Subtracting, we find that $a_{k+1} + \cdots + a_l = (c-b)m$ and thus $a_{k+1} + \cdots + a_l$ is divisible by m. □

Application 4. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of (consecutive) days during which the chess master will have played *exactly* 21 games.

Let a_1 be the number of games played on the first day, a_2 the total number of games played on the first and second days, a_3 the total number of games played on the first, second, and third days, and so on. The sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence[2] since at least one game is played each day. Moreover, $a_1 \geq 1$, and since at most 12 games are played during any one week, $a_{77} \leq 12 \times 11 = 132$.[3] Hence we have

$$1 \leq a_1 < a_2 < \cdots < a_{77} \leq 132.$$

The sequence $a_1+21, a_2+21, \ldots, a_{77}+21$ is also a strictly increasing sequence:

$$22 \leq a_1 + 21 < a_2 + 21 < \cdots < a_{77} + 21 \leq 132 + 21 = 153.$$

Thus each of the 154 numbers

$$a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$$

is an integer between 1 and 153. It follows that two of them are equal. Since no two of the numbers $a_1, a_2, \ldots, a_{77}$ are equal and no two of the numbers $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are equal, there must be an i and a j such that $a_i = a_j + 21$. Therefore on days $j+1, j+2, \ldots, i$ the chess master played a total of 21 games. □

Application 5. From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that among the integers chosen there are two such that one of them is divisible by the other.

[2] Each term of the sequence is larger than the one that precedes it.

[3] This is the only place where the assumption that at most 12 games are played during any of the 11 calendar weeks is used. Thus this assumption could be replaced by the assumption that at most 132 games are played in 77 days.

By factoring out as many 2's as possible, we see that any integer can be written in the form $2^k \times a$, where $k \geq 0$ and a is odd. For an integer between 1 and 200, a is one of the 100 numbers $1, 3, 5, \ldots, 199$. Thus among the 101 integers chosen, there are two having a's of equal value when written in this form. Let these two numbers be $2^r \times a$ and $2^s \times a$. If $r < s$, then the second number is divisible by the first. If $r > s$, then the first is divisible by the second. □

Let us note that the result of Application 5 is the best possible in the sense that one may select 100 integers from $1, 2, \ldots, 200$ in such a way that no one of the selected integers is divisible by any other, for instance, the 100 integers $101, 102, \ldots, 199, 200$.

We conclude this section with another application from number theory. First we recall that two positive integers m and n are said to be *relatively prime* if their greatest common divisor[4] is 1. Thus 12 and 35 are relatively prime, but 12 and 15 are not since 3 is a common divisor of 12 and 15.

Application 6. (*Chinese remainder theorem*) Let m and n be relatively prime positive integers, and let a and b be integers where $0 \leq a \leq m-1$ and $0 \leq b \leq n-1$. Then there is a positive integer x such that the remainder when x is divided by m is a, and the remainder when x is divided by n is b; that is, x can be written in the form $x = pm + a$ and also in the form $x = qn + b$ for some integers p and q.

To show this we consider the n integers

$$a, m+a, 2m+a, \ldots, (n-1)m+a.$$

Each of these integers has remainder a when divided by m. Suppose that two of them had the same remainder r when divided by n. Let the two numbers be $im + a$ and $jm + a$ where $0 \leq i < j \leq n-1$. Then there are integers q_i and q_j such that

$$im + a = q_i n + r$$

and

$$jm + a = q_j n + r.$$

Subtracting the first equation from the second, we get

[4] Also called *greatest common factor* or *highest common factor*.

$$(j-i)m = (q_j - q_i)n.$$

The preceding equation tells us that n is a factor of the number $(j-i)m$. Since n has no common factor other than 1 with m, it follows that n is a factor of $j-i$. However, $0 \le i < j \le n-1$ implies that $0 < j-i \le n-1$, and hence n cannot be a factor of $j-i$. This contradiction arises from our supposition that two of the numbers $a, m+a, 2m+a, \ldots, (n-1)m+a$ had the same remainder when divided by n. We conclude that each of these n numbers has a different remainder when divided by n. By the pigeonhole principle each of the n numbers $0, 1, \ldots, n-1$ occurs as a remainder; in particular, the number b does. Let p be the integer with $0 \le p \le n-1$ such that the number $x = pm + a$ has remainder b when divided by n. Then for some integer q,

$$x = qn + b.$$

So $x = pm + a$ and $x = qn + b$, and x has the required properties. □

The fact that a rational number a/b has a decimal expansion that eventually repeats is a consequence of the pigeonhole principle, and we leave a proof of this fact for the exercises.

For further applications we will need a stronger form of the pigeonhole principle.

2.2 Pigeonhole Principle: Strong Form

The following theorem contains Theorem 2.1.1 as a special case.

Theorem 2.2.1 *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects are put into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, . . . , or the nth box contains at least q_n objects.

Proof. Suppose that we distribute $q_1 + q_2 + \cdots + q_n - n + 1$ objects among n boxes. If for each $i = 1, 2, \ldots, n$ the ith box contains fewer than q_i objects, then the total number of objects in all boxes does not exceed

$$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n.$$

Since this number is one less than the number of objects distributed, we conclude that for some $i = 1, 2, \ldots, n$ the ith box contains at least q_i objects. □

Notice that it is possible to distribute $q_1 + q_2 + \cdots + q_n - n$ objects among n boxes in such a way that for no $i = 1, 2, \ldots, n$ is it true that the ith box contains q_i or more objects. We do this by putting $q_1 - 1$ objects into the first box, $q_2 - 1$ objects into the second box, and so on.

The simple form of the pigeonhole principle is obtained from the strong form by taking $q_1 = q_2 = \cdots = q_n = 2$. Then

$$q_1 + q_2 + \cdots + q_n - n + 1 = 2n - n + 1 = n + 1.$$

In terms of coloring, the strong form of the pigeonhole principle asserts that if each of $q_1 + q_2 + \cdots + q_n - n + 1$ objects is assigned one of n colors, then there is an i such that there are (at least) q_i objects of the ith color.

In elementary mathematics the strong form of the pigeonhole principle is most often applied in the special case when $q_1, q_2, \ldots, q_n$ are all equal to some integer r. In this case the principle reads as follows:

- If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more of the objects. Equivalently,

- If the average of n non-negative integers $m_1, m_2, \ldots, m_n$ is greater than $r-1$:

$$\frac{m_1 + m_2 + \cdots + m_n}{n} > r - 1,$$

then at least one of the integers is greater than or equal to r.

The connection between these two formulations is obtained by taking $n(r-1)+1$ objects and putting them into n boxes. For $i = 1, 2, \ldots, n$

once the values of $h_0, h_1, \ldots, h_{k-1}$, the so-called *initial values*, are prescribed. The recurrence relation (7.15) "kicks in" beginning with $n = k$. First, we ignore the initial values and look for solutions of (7.15) without prescribed initial values. It turns out that we can find "enough" solutions by considering solutions which form geometric sequences (and by suitably modifying them).

Example.[7] In this example we recall a method for solving linear homogeneous differential equations with constant coefficients. Consider the differential equation

$$y'' - 5y' + 6y = 0. \tag{7.16}$$

Here y is a function of a real variable x. We seek solutions of this equation among the basic exponential functions $y = e^{qx}$. Let q be a constant. Since $y' = qe^{qx}$ and $y'' = q^2e^{qx}$, we have that $y = e^{qx}$ is a solution of (7.16) if and only if

$$q^2e^{qx} - 5qe^{qx} + 6e^{qx} = 0.$$

Since the exponential function e^{qx} is never zero, it may be cancelled and we obtain the following equation which does not depend on x:

$$q^2 - 5q + 6 = 0.$$

This equation has two roots, namely, $q = 2$ and $q = 3$. Hence

$$y = e^{2x} \quad \text{and} \quad y = e^{3x}$$

are both solutions of (7.16). Since the differential equation is linear and homogeneous,

$$y = c_1e^{2x} + c_2e^{3x} \tag{7.17}$$

is also a solution of (7.16) for any choice of the constants c_1 and c_2.[8] Now we bring in initial conditions for (7.16). These are conditions which prescribe both the value of y and its first derivative when $x = 0$, and with the differential equation (7.16) uniquely determine y. Suppose we prescribe the initial conditions

$$y(0) = a, \quad y'(0) = b \tag{7.18}$$

[7] For those who have not studied differential equations, this example can be omitted.

[8] This can be verified by computing y' and y'' and substituting into (7.16).

where a and b are fixed but unspecified numbers. Then in order that the solution (7.17) of the differential equation (7.16) satisfy these initial conditions we must have

$$\begin{aligned} y(0) = a: &\quad c_1 + c_2 = a \\ y'(0) = b: &\quad 2c_1 + 3c_2 = b. \end{aligned}$$

This system of equations has a unique solution for each choice of a and b, namely,

$$c_1 = 3a - b, \quad c_2 = b - 2a. \tag{7.19}$$

Thus no matter what the initial conditions (7.18), we can choose c_1 and c_2 using (7.19) so that the function (7.17) is a solution of the differential equation (7.16). In this sense (7.17) is the *general solution* of the differential equation. Each solution of (7.16) with prescribed initial conditions can be written in the form (7.17) for suitable choice of the constants c_1 and c_2. □

The solution of linear homogeneous recurrence relations proceeds along similar lines with the role of the exponential function e^{qx} taken up by the discrete function q^n defined only for non-negative integers n (the geometric sequences).

(2)

Theorem 7.2.1 *Let q be a non-zero number. Then $h_n = q^n$ is a solution of the linear homogeneous recurrence relation*

$$h_n - a_1h_{n-1} - a_2h_{n-2} - \cdots - a_kh_{n-k} = 0, \quad (a_k \neq 0, n \geq k) \tag{7.20}$$

with constant coefficients if and only if q is a root of the polynomial equation

$$x^k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_k = 0. \tag{7.21}$$

If the polynomial equation has k distinct roots $q_1, q_2, \ldots, q_k$, then

$$h_n = c_1q_1^n + c_2q_2^n + \cdots + c_kq_k^n \tag{7.22}$$

is the general solution of (7.20) in the following sense: No matter what initial values for $h_0, h_1, \ldots, h_{k-1}$ are given, there are constants $c_1, c_2, \ldots, c_k$ so that (7.22) is the unique sequence which satisfies both the recurrence relation (7.20) and the initial conditions.

