $x = \beta_0$, so the basal area increment per hectare is

$$N \int_0^{\beta_0} \alpha \bar{f}(x/\beta)\,\mathrm{d}x = N\alpha\beta \int_0^{\beta_0/\beta} \bar{f}(u)\,\mathrm{d}u = ab \int_0^{(N/N_0)^{1/4}} \bar{f}(u)\,\mathrm{d}u \ .$$

The relationship between occupancy and closure is then

$$\Omega = \int_0^{R^{1/4}} \bar{f}(u)du \ . \tag{11}$$

For the parabolic model of example 3 we have

$$\Omega = R^{1/2}(3 - 2R^{1/4}) \ .$$

This function is plotted in Figure 4.2, together with that of [6]. Admittedly the agreement is not too good.

5 Where to from here?

It is not entirely clear to me if this line is worth pursuing. Perhaps some of the fundamental premises are wrong or too unrealistic. Maybe something

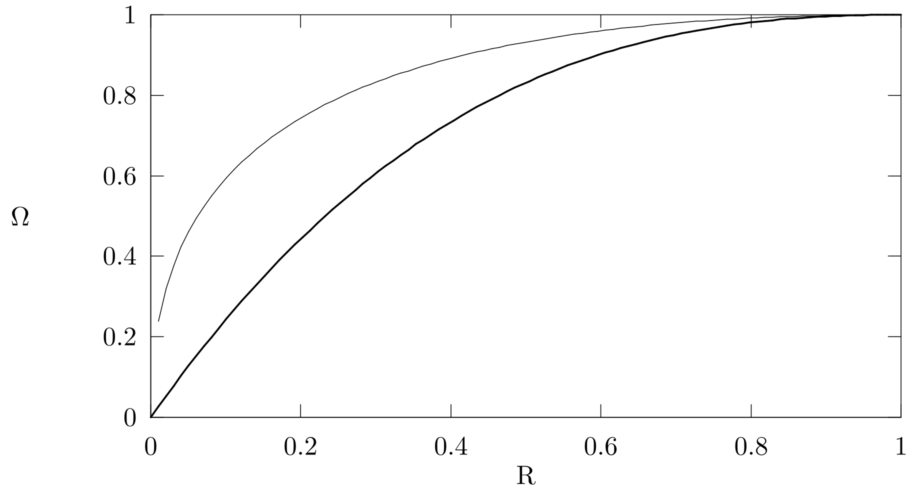

Figure 4: Occupancy *vs* closure. Lower curve is from [6]

similar has already been done by others. On the other hand, this approach might actually lead to compatible, integrated, well-behaved models. Opinions are sought.

If it is decided to continue, there are obviously some loose ends, and things that are not too clear and need more thinking. The modelling of open stands would have to be developed further. Also the isue of variability in sizes, although that does not seem too dificult. Some graphic simulations and testing against field data could be tried. Or the whole thing could just as well simply end here.

References

[1] J. Beekhuis. Crown depth of radiata pine in relation to stand density and height. *New Zealand Journal of Forestry*, 10(1):43–61, 1965.

[2] M. Busgen and E. Munch. *The structure and life of forest trees.* Chapman & Hall, London, 1929.

[3] F. Colin and F. Houllier. Branchiness of norway spruce in north-eastern france: modelling vertical trends in maximum nodal branch size. *Ann. Sci.For.*, 48:679–693, 1991.

[4] Francis Colin. *Analyse et modelisation de la branchaison de l'epicea commun en vue de predire la qualite de la ressource.* These, docteur en

sciences forestieres, Ecole Nationale du Génie rural, des Eaux et Forêts, 1992.

[5] Francis Colin. Modelling branching in the tree in relation to silvicultural practices. Final Task Report Task 5, EEC Forest Project, Contract No. MA2B-0024, 1994?

[6] Oscar García. Growth modelling — new developments. In H. Nagumo and Y. Konohira, editors, *Japan and New Zealand Symposium on Forestry Management Planning*, pages 152–158. Japan Association for Forestry Statistics, 1989.

[7] Oscar García. Growth of thinned and pruned stands. In R. N. James and G. L. Tarlton, editors, *New Approaches to Spacing and Thinning in Plantation Forestry: Proceedings of a IUFRO Symposium, Rotorua, New Zealand, 10–14 April 1989*, pages 84–97. Ministry of Forestry, FRI Bulletin No. 151, 1990.

