Functional and Logic Programming
Fall 2006
Winning and Losing Times for Nim

1 Computing the winning and losing times for Nim

In order to work out the running time of the Nim winning and losing functions we first must know what they compute.

Theorem 1. The functions `win` and `lose` are defined by

$$\texttt{win}\, n \Leftrightarrow n \bmod 5 \neq 1 \tag{1}$$

$$\texttt{lose}\, n \Leftrightarrow n \bmod 5 = 1 \tag{2}$$

Proof. We prove this by strong induction. For $n < 5$ these claims hold by examining the base cases of the corresponding SML code.

Now suppose that the result holds for all $n < k$.

First consider the case where $k\, \mathrm{m}od\, 5 = 1$. Then, by induction, for $1 \leq j \leq 4$, we have $\texttt{win}(k - j)$ is `true`, and $\texttt{lose}(k - j)$ is `false`. Again, by examining the code we see that we get $\texttt{win}(k)$ is `false`, and $\texttt{lose}(k)$ is `true`.

Next consider the case where $k\, \mathrm{m}od\, 5 \neq 1$. Set $j = (k - 1) \bmod 5$. This gives $1 \leq j \leq 4$, and, by induction, we have $\texttt{win}(k - j)$ is `false`, and $\texttt{lose}(k - j)$ is `true`. Again, by examining the code we see that we get $\texttt{win}(k)$ is `true`, and $\texttt{lose}(k)$ is `false`. Thus the claim holds for $n = k$, and by strong induction, it holds for all $n \geq 1$. □

From this we can work out that

Theorem 2. Up to constants, the running times of the functions `win` and `lose` are given by

$$R_w\, n = \begin{cases} 1 & \text{if } n < 5, \\ \sum_{j=1}^{4} R_\ell(n-j) & n \equiv 0 \text{ or } n \equiv 1 \pmod 5, \\ R_\ell(n-1) & n \equiv 2 \pmod 5, \\ R_\ell(n-1) + R_\ell(n-2) & n \equiv 3 \pmod 5, \\ R_\ell(n-1) + R_\ell(n-2) + R_\ell(n-3) & n \equiv 4 \pmod 5; \end{cases} \tag{3}$$

$$R_\ell\, n = R_w n \tag{4}$$

Proof. The fact that the losing times calculations are equal to the winning time calculations becomes evident when thinking about their dual nature. In either case, the computation short-circuits out of an `andalso` or an `orelse` when the argument of the recursive function is equal to 1 modulo 5. □

We can simplify this further by staring hard at the above formula and noticing that

$$R_w(5m + j) = 2^{j-1} R_w(5m) \quad \text{for } m \geq 1,\, 1 \leq j \leq 4. \tag{5}$$

We then have that

$$\begin{aligned} R_w(5m+5) &= \sum_{j=1}^{4} R_w(5m = 5 - j) = \sum_{j=1}^{4} R_w(5m + j) \\ &= (2^0 + 2^1 + 2^2 + 2^3) \cdot R_w(5m) = 15 \cdot R_w(5m). \end{aligned} \tag{6}$$

Putting this all together gives, for $0 \leq j \leq 4$,

$$R_w(5m + j) = \begin{cases} 1 & \text{if } m = 0, \\ 2^{\max(0, j-1)} \cdot 4 \cdot 15^{m-1} & \text{if } m > 0. \end{cases} \tag{7}$$

This function is definitely exponential, and approximately equal to $(0.13 \cdot 1.72^n)$.