Proof. We have that $h_n = q^n$ is a solution of (7.20) if and only if

$$q^n - a_1q^{n-1} - a_2q^{n-2} - \cdots - a_kq^{n-k} = 0$$

for all $n \geq k$. Since we assume $q \neq 0$, we may cancel q^{n-k}. Thus these equations (there is one for each $n \geq k$) are equivalent to the *one* equation

$$q^k - a_1q^{k-1} - a_2q^{k-2} - \cdots - a_k = 0.$$

We conclude that $h_n = q^n$ is a solution of (7.20) if and only if q is a root of the polynomial equation (7.21).

Since a_k is assumed to be different from zero, 0 is not a root of (7.21). Hence (7.21) has k roots, $q_1, q_2, \ldots, q_k$ all different from zero. These roots may be complex numbers. In general, $q_1, q_2, \ldots, q_k$ need not be distinct (the equation may have multiple roots), but we now assume that the roots $q_1, q_2, \ldots, q_k$ are distinct. Thus

$$h_n = q_1^n, \quad h_n = q_2^n, \quad \ldots, \quad h_n = q_k^n$$

are k different solutions of (7.20). The linearity and the homogeneity of the recurrence relation (7.20) imply that for any choice of constants $c_1, c_2, \ldots, c_k$

$$h_n = c_1q_1^n + c_2q_2^n + \cdots + c_kq_k^n \tag{7.23}$$

is also a solution of (7.20).[9] We now show that (7.23) is the general solution of (7.20) in the sense given in the statement of the theorem.

Suppose we prescribe the initial values

$$h_0 = b_0, \quad h_1 = b_1, \quad \ldots, \quad \text{and } h_{k-1} = b_{k-1}.$$

Can we choose the constants $c_1, c_2, \ldots, c_k$ so that h_n as given in (7.23) satisfies these initial conditions? Equivalently, can we always solve the system of equations (7.24) below no matter what the choice of $b_0, b_1, \ldots, b_{k-1}$?

$$\begin{array}{ll} (n=0) & c_1 + c_2 + \cdots + c_k = b_0 \\ (n=1) & c_1q_1 + c_2q_2 + \cdots + c_kq_k = b_1 \\ (n=2) & c_1q_1^2 + c_2q_2^2 + \cdots + c_kq_k^2 = b_2 \\ \vdots & \\ (n=k-1) & c_1q_1^{k-1} + c_2q_2^{k-1} + \cdots + c_kq_k^{k-1} = b_{k-1}. \end{array} \tag{7.24}$$

[9]This can be verified by direct substitution.

Now we shall rely on a little bit of linear algebra. The coefficient matrix of this system of equations is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix}. \tag{7.25}$$

The matrix in (7.25) is an important matrix called the *Vandermonde matrix*. The Vandermonde matrix is an invertible matrix if and only if $q_1, q_2, \ldots, q_k$ are distinct. Indeed its determinant equals

$$\prod_{1 \leq i < j \leq k} (q_j - q_i)$$

and hence is non-zero exactly when $q_1, q_2, \ldots, q_k$ are distinct.[10] Thus our assumption of the distinctness of $q_1, q_2, \ldots, q_k$ implies that the system (7.24) has a unique solution for each choice of $b_0, b_1, \ldots, b_{k-1}$. Therefore (7.23) is the general solution of (7.20) and the proof of the theorem is complete. □

The polynomial equation (7.21) is called the *characteristic equation* of the recurrence relation (7.20) and its k roots are the *characteristic roots*. By Theorem 7.2.1, if the characteristic roots are distinct, (7.22) is the general solution of (7.20).

Example. Solve the recurrence relation

$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}, \quad (n \geq 3)$$

subject to the initial values $h_0 = 1$, $h_1 = 2$, and $h_2 = 0$.

The characteristic equation of this recurrence relation is

$$x^3 - 2x^2 - x + 2 = 0,$$

and its three roots are $1, -1, 2$. By Theorem 7.2.1,

$$h_n = c_1 1^n + c_2(-1)^n + c_3 2^n = c_1 + c_2(-1)^n + c_3 2^n$$

[10]The proof of this fact is non-trivial.

of degree less than k and where $q(x)$ is a polynomial of degree k having constant term equal to 1. To find a general formula for the terms of the sequence, we first use the method of partial fractions to express $p(x)/q(x)$ as a sum of algebraic fractions of the form

$$\frac{c}{(1-rx)^t}$$

where t is a positive integer, r is a real number, and c is a constant. We then use (7.47) to find a power series for $1/(1-rx)^t$. Combining like terms, we obtain a power series for the generating function, from which we can read off the terms of the sequence.

Example. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be a sequence of numbers satisfying the recurrence relation

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0, \quad (n \geq 3)$$

where $h_0 = 0$, $h_1 = 1$ and $h_2 = -1$. Find a general formula for h_n.

Let $g(x) = h_0 + h_1x + h_2x^2 + \cdots + h_nx^n + \cdots$ be the generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$. Adding the four equations,

$$\begin{aligned}
g(x) &= h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_nx^n + \cdots,\\
xg(x) &= h_0x + h_1x^2 + h_2x^3 + \cdots + h_{n-1}x^n + \cdots,\\
-16x^2g(x) &= -16h_0x^2 - 16h_1x^3 - \cdots - 16h_{n-2}x^n - \cdots,\\
20x^3g(x) &= 20h_0x^3 + \cdots + 20h_{n-3}x^n + \cdots,
\end{aligned}$$

we obtain

$$\begin{aligned}
(1+x-16x^2+20x^3)g(x) &= h_0 + (h_1+h_0)x + (h_2+h_1-16h_0)x^2 + \\
&\quad (h_3+h_2-16h_1+20h_0)x^3 + \cdots + \\
&\quad (h_n+h_{n-1}-16h_{n-2}+20h_{n-3})x^n + \cdots.
\end{aligned}$$

Since $h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$, $(n \geq 3)$ and since $h_0 = 0$, $h_1 = 1$, and $h_2 = -1$, we get

$$(1+x-16x^2+20x^3)g(x) = x$$

and hence

$$g(x) = \frac{x}{1+x-16x^2+20x^3}.$$

We observe that $(1+x-16x^2+20x^3) = (1-2x)^2(1+5x)$. Hence for some constants c_1, c_2 and c_3,

$$\frac{x}{1+x-16x^2+20x^3} = \frac{c_1}{1-2x} + \frac{c_2}{(1-2x)^2} + \frac{c_3}{1+5x}.$$

To determine the constants, we multiply both sides of this equation by $1+x-16x^2+20x^3$ to get

$$x = (1-2x)(1+5x)c_1 + (1+5x)c_2 + (1-2x)^2c_3,$$

or, equivalently,

$$x = (c_1+c_2+c_3) + (3c_1+5c_2-4c_3)x + (-10c_1+4c_3)x^2.$$

Hence

$$\begin{aligned}
c_1 + c_2 + c_3 &= 0,\\
3c_1 + 5c_2 - 4c_3 &= 1,\\
-10c_1 + 4c_3 &= 0.
\end{aligned}$$

Solving these equations simultaneously, we find that

$$c_1 = -\frac{2}{49},\ c_2 = \frac{7}{49},\ \text{and } c_3 = -\frac{5}{49}.$$

Therefore

$$g(x) = \frac{x}{1+x-16x^2+20x^3} = -\frac{2/49}{1-2x} + \frac{7/49}{(1-2x)^2} - \frac{5/49}{1+5x}.$$

By (7.47)

$$\begin{aligned}
\frac{1}{1-2x} &= \sum_{k=0}^{\infty} 2^kx^k,\\
\frac{1}{(1-2x)^2} &= \sum_{k=0}^{\infty} \binom{k+1}{k} 2^kx^k = \sum_{k=0}^{\infty}(k+1)2^kx^k,\\
\frac{1}{1+5x} &= \sum_{k=0}^{\infty}(-5)^kx^k.
\end{aligned}$$

Hence

$$\begin{aligned}
g(x) &= -\frac{2}{49}\left(\sum_{k=0}^{\infty}2^kx^k\right) + \frac{7}{49}\left(\sum_{k=0}^{\infty}(k+1)2^kx^k\right) - \frac{5}{49}\left(\sum_{k=0}^{\infty}(-5)^kx^k\right)\\
&= \sum_{k=0}^{\infty}\left[-\frac{2}{49}2^k + \frac{7}{49}(k+1)2^k - \frac{5}{49}(-5)^k\right]x^k.
\end{aligned}$$

Since $g(x)$ is the generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$,

$$h_n = -\frac{2}{49}2^n + \frac{7}{49}(n+1)2^n - \frac{5}{49}(-5)^n, \quad (n = 0, 1, 2, \ldots). \ \square$$

The formula for h_n above should bring to mind the solution of recurrence relations, using the roots of the characteristic equation as described in section 7.2. Indeed, the formula above suggests that the roots of the characteristic equation for the given recurrence relation are 2, 2, and -5. The following discussion should clarify the relationship between the two methods.

In the preceding example we have expressed the generating function $g(x)$ in the form

$$g(x) = \frac{p(x)}{q(x)}$$

where

$$q(x) = 1 + x - 16x^2 + 20x^3.$$

Since the recurrence relation is

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0, \quad (n = 3, 4, 5, \ldots),$$

the associated characteristic equation is $r(x) = 0$ where

$$r(x) = x^3 + x^2 - 16x + 20.$$

If we replace x in $r(x)$ by $1/x$ (this amounts to the change in variable $y = 1/x$), we obtain

$$r(1/x) = \frac{1}{x^3} + \frac{1}{x^2} - 16\frac{1}{x} + 20,$$

or

$$x^3 r(1/x) = 1 + x - 16x^2 + 20x^3 = q(x).$$

The roots of the characteristic equation $r(x) = 0$ are $2, 2$, and -5. Since $r(x) = (x-2)^2(x+5)$, it follows that

$$q(x) = x^3\left(\frac{1}{x} - 2\right)^2\left(\frac{1}{x} + 5\right) = (1-2x)^2(1+5x),$$

which checks with our previous calculation.