[8] G. G. West, R. L. Knowles, and A. R. Koehler. Model to predict the effects of pruning and early thinning on the growth of radiata pine. FRI Bulletin No. 5, Forest Research Institute, New Zealand Forest Service, 1982.

Toward consistent branch and stem dynamic models.
II. Simulating branch and stem diameters.

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1 Introduction

In the previous report [3] tree basal area growth was derived from crown dimensions. Here, instead, basal area is assumed given, possibly from predictions with a growth model. Knowing also the base of the active crown (BAC) and the potential BAC, the branch and stem sectional areas along the stem are obtained through the pipe theory. Therefore, assumptions about crown effects on growth, crown efficiency, etc., are not needed. Total growth is known, and just needs to be distributed along the stem.

2 Data

For the simulation the model is discretized in time, for example using annual increments. For each year we know the tree dbh D, height H, BAC C, and potential BAC P.

The BAC may be somewhat higher than the actual green crown base, because there may be branches at the base of the crown that are not growing but are still alive (the BAC is the green crown base or the level of zero branch diameter growth, whichever is higher). This is actually necessary to explain the decreasing branch diameter toward the base of the crown that is often observed.

The potential BAC P is the level at which there would be zero branch growth. It may be equal to C, if the stand is fully closed and the canopy base is rising. Or it may be lower than C, if the stand has just been opened by a thinning, or in young stands, or if the tree has been pruned. It is a function of stand density, and may be estimated as a function of trees per hectare and top height [1].

3 Theory

As in (9) of [3], assume a branch sectional area growth distribution

$$\alpha f(x/\beta) ,$$

where x is distance from the tip, f is a normalized density distribution, and α and β are functions of time.

The normalization means that $f(0) = f(1) = 0$, and $\int_0^1 f(u)\,\mathrm{d}u = 1$. Therefore, $\beta = H - P$. Also, the total branch sectional area increment is

$$\int_0^{H-C} \alpha f(x/\beta)\,\mathrm{d}x = \alpha\beta \int_0^{(H-C)/\beta} f(u)\,\mathrm{d}u = \alpha(H-P)F(\frac{H-C}{H-P}) ,$$

writing $\int_0^v f(u)\,\mathrm{d}u = F(v)$ for the cummulative normalized distribution. According to the pipe theory this increment must equal the known tree basal area increment Δb, thus determining the value of α. The tree basal area increment is available from the D's.

We conclude, then, that the sectional area increment of the branches at height h is

$$\Delta y = \frac{\Delta b f(\frac{H-h}{H-P})}{(H-P)F(\frac{H-C}{H-P})} \tag{1}$$

for $h \geq C$, 0 otherwise. More precisely, this is the annual increment per unit of stem length. The sectional area for any year and any h can be obtained by adding the yearly increments. Knowing the number of branches per unit of stem length, the (quadratic) mean branch diameter may be calculated.

The increment in stem sectional area at a level h is equal to the increment of all branches above h, that is, $\Delta z(h) = \int_h^H \Delta y(u)\,\mathrm{d}u$. Therefore,

$$\Delta z = \frac{\Delta b F(\frac{H-h}{H-P})}{F(\frac{H-C}{H-P})} \tag{2}$$

if $h \geq C$, $\Delta z = \Delta b$ if $h \leq C$. As with the branches, by adding the increments the stem sectional areas and diameters at any level and age may be obtained.

4 Evidence for the pipe theory

The "pipe theory", or at least its name, is usually associated with the work of Shiniozaki and others in the early 60's, but it is actually much older. It was well-known for Jaccard around the beginning of the century [2], and Pressler had done some related work earlier. J-C.H̃ervé brought to my attention a paper that attributes it to Leonardo da Vinci! [6]

In a most interesting paper [5], Jacobs reports, among other things, on an experiment supporting the theory for radiata pine plantations in Australia.

The study involved the measurement of all the branches of 36 trees for three years (ages 8 to 11), evenly distributed in spacings of 6×6, 9×9, and 12×12 feet. Graphical and statistical analysis showed that the cross sectional area of the stem at any point was always consistently close to the total sectional area of the branches lying above it.