The relationships above hold in general. Let the sequence of numbers $h_0, h_1, h_2, \ldots, h_n, \ldots$ be defined by the recurrence relation of order k

$$h_n + a_1 h_{n-1} + \cdots + a_k h_{n-k} = 0, \ (n \geq k)$$

with initial values for $h_0, h_1, \ldots, h_{k-1}$. Recall that since the recurrence relation has order k, a_k is assumed to be different from 0. Let $g(x)$ be the generating function for our sequence. Using the method given in the examples, there are polynomials $p(x)$ and $q(x)$ such that

$$g(x) = \frac{p(x)}{q(x)}$$

where $q(x)$ has degree k and $p(x)$ has degree less than k. Indeed, we have

$$q(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_k x^k,$$

and

$$\begin{aligned} p(x) &= h_0 + (h_1 + a_1 h_0)x + (h_2 + a_1 h_1 + a_2 h_0)x^2 \\ &\quad + \cdots + (h_{k-1} + a_1 h_{k-2} + \cdots + a_{k-1} h_0)x^{k-1}. \end{aligned}$$

The characteristic equation for this recurrence relation is $r(x) = 0$ where

$$r(x) = x^k + a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k.$$

Hence

$$q(x) = x^k r(1/x).$$

Thus if the roots of $r(x) = 0$ are $q_1, q_2, \ldots, q_k$, then

$$r(x) = (x - q_1)(x - q_2) \cdots (x - q_k)$$

and

$$q(x) = (1 - q_1 x)(1 - q_2 x) \cdots (1 - q_k x).$$

Conversely, if we are given a polynomial

$$q(x) = b_0 + b_1 x + \cdots + b_k x^k$$

of degree k with $b_0 \neq 0$ and a polynomial

$$p(x) = d_0 + d_1 x + \cdots + d_{k-1} x^{k-1}$$

of degree less than k, then using partial fractions and (7.47), we can find a power series[14] $h_0 + h_1 x + \cdots + h_n x^n + \cdots$ such that

$$\frac{p(x)}{q(x)} = h_0 + h_1 x + \cdots + h_n x^n + \cdots.$$

[14]This power series will converge to $p(x)/q(x)$ for all x with $|x| < t$ where t is the smallest absolute value of a root of $q(x) = 0$. Since we assume that $b_0 \neq 0$, 0 is not a root of $q(x) = 0$.

We can write the above equation in the form

$$d_0 + d_1 x + \cdots + d_{k-1}x^{k-1} = (b_0 + b_1 x + \cdots + b_k x^k) \times (h_0 + h_1 x + \cdots + h_n x^n + \cdots).$$

Multiplying out the right side and comparing coefficients, we obtain

$$\begin{aligned} b_0 h_0 &= d_0, \\ b_0 h_1 + b_1 h_0 &= d_1, \\ &\vdots \\ b_0 h_{k-1} + b_1 h_{k-2} + \cdots + b_{k-1} h_0 &= d_{k-1}, \end{aligned} \qquad (7.48)$$

and

$$b_0 h_n + b_1 h_{n-1} + \cdots + b_k h_{n-k} = 0, \qquad (n \geq k), \qquad (7.49)$$

Since $b_0 \neq 0$, equation (7.49) can be written in the form

$$h_n + \frac{b_1}{b_0} h_{n-1} + \cdots + \frac{b_k}{b_0} h_{n-k} = 0, \qquad (n \geq k).$$

This is a linear homogeneous recurrence relation with constant coefficients which is satisfied by $h_0, h_1, h_2, \ldots, h_n, \ldots$. The initial values $h_0, h_1, \ldots, h_{k-1}$ can be determined by solving the triangular system of equations (7.48), using the fact that $b_0 \neq 0$. We summarize in the following theorem.

Theorem 7.5.1 *Let*

$$h_0, h_1, h_2, \ldots, h_n, \ldots$$

be a sequence of numbers which satisfies the linear homogeneous recurrence relation

$$h_n + c_1 h_{n-1} + \cdots + c_k h_{n-k} = 0, \quad c_k \neq 0, \quad (n \geq k) \qquad (7.50)$$

of order k with constant coefficients. Then its generating function $g(x)$ is of the form

$$g(x) = \frac{p(x)}{q(x)} \qquad (7.51)$$

where $q(x)$ is a polynomial of degree k with a non-zero constant term and $p(x)$ is a polynomial of degree less than k. Conversely, given such polynomials $p(x)$ and $q(x)$, there is a sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ satisfying a linear homogeneous recurrence relation with constant coefficients of order k of the type (7.50) whose generating function is given by (7.51).

7.6 A Geometry Example

A set K of points in the plane or in space is said to be *convex*, provided that for any two points p and q in K, all the points on the line segment joining p and q are in K. Triangular regions, circular regions, and rectangular regions in the plane are all convex sets of points. On the other hand, the region on the left in Figure 7.2 is not convex since, for the two points p and q shown, the line segment joining p and q goes outside the region.

The regions in Figure 7.2 are examples of a *polygonal regions*, that is, regions whose boundaries consist of a finite number of line segments, called their *sides*. Triangular regions and rectangular regions are polygonal, but circular regions are not. Any polygonal region must have at least three sides. The region on the right in Figure 7.2 is a convex polygonal region with six sides.

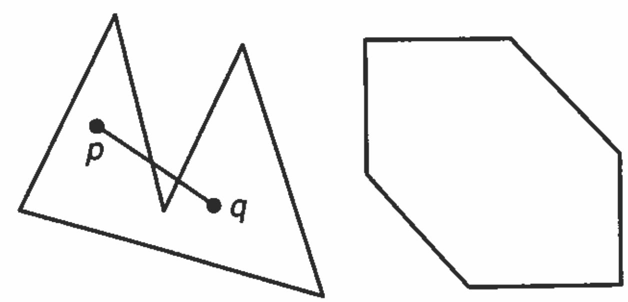


Figure 7.2

In a polygonal region the points at which the sides meet are called *corners* (or *vertices*). A *diagonal* is a line segment joining two non-consecutive corners.

Let K be a polygonal region with n sides. We can count the number of its diagonals as follows. Each corner is joined by a diagonal to $n-3$ other corners. Thus counting the number of diagonals at each corner and summing, we get $n(n-3)$. Since each diagonal has two corners, each diagonal is counted twice in this sum. Hence the number of diagonals is $n(n-3)/2$. We can arrive at this same number indirectly in the following way. There are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

line segments joining the n corners. Of these, n are sides of the polygonal region. The remaining ones are diagonals. Hence there

Chapter 8

Special Counting Sequences

We have already considered several special counting sequences in the previous chapters. The counting sequence for permutations of a set of n elements is

$$0!, 1!, 2!, \ldots, n!, \ldots.$$

The counting sequence for derangements of a set of n elements is

$$D_0, D_1, D_2, \ldots, D_n, \ldots$$

where D_n has been evaluated in Theorem 6.3.1. In addition we have investigated the Fibonacci sequence

$$f_0, f_1, f_2, \ldots, f_n, \ldots,$$

and a formula for f_n has been given in Theorem 7.1.1. In this chapter we study primarily four famous and important counting sequences, the sequence of Catalan numbers, the sequences of the Stirling numbers of the first and second kind, and the sequence of the number of partitions of a positive integer n.

8.1 Catalan Numbers

The *Catalan sequence*[1] is the sequence

$$C_0, C_1, C_2, \ldots, C_n, \ldots$$

[1] After Eugène Catalan (1814-1894).

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad (n = 0, 1, 2, \ldots)$$

is the nth *Catalan number*. The first few Catalan numbers are

$$\begin{array}{ll} C_0 = 1 & C_5 = 42 \\ C_1 = 1 & C_6 = 132 \\ C_2 = 2 & C_7 = 429 \\ C_3 = 5 & C_8 = 1430 \\ C_4 = 14 & C_9 = 4862 \end{array}$$

The Catalan number

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$$

arose in section 7.6 as the number of ways to divide a convex polygonal region with $n+1$ sides into triangles by inserting diagonals which do not intersect in the interior. The Catalan numbers occur in several seemingly unrelated counting problems and we discuss some of them in this section.

Theorem 8.1.1 *The number of sequences*

$$a_1, a_2, \ldots, a_{2n} \tag{8.1}$$

of $2n$ terms that can be formed by using n $+1$'s and n -1's whose partial sums satisfy

$$a_1 + a_2 + \cdots + a_k \geq 0, \qquad (k = 1, 2, \ldots, 2n) \tag{8.2}$$

equals the nth Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad (n \geq 0).$$

Proof. We call a sequence (8.1) of n $+1$'s and n -1's *acceptable* if it satisfies (8.2) and *unacceptable* otherwise. Let A_n denote the number of acceptable sequences of n $+1$'s and n -1's, and let U_n denote the number of unacceptable ones. The total number of sequences of n $+1$'s and n -1's is

$$\binom{2n}{n} = \frac{(2n)!}{n!n!}$$

since such sequences can be regarded as the permutations of objects of two different types with n objects of one type (the $+1$'s) and n of the other (the -1's). Hence

$$A_n + U_n = \binom{2n}{n},$$

and we evaluate A_n by first evaluating U_n and then subtracting from $\binom{2n}{n}$.

Consider an unacceptable sequence (8.1) of n $+1$'s and n -1's. Because the sequence is unacceptable there is a *smallest* k such that the partial sum

$$a_1 + a_2 + \cdots + a_k$$

is negative. Because k is smallest there is an equal number of $+1$'s and -1's preceding a_k, and we have

$$a_1 + a_2 + \cdots + a_{k-1} = 0,$$

and

$$a_k = -1.$$

In particular, k is an odd integer. We now reverse the signs of each of the first k terms; that is, we replace a_i by $-a_i$ for each $i = 1, 2, \ldots, k$ and leave unchanged the remaining terms. The resulting sequence

$$a'_1, a'_2, \ldots, a'_{2n}$$

is a sequence of $(n+1)$ $+1$'s and $(n-1)$ -1's. This process is reversible: Given a sequence of $(n+1)$ $+1$'s and $(n-1)$ -1's, there is a first instance when the number of $+1$'s exceeds the number of -1's (since there are more $+1$'s than -1's). Reversing the $+1$'s and -1's up to that point results in an unacceptable sequence of n $+1$'s and n -1's. Thus there are as many unacceptable sequences as there are sequences of $(n+1)$ $+1$'s and $(n-1)$ -1's. The number of sequences of $(n+1)$ $+1$'s and $(n+1)$ -1's is the number

$$\frac{(2n)!}{(n+1)!(n-1)!}$$

of permutations of objects of two types, with $n+1$ objects of one type and $n-1$ of the other. Hence

$$U_n = \frac{(2n)!}{(n+1)!(n-1)!}$$

and therefore

$$\begin{aligned} A_n &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n(n+1)}\right) \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

□

There are many different interpretations of Theorem 8.1.1. We discuss two of them in the next examples.

5

Example. There are $2n$ people in line to get into a theatre. Admission is 50 cents.[2] Of the $2n$ people, n have a 50 cent piece and n have a 1 dollar bill. The box office at the theatre rather foolishly begins with an empty cash register. In how many ways can the people line up so that whenever a person with a \$1 dollar bill buys a ticket, the box office has a 50 cent piece in order to make change?