In Scots pine (20 trees in one stand), Hari *et al* [4] found proportionality between the cross sectional area of the stem at the base of the crown and the total cross sectional area of the branches (and that of the roots, and also for main and their secondary branches). The proportionality factor was not exactly one, however, what might reflect different water conducting efficiencies or might be due to the measuring procedures. Such a coefficient could be easily incorporated in the simulations shown here. They cite several similar studies.

Many other papers have been published on the subject, usually showing the pipe model to be a reasonable approximation. Its accuracy, however, is not always good. For example, it is well-known that the area of growth rings below the crown can be far from uniform ("Pressler law"). Of course, the butt swell, related to mechanical factors different from the water conducting principles inspiring the pipe theory, is not modelled at all.

5 Examples

The simulation described above was implemented in *APL*. The program takes series of values of H, D, C and P, and generates data for plotting branch and stem sectional area and diameter curves.

A stand growth model for radiata pine in New Zealand was used to generate some plausible test data. Three silvicultural regimes were tried for a stand of site index 30 (meters at age 20). One was planted at 1600 stems per hectare and left unthinned. The second one was a typical New Zealand heavy early heavy thinning and pruning regime. The third regime applied thinnings of 30% of basal area at intervals of three years, being probably closer to some European practices. The stand densities are compared in Figure 5. The model generated annual (and before and after thinning) values of top height, number of trees, basal area per hectare, and green crown level. The potential green crown level was also calculated, with a regression on height and stand density used internally by the model.

The basal area divided by the number of trees was used to obtain a "mean tree basal area". Actually this does not corresponds exactly to any tree, because the "mean tree" changes with time and with thinning. However, it is probably good enough for illustration purposes, and a more sophisticated procedure was not deemed necessary. Before and after thinning values were averaged. The tree diameters thus obtained are shown in Figure 5.

The mean tree height was taken as the top height minus 0.5 m. For the

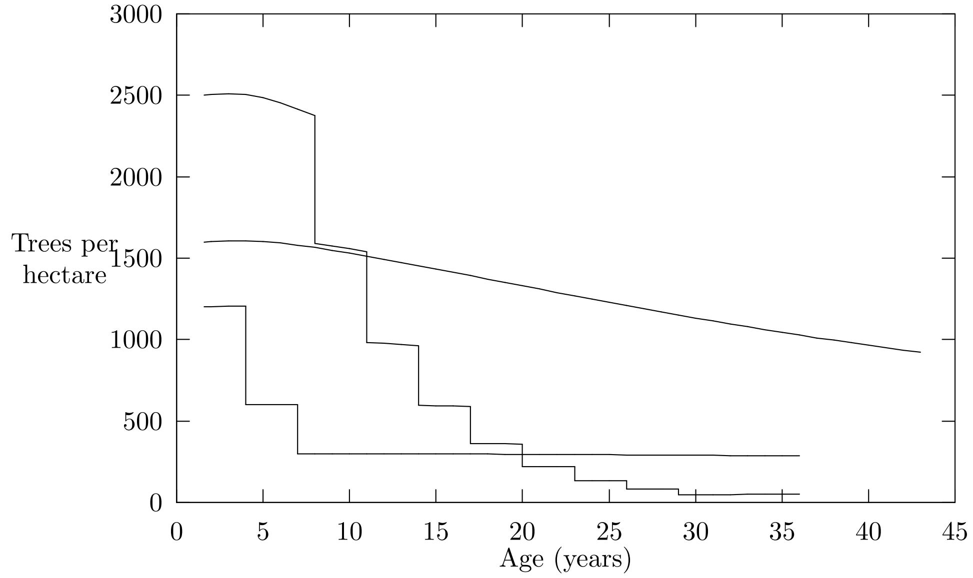

Figure 1: Stand densities for example trees: Unthinned, heavy thinning and pruning, light and frequent thinnings

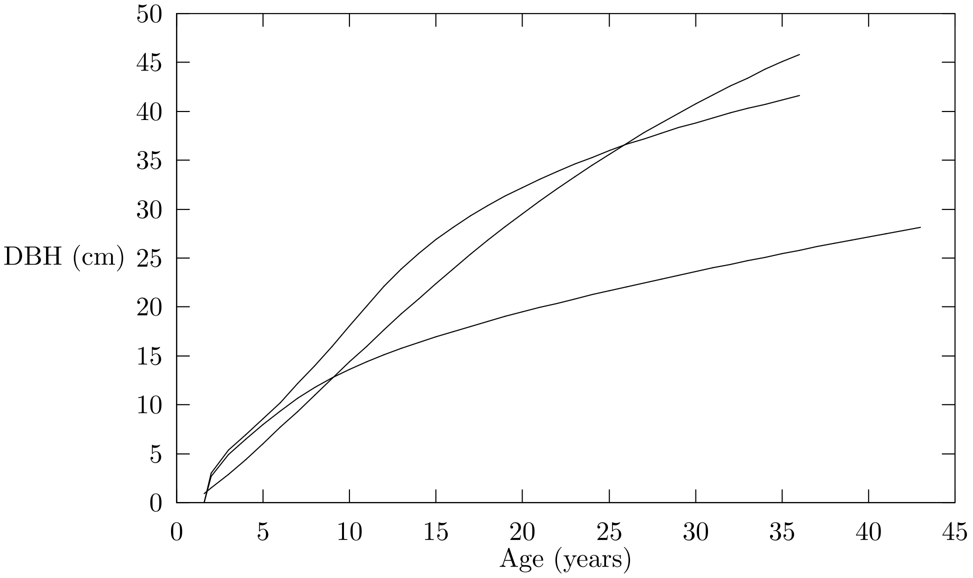

Figure 2: DBH for example trees

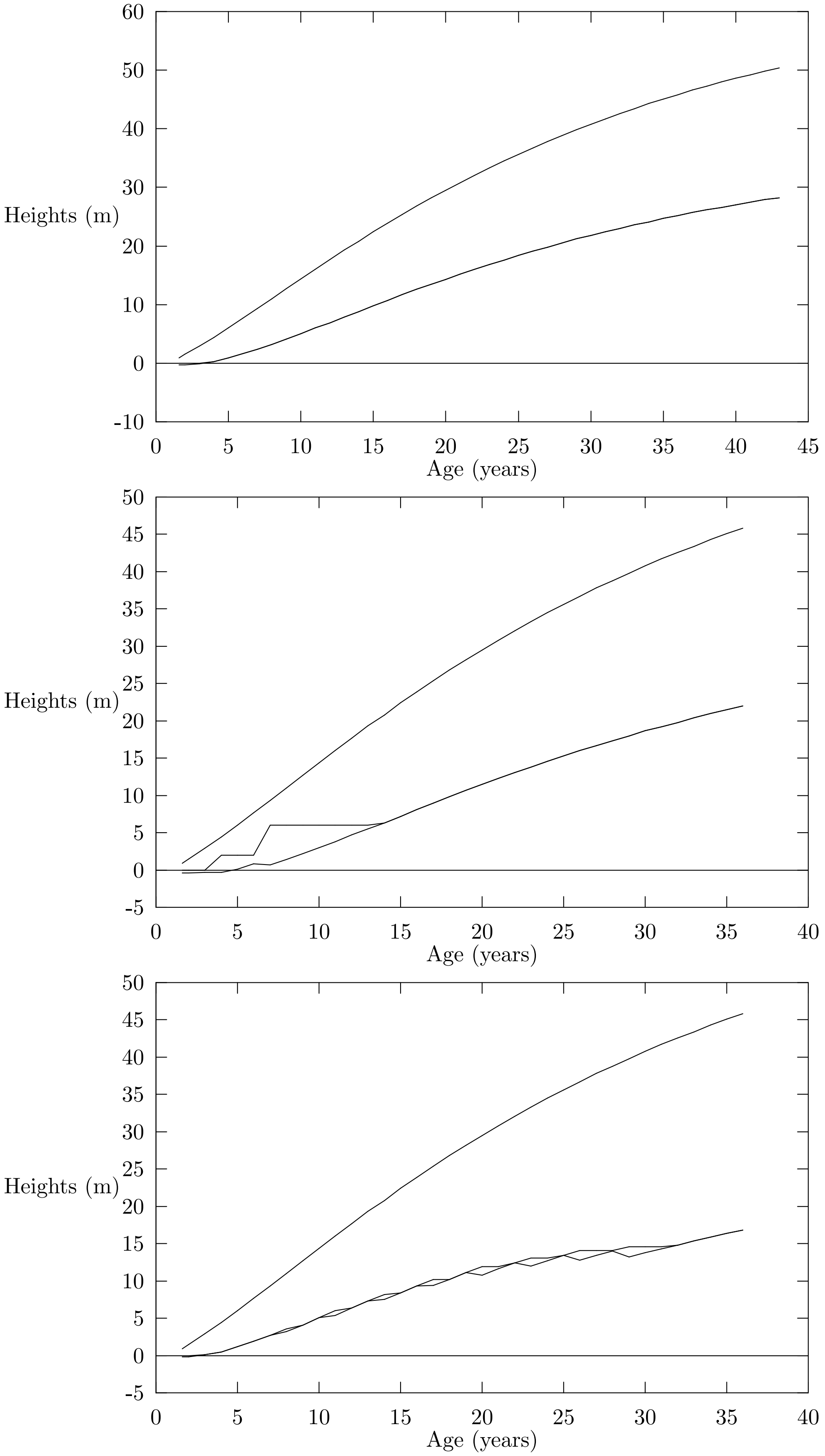

Figure 3: Total height, BAC, and potential BAC, for each of the three example trees

actual and potential BACs, 1.5 m was added to the estimated green crown level, except before the start of crown base rising and in the case of pruning. Figure 5 displays the various height trajectories for each of the three trees (regimes).

A simple quadratic distribution was used for the simulation:

$$f(x) = 6x(1 - x)$$

if $0 \leq x \leq 1$, and 0 otherwise.

The annual branch and stem sectional area increments were computed and accumulated. For the graphs of Fig. 5 a constant number of 10 branches per meter was assumed to display an average branch diameter. Each curve gives the mean branch diameters at every height for each age (ages 1.6, 2, 3, 4, 5,... years). The stem diameters are in Figure 5.

6 Discussion

This approach gives results consistent with growth model projections. Given the total increments, only their distribution along the stem needs to be modelled. The correctness of the procedure depends on the validity of the pipe theory and on the distribution function used. The effect of using different distribution functions is not obvious, but can easily be investigated. Clearly, the butt swell is not represented and would have to be modelled separately.

It is not clear to me how plausible the predictions of Fig. 4 are. At least those for the light and frequent thinnings look similar to some of those reported by F. Colin. It might be interesting to simulate the development of trees for which data is available.

Some irregularities are visible in the graphs. Apparently some are due to the discretization, especially for the fast developing crowns in young and pruned trees, and others to rounding in the input data. The growth model projections were rounded to one decimal place (or to the nearest integer in the stems per hectare). I believe that the fact that such minor perturbations in the input produce highly visible effects in predictions should be taken as a warning. It seems to me unavoidable that the results of trying to model at a high level of detail must be unreliable. It is not the first time that I find small input rounding errors amplified by a model, a nice sensitivity demonstration often masked by accurate computing!

I do not know if data exists or could be obtained to verify branch increment distributions. With only final branch diameter data it would be possible, in principle, to estimate indirectly the model components. However, that might well be an ill-defined and badly conditioned problem if the models happen to be very sensitive to noise and relatively insensitive to the underlying relationships. If my suspicions are true, then (a) the exact shape of $f(x)$ may not make much difference, (b) it may be impossible to determine