If we regard the people as "indistinguishable" and identify a 50 cent piece with a $+1$ and a dollar bill with a -1, then the answer is the number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

of acceptable sequences as defined in Theorem 8.1.1. If the people are regarded as "distinguishable", the answer is

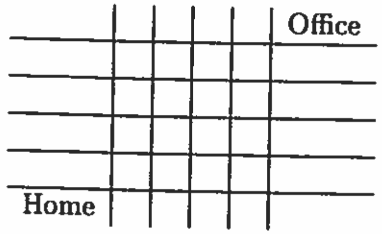
$$(n!)(n!)\frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n+1}.$$

□

(See Example 4.10, p. 46 of Bogart's text)

Example. A big city lawyer works n blocks north and n blocks east of her place of residence. Every day she walks $2n$ blocks to work. (See the map below for $n = 4$.) How many routes are possible if she never crosses (but may touch) the diagonal line from home to office?

[2] This problem shows its age!



Each acceptable route is either above the diagonal or below the diagonal. We find the number of acceptable routes above the diagonal, and multiply by 2. Each route is a sequence of n northerly blocks and n easterly blocks. We identify north with $+1$ and east with -1. Thus each route corresponds to a sequence

$$a_1, a_2, \ldots, a_{2n}$$

of n $+1$'s and n -1's, and in order that the route not dip below the diagonal we must have

$$\sum_{i=1}^{k} a_i \geq 0, \qquad (k = 1, \ldots, 2n).$$

Hence by Theorem 8.1.1 the number of acceptable routes above the diagonal equals the nth Catalan number and the total number of acceptable routes is

$$2C_n = \frac{2}{n+1}\binom{2n}{n}.$$

□

We next show that the Catalan numbers satisfy a homogeneous recurrence relation of order 1 (but with a non-constant coefficient). We have

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{n!n!}$$

and

$$C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} = \frac{1}{n}\frac{(2n-2)!}{(n-1)!(n-1)!}.$$

Dividing, we obtain

$$\frac{C_n}{C_{n-1}} = \frac{4n-2}{n+1}.$$

Therefore the Catalan sequence is determined by the following recurrence relation and initial condition:

$$\begin{aligned} C_n &= \frac{4n-2}{n+1}C_{n-1}, \quad (n \geq 1) \\ C_0 &= 1. \end{aligned} \tag{8.3}$$

We have previously noted that $C_9 = 4862$. It follows from the recurrence relation (8.3) that

$$C_{10} = \frac{38}{11}\, C_9 = \frac{38}{11}(4862) = 16,796.$$

We now define a new sequence of numbers

$$C_1^*, C_2^*, \ldots, C_n^*, \ldots$$

which, in order to refer to them by name, we call the *pseudo-Catalan numbers*. We let

$$C_n^* = n!C_{n-1}, \qquad (n = 1, 2, 3, \ldots).$$

We have

$$C_1^* = 1!(1) = 1$$

and using (8.3) with n replaced by $n-1$, we obtain

$$\begin{aligned} C_n^* &= n!C_{n-1} \\ &= n!\frac{4n-6}{n}C_{n-2} \\ &= (4n-6)(n-1)!C_{n-2} \\ &= (4n-6)C_{n-1}^*. \end{aligned}$$

Thus the pseudo-Catalan numbers are determined by the following recurrence relation and initial condition:

$$\begin{aligned} C_n^* &= (4n-6)C_{n-1}^*, \quad (n \geq 2) \\ C_1^* &= 1. \end{aligned} \tag{8.4}$$

Using this recurrence relation we calculate the first few pseudo-Catalan numbers:

$$\begin{array}{ll} C_1^* = 1 & C_4^* = 120 \\ C_2^* = 2 & C_5^* = 1680 \\ C_3^* = 12 & C_6^* = 30240. \end{array}$$

We can write the above equation in the form

$$d_0 + d_1x + \cdots + d_{k-1}x^{k-1} = (b_0 + b_1x + \cdots + b_kx^k) \times (h_0 + h_1x + \cdots + h_nx^n + \cdots).$$

Multiplying out the right side and comparing coefficients, we obtain

$$\begin{aligned} b_0h_0 &= d_0, \\ b_0h_1 + b_1h_0 &= d_1, \\ &\vdots \\ b_0h_{k-1} + b_1h_{k-2} + \cdots + b_{k-1}h_0 &= d_{k-1}, \end{aligned} \qquad (7.48)$$

and

$$b_0h_n + b_1h_{n-1} + \cdots + b_kh_{n-k} = 0, \qquad (n \geq k), \qquad (7.49)$$

Since $b_0 \neq 0$, equation (7.49) can be written in the form

$$h_n + \frac{b_1}{b_0}h_{n-1} + \cdots + \frac{b_k}{b_0}h_{n-k} = 0, \qquad (n \geq k).$$

This is a linear homogeneous recurrence relation with constant coefficients which is satisfied by $h_0, h_1, h_2, \ldots, h_n, \ldots$. The initial values $h_0, h_1, \ldots, h_{k-1}$ can be determined by solving the triangular system of equations (7.48), using the fact that $b_0 \neq 0$. We summarize in the following theorem.

Theorem 7.5.1 *Let*

$$h_0, h_1, h_2, \ldots, h_n, \ldots$$

be a sequence of numbers which satisfies the linear homogeneous recurrence relation

$$h_n + c_1h_{n-1} + \cdots + c_kh_{n-k} = 0, \quad c_k \neq 0, \quad (n \geq k) \qquad (7.50)$$

of order k with constant coefficients. Then its generating function $g(x)$ is of the form

$$g(x) = \frac{p(x)}{q(x)} \qquad (7.51)$$

where $q(x)$ is a polynomial of degree k with a non-zero constant term and $p(x)$ is a polynomial of degree less than k. Conversely, given such polynomials $p(x)$ and $q(x)$, there is a sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ satisfying a linear homogeneous recurrence relation with constant coefficients of order k of the type (7.50) whose generating function is given by (7.51).

7.6 A Geometry Example

(See also Exercise 23 Chapter 3, Section 4 of Bogart's text)

A set K of points in the plane or in space is said to be *convex*, provided that for any two points p and q in K, all the points on the line segment joining p and q are in K. Triangular regions, circular regions, and rectangular regions in the plane are all convex sets of points. On the other hand, the region on the left in Figure 7.2 is not convex since, for the two points p and q shown, the line segment joining p and q goes outside the region.

The regions in Figure 7.2 are examples of a *polygonal regions*, that is, regions whose boundaries consist of a finite number of line segments, called their *sides*. Triangular regions and rectangular regions are polygonal, but circular regions are not. Any polygonal region must have at least three sides. The region on the right in Figure 7.2 is a convex polygonal region with six sides.

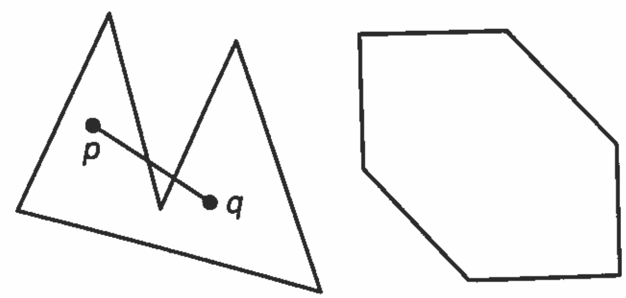


Figure 7.2

In a polygonal region the points at which the sides meet are called *corners* (or *vertices*). A *diagonal* is a line segment joining two non-consecutive corners.

Let K be a polygonal region with n sides. We can count the number of its diagonals as follows. Each corner is joined by a diagonal to $n-3$ other corners. Thus counting the number of diagonals at each corner and summing, we get $n(n-3)$. Since each diagonal has two corners, each diagonal is counted twice in this sum. Hence the number of diagonals is $n(n-3)/2$. We can arrive at this same number indirectly in the following way. There are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

line segments joining the n corners. Of these, n are sides of the polygonal region. The remaining ones are diagonals. Hence there are

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

diagonals.

Now assume that K is convex. Then each diagonal of K lies wholly within K. Thus each diagonal of K divides K into one convex polygonal region with k sides and another with $n-k+2$ sides for some $k = 3, 4, \ldots, n-1$.

We can draw $n-3$ diagonals meeting a particular corner of K, and in doing so divide K into $n-2$ triangular regions. But there are other ways of dividing the region into triangular regions by inserting $n-3$ diagonals no two of which intersect in the interior of K, as the example in Figure 7.3 shows for $n = 8$.

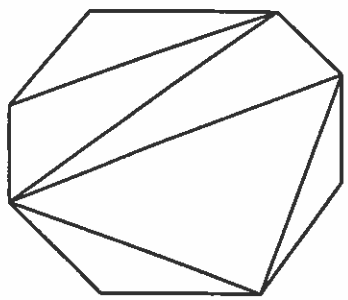


Figure 7.3

In the following theorem we determine the number of different ways to divide a convex polygonal region into triangular regions by drawing diagonals which do not intersect in the interior. For notational convenience we deal with a convex polygonal region of $n+1$ sides which is then divided into $n-1$ triangular regions by $n-2$ diagonals.

Theorem 7.6.1 *Let h_n denote the number of ways of dividing a convex polygonal region with $n+1$ sides into triangular regions by inserting diagonals which do not intersect in the interior. Define $h_1 = 1$. Then h_n satisfies the recurrence relation*

$$\begin{aligned} h_n &= h_1 h_{n-1} + h_2 h_{n-2} + \cdots + h_{n-1} h_1 \\ &= \sum_{k=1}^{n-1} h_k h_{n-k}, \quad (n \geq 2). \end{aligned} \tag{7.52}$$

The solution of this recurrence relation is

$$h_n = \frac{1}{n}\binom{2n-2}{n-1}, \qquad (n = 1, 2, 3, \ldots). \tag{7.53}$$

Proof. We have defined $h_1 = 1$, and we think of a line segment as a polygonal region with two sides and no interior. We have $h_2 = 1$ since a triangular region has no diagonals, and it cannot be further subdivided. The recurrence relation (7.52) holds for $n = 2$,[15] since

$$\sum_{k=1}^{2-1} h_k h_{2-k} = \sum_{k=1}^{1} h_k h_{2-k} = h_1 h_1 = 1.$$

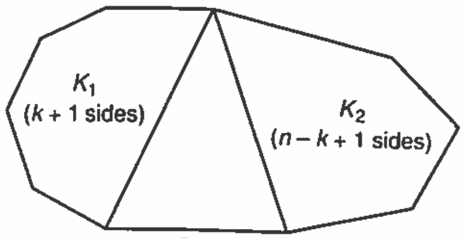
Now let $n \geq 3$. Consider a convex polygonal region K with $n+1 \geq 4$ sides. We distinguish one side of K and call it the *base*. In each division of K into triangular regions, the base is a side of one of the triangular regions T, and this triangular region divides the remainder of K into a polygonal region K_1 with $k+1$ sides and a polygonal region K_2 with $n-k+1$ sides, for some $k = 1, 2, \ldots, n-1$ (see Figure 7.4).

The further subdivision of K is accomplished by dividing K_1 and K_2 into triangular regions by inserting diagonals of K_1 and K_2, respectively, which do not intersect in the interior. Since K_1 has $k+1$ sides, K_1 can be divided into triangular regions in h_k ways. Since K_2 has $n-k+1$ sides, K_2 can be divided into triangular regions in h_{n-k} ways. Hence, for a particular choice of the triangular region T containing the base, there are $h_k h_{n-k}$ ways of dividing K into triangular regions by diagonals that do not intersect in the interior. Hence there are a total of

$$h_n = \sum_{k=1}^{n-1} h_k h_{n-k}$$

ways to divide K into triangular regions in this way. This establishes the recurrence relation (7.52).

[15] This is why we defined $h_1 = 1$.



Polygonal region with $n+1$ sides

Figure 7.4

We now turn to the solution of (7.52) with the initial condition $h_1 = 1$. This recurrence relation is not linear. Moreover, h_n does not depend on a fixed number of values that come before it, but on all the values $h_1, h_2, \ldots, h_{n-1}$ that come before it. Thus none of our methods for solving recurrence relations apply. Let

$$g(x) = h_1x + h_2x^2 + \cdots + h_nx^n + \cdots$$

be the generating function for the sequence $h_1, h_2, h_3, \ldots, h_n, \ldots$. Multiplying $g(x)$ by itself, we find that

$$\begin{aligned}(g(x))^2 &= h_1^2x^2 + (h_1h_2 + h_2h_1)x^3 + (h_1h_3 + h_2h_2 + h_3h_1)x^4 \\ &\quad + \cdots + (h_1h_{n-1} + h_2h_{n-2} + \cdots + h_{n-1}h_1)x^n + \cdots.\end{aligned}$$

Using (7.52) and the fact that $h_1 = h_2 = 1$, we obtain

$$\begin{aligned}(g(x))^2 &= h_1^2x^2 + h_3x^3 + h_4x^4 + \cdots + h_nx^n + \cdots \\ &= h_2x^2 + h_3x^3 + h_4x^4 + \cdots + h_nx^n + \cdots \\ &= g(x) - h_1x = g(x) - x.\end{aligned}$$

Thus $g(x)$ satisfies the equation

$$(g(x))^2 - g(x) + x = 0.$$

This is a quadratic equation for $g(x)$, and so by the quadratic formula[16] $g(x) = g_1(x)$ or $g(x) = g_2(x)$ where

$$g_1(x) = \frac{1+\sqrt{1-4x}}{2} \text{ and } g_2(x) = \frac{1-\sqrt{1-4x}}{2}.$$

[16] But omitting some subtleties.

From the definition of $g(x)$, it follows that $g(0) = 0$. Since $g_1(0) = 1$ and $g_2(0) = 0$, we conclude that

$$g(x) = g_2(x) = \frac{1-\sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}.$$

By Newton's binomial theorem (see, in particular, the calculation done at the end of section 5.6),

$$(1+z)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \times 2^{2n-1}} \binom{2n-2}{n-1} z^n, \qquad (|z| < 1).$$

If we replace z by $-4x$, we get

$$\begin{aligned}(1-4x)^{1/2} &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \times 2^{2n-1}} \binom{2n-2}{n-1} (-1)^n 4^n x^n \\ &= 1 + \sum_{n=1}^{\infty} (-1)^{2n-1} \frac{2}{n} \binom{2n-2}{n-1} x^n \\ &= 1 - 2\sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n, \quad (|x| < \frac{1}{4}).\end{aligned}$$

Thus

$$g(x) = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2} = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n \qquad (7.54)$$

and hence

$$h_n = \frac{1}{n} \binom{2n-2}{n-1}, \qquad (n \geq 1).$$

□

The numbers

$$\frac{1}{n} \binom{2n-2}{n-1}$$

in the previous theorem are the Catalan numbers, and these will be investigated more throughly in Chapter 8.

The defining formula for the Catalan numbers and the definition of the pseudo-Catalan numbers imply the formula

$$C_n^* = (n-1)!\binom{2n-2}{n-1}, \qquad (n \geq 1)$$

for the pseudo-Catalan numbers. This formula can also be derived from the recurrence relation (8.4).

Example. Let $a_1, a_2, \ldots, a_n$ be n numbers. By a *multiplication scheme* for these numbers we mean a scheme for carrying out the multiplication of $a_1, a_2, \ldots, a_n$. A multiplication scheme requires $n-1$ multiplications between two numbers, each of which is either one of $a_1, a_2, \ldots, a_n$ or a partial product of them. Let h_n denote the number of multiplication schemes for n numbers. We have $h_1 = 1$ (this can be taken as the definition of h_1) and $h_2 = 2$ since

$$(a_1 \times a_2) \quad \text{and} \quad (a_2 \times a_1)$$

are two possible schemes. This example serves to show that the order of the numbers in the multiplication scheme is taken into consideration.[3] If $n = 3$, there are 12 schemes:

$$\begin{array}{lll} (a_1 \times (a_2 \times a_3)) & (a_2 \times (a_1 \times a_3)) & (a_3 \times (a_1 \times a_2)) \\ ((a_2 \times a_3) \times a_1) & ((a_1 \times a_3) \times a_2) & ((a_1 \times a_2) \times a_3) \\ (a_1 \times (a_3 \times a_2)) & (a_2 \times (a_3 \times a_1)) & (a_3 \times (a_2 \times a_1)) \\ ((a_3 \times a_2) \times a_1) & ((a_3 \times a_1) \times a_2) & ((a_2 \times a_1) \times a_3). \end{array}$$

Thus $h_3 = 12$. Each multiplication scheme for three numbers requires two multiplications and each multiplication corresponds to a set of parentheses. The outside parentheses allows us to identify each multiplication $\times$ with a set of parentheses. In general, each multiplication scheme can be obtained by listing $a_1, a_2, \ldots, a_n$ in some order and then inserting $n-1$ pairs of parentheses so that each pair of parentheses designates a multiplication of two factors. But in order to derive a recurrence relation for h_n we look at it in an inductive way. Each scheme for $a_1, a_2, \ldots, a_n$ can be gotten from a scheme for $a_1, a_2, \ldots, a_{n-1}$ in exactly one of the following ways:

[3] In more algebraic language, we are not assuming that the commutative law holds.

(i) Take a multiplication scheme for $a_1, a_2, \ldots, a_{n-1}$ (which has $n-2$ multiplications and $n-2$ sets of parentheses) and insert a_n on either side of either factor in one of the $n-2$ multiplications. Thus each scheme for $n-1$ numbers gives $2 \times 2 \times (n-2) = 4(n-2)$ schemes for n numbers in this way.

(ii) Take a multiplication scheme for $a_1, a_2, \ldots, a_{n-1}$ and multiply it on the left or right by a_n. Thus each scheme for $n-1$ numbers gives two schemes for n numbers in this way.

To illustrate, let $n = 6$ and consider the multiplication scheme

$$((a_1 \times a_2) \times ((a_3 \times a_4) \times a_5)).$$

for a_1, a_2, a_3, a_4, a_5.[4] There are 4 multiplications in this scheme. We take any one of them, say the multiplication of $(a_3 \times a_4)$ and a_5, and insert a_6 on either side of either of these two factors to get

$$\begin{array}{l} ((a_1 \times a_2) \times (((a_6 \times (a_3 \times a_4)) \times a_5)) \\ ((a_1 \times a_2) \times (((a_3 \times a_4) \times a_6) \times a_5)) \\ ((a_1 \times a_2) \times ((a_3 \times a_4) \times (a_6 \times a_5))) \\ ((a_1 \times a_2) \times ((a_3 \times a_4) \times (a_5 \times a_6))). \end{array}$$

There are $4 \times 4 = 16$ schemes for $a_1, a_2, a_3, a_4, a_5, a_6$ obtained in this way. Besides these we have two additional schemes in which a_6 enters into the final multiplication, namely,

$$(a_6 \times ((a_1 \times a_2) \times ((a_3 \times a_4) \times a_5))), \quad (((a_1 \times a_2) \times ((a_3 \times a_4) \times a_5)) \times a_6).$$

Thus each multiplication scheme for five numbers gives 18 schemes for six numbers; and we have $h_6 = 18h_5$.

Let $n \geq 2$. Then generalizing the analysis above we see that each of the h_{n-1} multiplication schemes for $n-1$ numbers gives $4(n-2) + 2 = 4n - 6$ schemes for n numbers We thus obtain the recurrence relation

$$h_n = (4n-6)h_{n-1}, \quad (n \geq 2)$$

which together with the initial value $h_1 = 1$ determines the entire sequence $h_1, h_2, \ldots, h_n, \ldots$. This is the same type of recurrence

[4] Which multiplication $\times$ corresponds to each set of parentheses above?

relation with the same initial value satisfied by the pseudo-Catalan numbers (8.4). Hence

$$h_n = C_n^* = (n-1)!\binom{2n-2}{n-1}, \quad (n \geq 1).$$

□

In the preceding example, suppose that we count only those multiplication schemes in which the n numbers are listed in the order $a_1, a_2, \ldots, a_n$. Thus, for instance, $((a_2 \times a_1) \times a_3)$ is no longer counted. Let g_n denote the number of multiplication schemes with this additional restriction. Then $h_n = n! g_n$ and hence

$$g_n = \frac{h_n}{n!} = \frac{C_n^*}{n!} = \frac{1}{n}\binom{2n-2}{n-1}, \quad (n \geq 1).$$

since we consider only one of $n!$ possible orders.