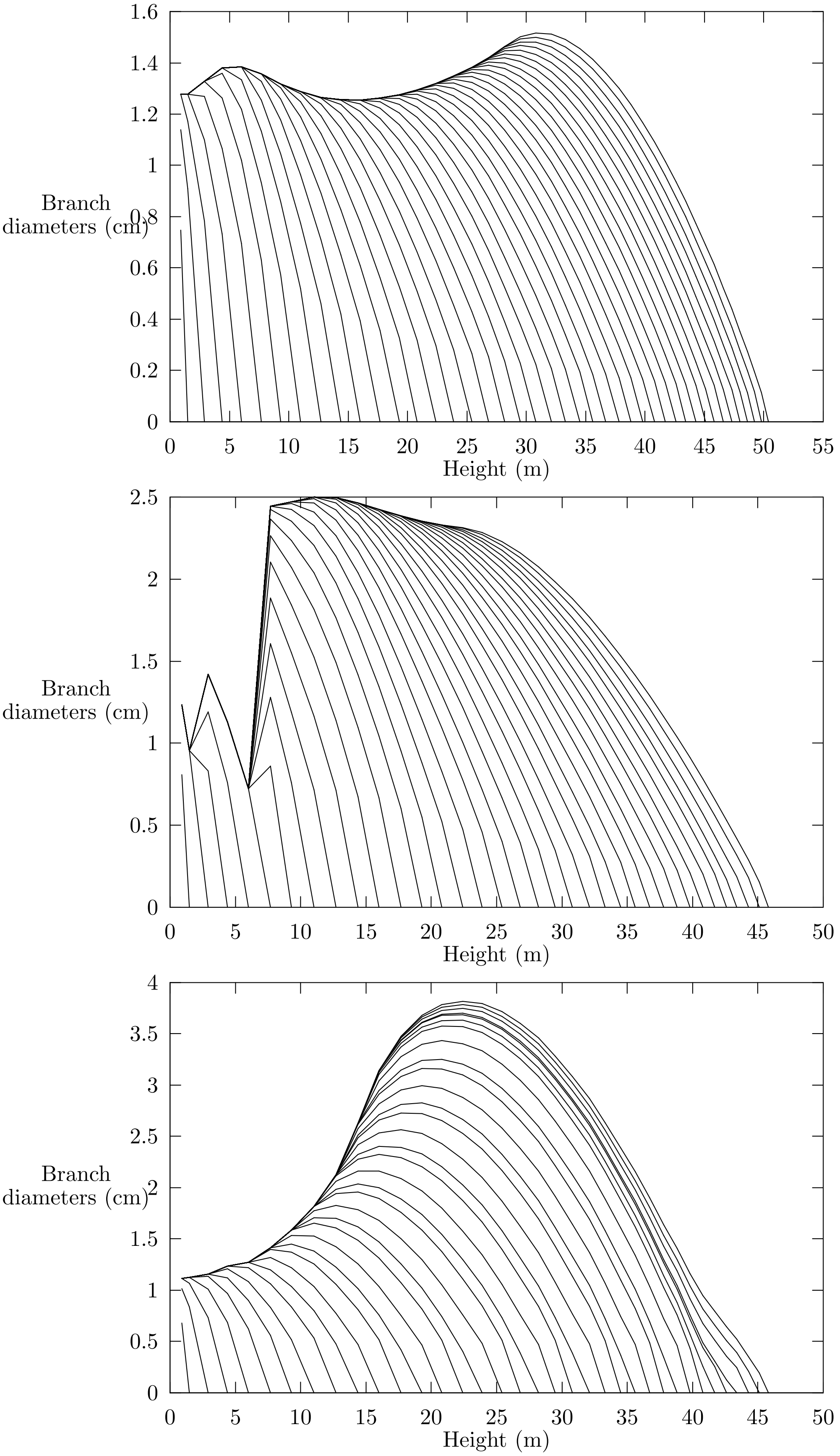

Figure 4: Simulated mean branch diameters: Unthinned, heavy thinning/pruning, light frequent thinnings