We can also derive a recurrence relation for g_n, using its definition as follows. In each scheme for $a_1, a_2, \ldots, a_n$ there is a final multiplication $\times$ (corresponding to the outer parentheses):

$$((\text{scheme for } a_1, \ldots, a_k) \times (\text{scheme for } a_{k+1}, \ldots, a_n)).$$

The multiplication scheme for $a_1, \ldots, a_k$ can be chosen in g_k ways, and the multiplication scheme for $a_{k+1}, \ldots, a_n$ can be chosen in g_{n-k} ways. Since k can be any of the numbers $1, 2, \ldots, n-1$ we have

$$g_n = g_1 g_{n-1} + g_2 g_{n-2} + \cdots + g_{n-1} g_1, \quad (n \geq 2). \qquad (8.5)$$

This recurrence relation, along with the initial condition $g_1 = 1$, uniquely determines the counting sequence

$$g_1, g_2, g_3, \ldots, g_n, \ldots .$$

Thus the solution of the recurrence relation (8.5) which satisfies the initial condition $g_1 = 1$ is

$$g_n = \frac{C_n^*}{n!} = \frac{1}{n}\binom{2n-2}{n-1}, \quad (n \geq 1).$$

The recurrence relation (8.5) is the same recurrence relation that occurred in section 7.6 in connection with the problem of dividing a convex polygonal region into triangles by means of its diagonals. Thus we have a purely combinatorial derivation[5] of the formula obtained in section 7.6, and we conclude that the number of ways to divide a convex polygonal region with $n+1$ sides into triangular regions by inserting diagonals which do not intersect in the interior is the same as the number of multiplication schemes for n numbers given in a specified order!

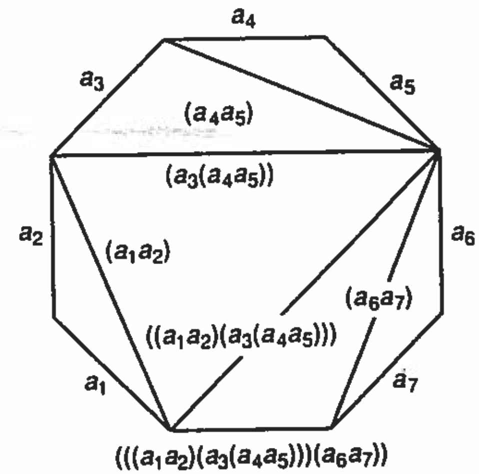


Figure 8.1

The correspondence between the multiplication schemes for the n numbers $a_1, a_2, \ldots, a_n$ and triangularizations of a convex polygonal regions of $n+1$ sides is indicated, in the Figure 8.1, for $n = 7$. Each diagonal corresponds to one of the multiplications other than the last, with the base of the polygon corresponding to the last multiplication.

8.2 Difference Sequences and Stirling Numbers

Let

$$h_0, h_1, h_2, \ldots, h_n, \ldots \qquad (8.6)$$

be a sequence of numbers. We define a new sequence

$$\Delta h_0, \Delta h_1, \Delta h_2, \ldots, \Delta h_n, \ldots, \qquad (8.7)$$

[5] The derivation in section 7.6 is analytic in nature.

30. Prove that

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \cdot \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{4.15}$$

31. Let us enumerate integer partitions whose Ferrers shape fits within an $m \times n$ rectangle according to their sizes. That is, let

$$p(m,n,q) = \sum_a q^{|a|},$$

where a ranges over all integer partitions that have at most m rows and at most n columns, and $|a|$ denotes the integer of which a is a partition. For instance, $p(2,2,q) = 1 + q + 2q^2 + q^3 + q^4$.

Prove that $p(m,n,q) = \begin{bmatrix} m+n \\ m \end{bmatrix}$.

32. We say that a permutation *avoids* the pattern 132 if it does not have three elements that relate to each other the same way as 1, 3, and 2. That is, if $p = p_1p_2\cdots p_n$, then p is 132-avoiding if there are no three indices $i < j < k$ so that $p_j > p_k > p_i$. That is, there are no three entries in p among which the leftmost is the smallest and the one in the middle is the largest, just as in 132. For instance, 42351 is 132-avoiding, but 35241 is not, for the three entries 3, 5, and 4. So we will say that 35241 *contains* 132.

Prove that the number of 132-avoiding n-permutations is the nth Catalan number.

33. Let q be any permutation of length k, and define q-avoiding permutations in an analoguous way to 132-avoiding permutations, with q playing the role of 132. Let $S_n(q)$ be the number of n-permutations avoiding the pattern q. For what permutations q does the result of the previous exercise immediately imply that $S_n(q) = C_n$?

34. Prove that if the n-permutation $p = p_1p_2\cdots p_n$ contains a 312-pattern, then it must contain a 312-pattern in which entries playing the role of the entries 3 and 1 of the 312-pattern are *consecutive entries* in p.

35. Find a non-generating function proof for the result of Example 3.34.

36. A *regular tetrahedron* is a solid with four vertices, six edges, and three faces, so that each edge is of the same length. See Figure 4.5 for an illustration.

Figure 4.

(a) Find the numbe
(b) Find the numb that can be obt
(c) Symmetries of vertex set. Does us whether the series of rotatio

4.8 Solutions to

1. Recall that $A(n,k)$ descents. If $p = p_1$ $p^r = p_np_{n-1}\cdots p_1$, ha descents. In other w set of permutations counted by $A(n, n+$

Alternatively, instea That is, for $p = p_1p_2$ entry is $n+1-p_i$. I has $k-1$ ascents, ar

2. In a permutation co an increasing sequen increasing sequence. entries, we know th easy to prove compu is taken for $i = \lfloor n/2$

3. Let $\pi = \{B_1, B_2, B_3$ blocks B_1, B_2, B_3 ha

vious that then all a_i have
permutations in which both
mutations in which neither
w is that there are as many
$p_{i+1} < p_{i+2}$ (set A) as there
$p_i < p_{i+1} > p_{i+2}$ (set B).
Leave the entries on the left
. Then rearrange the entries
d then taking complements
nology of Exercise 32, if the
turn it into 231, and if it
number of total inversions
new permutation is still in
nap is bijective, since taking
ation.

ns 1122, 1212, 1221, 2112,

$+ q^3 + q^4$.

ations 12222, 21222, 22122,

$+ q^3 + q^4$.

permutations 11122, 11212,
, 21121, 21211, and 22111.

$+ 2q^4 + q^5 + 1$.

iset K. Then reverse p, and
Then the new permutation
consisting of k copies of 2
nutation counted by $\left[{n \atop n-k}\right]$.
our claim.

ns that end in a 2, and the
ns that end in a 1.

31. It is easy to prove that $p(m,n,q)$ satisfies the same recursive relation as $\left[{m+n \atop m}\right]$. Indeed, if the Ferrers shape of a partition fits into an $m \times n$ rectangle, then there are two possibilities: Either the partition has at most $m-1$ parts, and then its Ferrers shape fits even into an $(m-1) \times n$ rectangle, or the partition has m parts, and then, after removing the first column of its Ferrers shape, its remaining shape fits into an $m \times (n-1)$ rectangle. These two cases correspond to the two summands in the recursive relation satisfied by the Gaussian coefficients.

32. We prove the statement by induction on n, the initial case of $n = 1$ being trivial. We will say that there is one permutation of length 0 that avoids 132.

Now assume that we know the statement for all nonnegative integers less than n. Suppose we have a 132-avoiding n-permutation in which the entry n is in the ith position. Then it is clear that any entry to the left of n must be larger than any entry to the right of n, otherwise the two entries violating this condition and the entry n, would form a 132-pattern. Moreover, by our induction hypothesis, there are C_{i-1} possibilities for the substring of entries to the left of n, and C_{n-i} possibilities for that to the right of n. Summing over all allowed i, we get the following recursion:

$$C_n = \sum_{i=0}^{n-1} C_{i-1} C_{n-i}, \qquad (4.16)$$

and we know from (3.22) that this is the recursion of the Catalan numbers.

33. If the n-permutation p contains the pattern q, then the reverse of p will contain the reverse of q. This implies $S_n(132) = S_n(231)$. Similarly, if p contains the pattern q, then the complement of p will contain the complement of q. Therefore, $S_n(132) = S_n(213)$. Finally, $S_n(213) = S_n(312)$ by taking reverses. So all four patterns 132, 213, 231, 312 are avoided by C_n permutations of length n.

This is not even the end of the story, since we also have $S_n(123) = S_n(321) = C_n$. The latter is somewhat harder to prove (Supplementary Exercise 39).

34. Let p_i, p_j, and p_k form a 312-pattern in p. If there are several 312-patterns in p, choose the one for which $j - i$ is minimal. In that

 # Pr. (11) Let a_n be the number of strings of length n formed by the letters of $\Sigma = \{a, b, c\}$ so that they have an even # of a's. Find a_n as a function of n.

Sol. Consider parallely

d_n = # of strings as requested, but with an odd # of a's.

Then $a_1 = 2$ b, c

$a_2 = 5$ aa, bc, cb, bb, cc

$a_3 = 14$ $aa{b \atop c}, a{b \atop c}a, {b \atop c}aa$, $+ 2^3$ words of length 3 over $\{b, c\}$

Also, we have

$a_n = 2a_{n-1} + d_{n-1}$ (counting if the first symbol is b or c + first symbol a)

and

$d_n = 2d_{n-1} + a_{n-1}$ (again, if the first symbol is b or c + first symbol a)

$d_{n-1} = a_n - 2a_{n-1}$

$d_n = a_{n+1} - 2a_n$

$a_{n+1} - 2a_n = 2(a_n - 2a_{n-1}) + a_{n-1}$

$a_{n+1} - 4a_n + 3a_{n-1} = 0$

$a_{n+2} - 4a_{n+1} + 3a_n = 0$ with $a_1 = 2$, $a_2 = 5$

(ix) Show that if we choose an integer less than 16 and another 99 integers from the numbers $1, 2, 3, \ldots, 200$, then among these 100 integers there are two different ones whose quotient is an integer.

(x) Can the number 16 in the statement of (ix) be replaced by 17?

*(xi) Suppose that n different numbers are chosen from the set $\{1, 2, \ldots, 2n-1\}$ such that every chosen number is not divisible by any other one of the chosen numbers. Show that none of these numbers can be less than 2^k, where the integer k is uniquely determined by the condition $3^k < 2n < 3^{k+1}$.

*(xii) A generalization of 4.13.(i). Given the collection of n real numbers $x_1, x_2, \ldots, x_n$, form the $N = \binom{n}{2}$ sums $x_i + x_j$ $(1 \le i < j \le n)$, and denote them by $y_1, y_2, \ldots, y_N$ (in any order). Show that if $n \ne 2^k$, $k \in \mathbb{N}$, then the original numbers $x_1, x_2, \ldots, x_n$ are uniquely determined by the collection of numbers $y_1, y_2, \ldots, y_N$. Furthermore, show that this conclusion is not valid for any number $n = 2^k$.