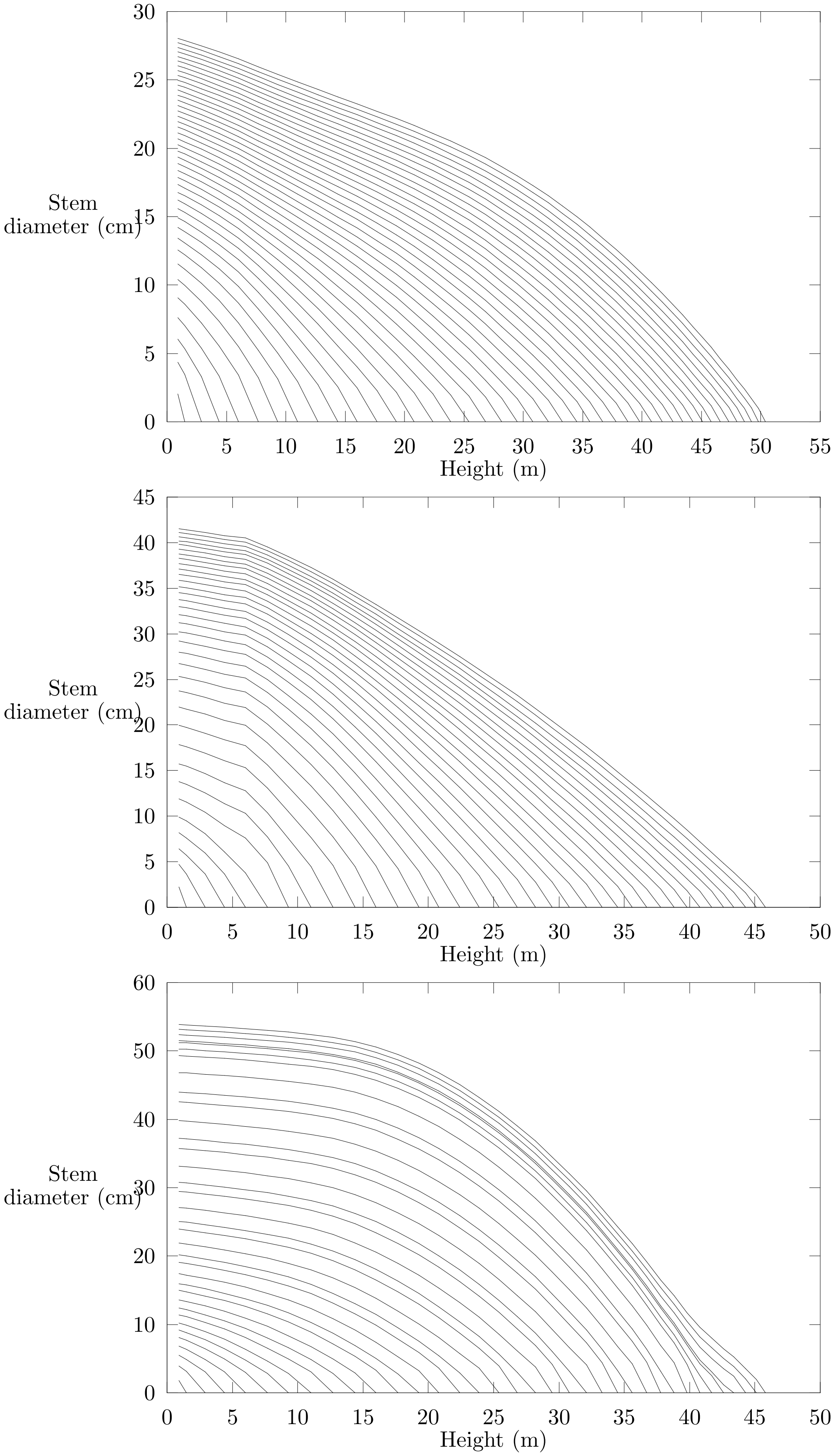

Figure 5: Simulated stem diameters: Unthinned, heavy thinning/pruning, light frequent thinnings

it from real data, and (c) whatever we do, predictions will be very rough anyway. Warning: I have not really done much thinking about all this.

References

[1] J. Beekhuis. Crown depth of radiata pine in relation to stand density and height. *New Zealand Journal of Forestry*, 10(1):43–61, 1965.

[2] M. Busgen and E. Munch. *The structure and life of forest trees.* Chapman & Hall, London, 1929.

[3] Oscar García. Toward consistent branch and stem dynamic models. working progress draft. (unpublished), December 1995.

[4] Pertti Hari, Pirkko Heikinheimo, Annikki Mäkelä, Leo Kaipiainen, Eeva Korpilahti, and Juka Samela. Trees as a water transport system. *Silva Fennica*, 20(3):205–210, 1986.

[5] M. R. Jacobs. Notes on pruning *pinus radiata.* Part I. observations on features which influence pruning. Bulletin 23, Forestry Bureau, Canberra, 1938.

[6] Michel Mendes France. De l'arbre de Leonardo da Vinci á la théorie de la dimension. *Revue du Palais de la Discoverie*, 10(91):52–60, 1981?