5 Iterations

By the term *iteration* we usually understand in mathematics the result of some repetition of the same mathematical operation, algorithm, rule, etc. In Section 3.9 of this chapter we have already solved several problems concerning iterations of certain operations on arrays. Before turning to further problems on numerical configurations, we present a description of such problems in a general setting. This will enable us in the following sections to formulate in general some possible approaches to their solutions.

The letter X will denote the set of all configurations to be considered in a given iterative problem. This may be, for instance, the set of all 4×4 integer arrays, or the set of all real sequences of a given length. The given "rule" for the individual elements $x \in X$ that we will iterate in the problem then introduces a certain *relation* Ω on X, that is, a (nonempty) subset of the Cartesian product $X \times X$ $(\Omega \subseteq X \times X)$. In all problems the relation Ω can be described as follows: We have a rule that, given an element $x \in X$, allows us to form all elements of the set

$$\Omega(x) \overset{\text{def}}{=} \{y \in X : (x, y) \in \Omega\}$$

(if the set $\Omega(x)$ has only one element for all $x \in X$, then the relation Ω is a *mapping* $\Omega: X \to X$). If the set X does not have too many elements, then it is convenient to visualize the relation Ω by way of a *directed graph* (see Example 5.1.(i)).

A finite or infinite sequence $x_1, x_2, \ldots$ of elements of the set X will be called *iterative* (with the relation Ω) if for any two neighboring terms x_i, x_{i+1} we have $x_{i+1} \in \Omega(x_i)$. We now state the three most frequent questions that occur in the study of iterative sequences.

(a) The *attainability problem*. Given two sets $A \subseteq X$ and $B \subseteq X$, decide whether there is an iterative sequence $x_1, x_2, \ldots, x_n$ such that $x_1 \in A$ and $x_n \in B$. Of special importance is the case where one or both of the sets A, B have only one element. We say that the element b is *attainable* from the element a if there is an iterative sequence $a = x_1, x_2, \ldots, x_n = b$; if at the same time a is also attainable from b, then we call a, b a pair of mutually attainable elements.

(b) The *finiteness problem*. Decide whether there are infinite iterative sequences; if the answer is negative, establish or estimate an upper bound for the maximal length of these sequences. (In this formulation we have to exclude from Ω all pairs of the form (x, x); otherwise, we would have to talk about the *stabilization problem* of infinite iterative sequences.)

(c) The *periodicity problem*. Decide whether there exist periodical iterative sequences; possibly describe all of them or find their periods.

5.1 Introductory Examples

We begin by illustrating the problems surrounding iterative sequences, which we just sketched, with a few examples.

(i) On a table there are 6 pebbles, divided into several piles. From each pile we take one pebble and form a new pile with them. We keep repeating this operation. Decide how many piles there will be on the table after 30 steps (the initial distribution of the pebbles is not known).

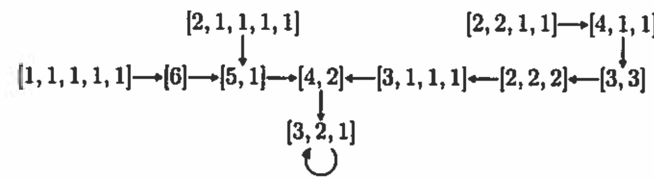


Figure 11

SOLUTION. We describe each distribution of the pebbles into piles by a collection of numbers, each giving the number of pebbles in a pile. Since the order clearly does not matter, each distribution is described by one of the collections [6], [5,1], [4,2], [3,3], [4,1,1], [3,2,1], [2,2,2], [3,1,1,1], [2,2,1,1], [2,1,1,1,1], [1,1,1,1,1,1]. The course of the operations can then be visualized by the directed graph of Figure 11, in which the arrows represent the change of the composition of the piles in one step. We easily see from the graph that after at most 6 steps we obtain the distribution [3,2,1], which will not change any further (and thus after 30 steps there will be three piles on the table). □

We note that the final sections, 5.11 and 5.12, will be devoted to a generalization of Problem (i) to the case of an arbitrary number of pebbles.

(ii) From a quadruple of positive numbers (a,b,c,d) we form a new quadruple (ab,bc,cd,da), and keep repeating this operation. Show that in the resulting iterative sequence of quadruples the original quadruple (a,b,c,d) does not occur again, with the exception of the case where $a=b=c=d=1$.

SOLUTION. Suppose that after several steps we obtain the original quadruple, and we set $s=abcd$. Since $(ab)(bc)(cd)(da)=s^2$, we can easily deduce by induction that after k steps we obtain a quadruple of numbers whose product is s^{2^k}. Therefore, it follows from our assumption that $s^{2^k}=s$ for some $k\ge 1$, and thus $s=1$. Let us now see what in the case $abcd=1$ the fourth quadruple looks like:

$$(a,b,c,d)\to(ab,bc,cd,da)\to\left(ab^2c,bc^2d,cd^2a,da^2b\right)\to$$
$$\to\left(ab^3c^3d,bc^3d^3a,cd^3a^3b,da^3b^3c\right)=\left(b^2c^2,c^2d^2,d^2a^2,a^2b^2\right).$$

Thus the fourth quadruple comes from the second quadruple by squaring the elements and then changing their orders; in a similar way the sixth quadruple comes from the fourth, the eighth from the sixth, etc. Therefore, the largest number in the $2k$th quadruple is equal to $t^{2^{k-1}}$, where t is the largest one of the numbers ab,bc,cd,da. Since by assumption the iterative sequence is periodic, the sequence of numbers $t,t^2,t^4,t^8,\dots$ can have only finitely many different terms; this is possible only when $t=1$. On the other hand, we have

$$1=a^2b^2c^2d^2=(ab)(bc)(cd)(da)\le t\cdot t\cdot t\cdot t=t^4.$$

From this it follows that $ab=bc=cd=da=1$. The second quadruple is then $(1,1,1,1)$; therefore, all following quadruples have the same form, and finally the original one does as well. □

(iii) From the n-tuple of numbers $x_1,x_2,\dots,x_n$ consisting of $+1$ and -1 we form the new n-tuple $(x_1x_2,x_2x_3,\dots,x_nx_1)$, and keep repeating this operation. Show that if $n=2^k$ for some integer $k\ge 1$, then after a certain number of steps we obtain the n-tuple $(1,1,\dots,1)$.

SOLUTION. We prove the assertion, which is clear for $k=1$, by induction on k. We assume that it is true for some $k\ge 1$ and consider an arbitrary sequence $(x_1,x_2,\dots,x_n)$ of the numbers ±1 of length $n=2^{k+1}$. Since $x_i^2=1$ for all i, the second iteration

$$\left(x_1x_2^2x_3,x_2x_3^2x_4,\dots,x_{n-1}x_n^2x_1,x_nx_1^2x_2\right)$$

can be written as the n-tuple $(x_1x_3,x_2x_4,\dots,x_{n-1}x_1,x_nx_2)$, which arises as a regular interlacing of the terms of the two 2^k-tuples

$$(x_1x_3,x_3x_5,x_5x_7,\dots,x_{n-1}x_1)\quad\text{and}\quad(x_2x_4,x_4x_6,\dots,x_nx_2).\qquad(29)$$

The same rule can also be used for obtaining the fourth iteration from the second, the sixth from the fourth, etc. Therefore, after $2j$ steps ($j\ge 2$) we obtain from the original n-tuple one in which the terms of the $(j-1)$th iterations of both sequences in (29) are regularly interlaced. But by the induction hypothesis these iterations consist only of ones for sufficiently large j. This completes the proof by induction. □

(iv) In a triple of positive integers we replace one of them by the sum of the remaining two, decreased by 1, where this transformation is considered as an iterative step only when the original triple becomes in fact a different one. Show that in any finite iterative sequence

$$[a_0,b_0,c_0]\to[a_1,b_1,c_1]\to\cdots\to[a_n,b_n,c_n]\qquad(30)$$

one can determine from the final iteration $[a_n,b_n,c_n]$ all preceding ones with the exception of the original triple $[a_0,b_0,c_0]$. (The order of the numbers in a triple is irrelevant.)

SOLUTION. We note that in contrast to the previous three examples the rule under consideration does not determine a mapping, since, for example,

$$\Omega([1,2,3])=\{[4,2,3],\ [1,3,3],\ [1,2,2]\}.$$

Because the triple $[1,2,2]$ is at the same time the first iteration of all triples of the form $[1,2,n]$, we don't even have the assertion that each triple is determined by its first iteration. Nevertheless, we will show that in any iterative sequence (30) and for every index $k>1$ the triple $[a_k,b_k,c_k]$, where $a_k\le b_k\le c_k$, is necessarily preceded by a triple of the form $[a_k,b_k,b_k-a_k+1]$. Indeed, from the rule $[a_{k-1},b_{k-1},c_{k-1}]\to[a_k,b_k,c_k]$ it follows that one of the equalities

$$a_k=b_k+c_k-1,\quad b_k=a_k+c_k-1,\quad c_k=a_k+b_k-1$$

must hold. We show that only the third one can hold: Since $a_k\le b_k\le c_k$, the first (respectively second) equality is possible only if $a_k=b_k=c_k=1$ (respectively $a_k=1$ and $b_k=c_k$), which in both cases is a contradiction to $k>1$. This means that we have $[a_{k-1},b_{k-1},c_{k-1}]=[a_k,b_k,x]$ for some $x\in\mathbb{N}$. To determine the number x, we repeat the previous consideration: Since $k>1$, one of the equalities

$$a_k=b_k+x-1,\quad b_k=a_k+x-1,\quad x=a_k+b_k-1\qquad(31)$$

is true, where the third one means that $x=c_k$, and thus $[a_{k-1},b_{k-1},c_{k-1}]=[a_k,b_k,c_k]$, which is a contradiction. From $a_k\le b_k\le b_k+x-1$ it follows that the first equality in (31) is possible only when $a_k=b_k$; in any case we therefore have the middle equality, from which we obtain $x=b_k-a_k+1$. This completes the proof. □

5.2 *Exercises*

(i) Suppose that k even and $k+1$ odd numbers are written on the circumference of a circle in some order. Between any two neighboring numbers we write their sum, then remove the original numbers, and repeat the process with the new $2k+1$ numbers. Show that after any number of steps at least one of the $2k+1$ numbers will be odd.

(ii) Continue to study the operation described in (i), when in the starting position there are 25 numbers on the circumference of the circle: 12-times the number $+1$ and 13-times -1. Show that after 100 steps one of the 25 numbers will be less than -10^{28}.

(iii) Return to the situation of 5.1.(iii), with a general n. Show that for $n = 2^k \cdot \ell$, where $k \geq 0$ and $\ell \geq 3$ is odd, after a finite number of steps you obtain the n-tuple $(1, 1, \ldots, 1)$ if and only if the elements of the original n-tuple $(x_1, x_2, \ldots, x_n)$ satisfy $x_i = x_{i+2^k}$ for all $i = 1, 2, \ldots, n - 2^k$.

5.3 *The Method of Invariants*

We now describe an important concept that is often useful in the solution of iterative problems. An *invariant* of a given relation Ω on X with values in K is any mapping $I\colon X \to K$ that is nonconstant (that is, $I(x) \neq I(y)$ for some distinct elements $x, y \in X$) and that has the property that $I(x) = I(y)$ for any pair $(x, y) \in \Omega$. In our problems K will always be some set of numbers. Since the value of the invariant does not change on the elements of an arbitrary iterative sequence, we conclude that the element y is not attainable from the element x if there exists an invariant I such that $I(x) \neq I(y)$. We will illustrate this *nonattainability rule* by way of four examples.

(i) On the circumference of a circle there are 2 ones and 48 zeros in the order $1, 0, 1, 0, \ldots, 0$. It is allowed to change any pair of neighboring numbers x, y by the pair $x+1, y+1$ (in this order). Show that by repeating this operation we cannot end up with all 50 numbers being identical.

Solution. We denote the numbers on the circumference of the circle by $x_1, x_2, \ldots, x_{50}$, counting in a certain direction. Since we have the identity $(x+1) - (y+1) = x - y$, an invariant of the operation in question is an expression that for every pair of neighboring terms x_i, x_{i+1} depends only on the difference $x_i - x_{i+1}$. It is not difficult to guess that the expression

$$I = x_1 - x_2 + x_3 - x_4 + \cdots + x_{49} - x_{50}$$

is of this kind (in checking this, you should not forget that x_{50}, x_1 are also a pair of neighboring terms). For the original sequence we have $I = 1 - 0 + 1 = 2$, while any sequence of 50 identical numbers gives $I = 0$. This completes the proof of nonattainability. □

(ii) Suppose that an $n \times n$ array consists of the signs $+$ and $-$. It is allowable to change all the signs that lie on the fields of the same row, or the same column, or on the fields of a "slanted row" parallel to one of the two diagonals (such a "row" is also formed by each of the corner fields of the array). For each of the three arrays in Figure 12, where $n = 4$, 5, respectively 6, decide whether upon repeating the operations described above one can transform them into arrays consisting of n^2 copies of the $+$ sign.

+	+	−	+
+	+	+	+
+	+	+	+
+	+	+	+

+	−	−	−	+
+	+	+	+	+
−	+	+	+	−
+	+	−	+	+
+	−	+	+	+

+	+	+	+	+	+
+	+	+	+	+	+
+	+	+	−	−	+
+	+	+	+	+	+
+	+	+	+	+	−
+	+	+	+	+	+

Figure 12

Solution. We change the signs in the arrays to the numbers $+1$ and -1 and denote them in the obvious way by a_{ij}, $i, j \in \{1, 2, \ldots, n\}$. Then the product of some factors a_{ij} is an invariant of the transformation if and only if any row, column, or slanted row contains an even number of the factors in question. Try to find such a product for $n = 4$; you will notice that it is unique and has the form

$$I = a_{12}a_{13}a_{21}a_{24}a_{31}a_{34}a_{42}a_{43}.$$

Since for the first array in Figure 12 we have $I = -1$, it cannot be transformed into an array with the value $I = 1$. For the other two arrays it is not necessary to look for further invariants; it suffices to consider the same invariant constructed for their 4×4 subarrays placed at their lower right corners. Since in both cases we have again $I = -1$, it is not possible to transform these two to the desired arrays without minus signs. □

(iii) In the set X_n of all sequences $a = (a_1, a_2, \ldots, a_n)$ consisting of the numbers $0, 1$, the following transformation is allowed: In each sequence $(a_1, \ldots, a_n)$ we may interchange any two neighboring triples of elements (a_i, a_{i+1}, a_{i+2}) and $(a_{i+3}, a_{i+4}, a_{i+5})$, where $1 \leq i \leq n-5$, that is, transform it to the sequence

$$a' = (a_1, \ldots, a_{i-1}, a_{i+3}, a_{i+4}, a_{i+5}, a_i, a_{i+1}, a_{i+2}, a_{i+6}, \ldots, a_n).$$

Show that from the set X_n one can choose at least p sequences that are pairwise nonattainable by the transformation described, where the number

p is given by

$$p = \begin{cases} (k+1)^3 & \text{if } n = 3k, \\ (k+2)(k+1)^2 & \text{if } n = 3k+1, \\ (k+2)^2(k+1) & \text{if } n = 3k+2. \end{cases}$$

SOLUTION. For each $a = (a_1, a_2, \dots, a_n) \in X_n$ we set

$$S_1(a) = a_1 + a_4 + a_7 + \cdots,$$
$$S_2(a) = a_2 + a_5 + a_8 + \cdots,$$
$$S_3(a) = a_3 + a_6 + a_9 + \cdots.$$

The sums S_1, S_2, S_3 are clearly invariants, since any allowable interchange of triples causes only the switching of two neighboring summands in each of the sums S_i. In the case $n = 3k$ for any one of $p = (k+1)^3$ triples of numbers chosen from $\{0, 1, \dots, k\}$ we can easily find a sequence $a \in X_n$ such that $S_i(a) = \alpha_i$ $(i = 1, 2, 3)$, so these p sequences are pairwise nonattainable. Similarly, the appropriate number p of sequences can also be found in the cases $n = 3k + 1$ and $n = 3k + 2$, where the first, respectively both, of the numbers α_1, α_2 can also take on the value $k + 1$. □

(iv) Suppose that the number -1 is written on the corner A_1 of the regular 12-gon $A_1A_2 \dots A_{12}$, and the other corners have the number $+1$. It is allowed to simultaneously change the signs of the numbers at six arbitrary neighboring corners. Show that no repetition of this rule can make the corner A_2 to be -1 and the remaining corners $+1$. Also prove the same assertion for the case where not six but four signs of the numbers at any four neighboring corners can be changed.

SOLUTION. Let a_k be the number written on the corner A_k $(1 \le k \le 12)$. In the case of changing the signs of the numbers on six neighboring corners the sign of each of the products a_1a_7, a_2a_8, a_3a_9, $\dots$, a_6a_{12} will change. Therefore, $I = (a_2a_8)(a_3a_9)$ is an invariant that in the starting position has the value $I = 1$. Hence no number of operations can lead to $a_2 = -1$ and $a_3 = a_8 = a_9 = 1$, since this would mean that $I = -1$. In the case of changing the signs on four neighboring corners, the signs of all of the products $a_1a_5a_9, a_2a_6a_{10}, a_3a_7a_{11}, a_4a_8a_{12}$ will change, and the desired assertion follows from considering the invariant $I = (a_1a_5a_9)(a_3a_7a_{11})$. □

5.4 *Exercises*

(i) Suppose that in a 4×4 array consisting of the signs + and − it is allowed to change all the signs in any row or in any column. Determine whether the arrays in Figure 13 are mutually attainable.

+	−	+	−
−	+	−	+
+	−	+	−
−	+	−	+

−	+	−	+
+	−	−	−
+	+	+	+
+	+	+	−

Figure 13

(ii) Does the assertion from 5.3.(iv) hold if the allowable operation consists of changing signs of the numbers in any triple of neighboring corners?

(iii) Answer the question from (ii) for changing signs in the corners that form an isosceles but not a right triangle.

(iv) Let M be an arbitrary finite subset of $\mathbb{R} \times \mathbb{R}$. If $(x, y) \in M$ is any pair such that $(x + 1, y) \notin M$ and $(x, y + 1) \notin M$, then we may exchange the pair (x, y) in the set M with the two pairs $(x + 1, y)$ and $(x, y + 1)$. By repeating these operations, can an initial set $M = \{(1, 1)\}$ be changed into a set M' that is such that $x + y > 4$ whenever $(x, y) \in M'$?

***(v)** A generalization of 5.3.(iv). Suppose that the numbers $a_1, a_2, \dots, a_n$, where $a_k \in \{-1, 1\}$, $1 \le k \le n$, are written next to each other on the corners of a regular n-gon. For a fixed integer p $(1 \le p \le n)$ it is allowed to simultaneously change the signs of the numbers a_k in any p neighboring corners. To the n-tuple of numbers $(a_1, a_2, \dots, a_n)$ we associate the d-tuple $(s_1, s_2, \dots, s_d)$, where d is the greatest common divisor of the integers n and p, and where $s_k = a_k a_{k+d} a_{k+2d} \cdots a_{k+n-d}$ $(1 \le k \le d)$. Show that two n-tuples $(a_1, a_2, \dots, a_n)$ and $(a'_1, a'_2, \dots, a'_n)$ are mutually attainable through repeated use of the operation described above if and only if either $s_k = s'_k$ $(1 \le k \le d)$, or $s_k = -s'_k$ $(1 \le k \le d)$ and the integer $\frac{p}{d}$ is odd. (The numbers s'_k are formed from the a'_k in the same way as the numbers s_k are formed from the a_k.)

5.5 *Invariants in Residue Classes*

For solving problems concerning operations on integer configurations it is often useful to find invariants with values in the set $\{0, 1, 2, \dots, m-1\}$ of residue classes with an appropriately chosen modulus m. We illustrate this with the following five examples.

(i) Suppose that a 10×10 array consists of integers, and we are allowed to choose any 3×3 or 4×4 subarray and increase every number in it by 1. Is it always possible to appropriately repeat this operation such that we obtain a new array all of whose numbers are divisible by 3?

SOLUTION. If $r_1, r_2, \dots, r_{10}$ are the row sums of the array, then none of the allowed operations with a 3×3 subarray changes the remainder that the sums r_i leave upon division by 3; an operation with a 4×4 subarray changes exactly four of the sums r_i, which are increased by 4. Among four sums changed in this way there is exactly one of the sums r_4, r_8, and therefore exactly three of the other sums. Hence as an invariant we can take the remainder upon division by 3 of the sum

$$I = r_1 + r_2 + r_3 + r_5 + r_6 + r_7 + r_9 + r_{10}.$$

If we choose an initial array such that, for example, $r_1 = 1$ and $r_i = 0$ $(1 < i \le 10)$, then it is not possible to repeat the allowable operations in such a way that the sum I is an integer multiple of 3, which means that not every number in the final array can be a multiple of 3.

We note that in solving this problem it was impossible to use the approach of 3.9.(iv), since the number of subarrays that can be changed is

$$(10-3+1)^2 + (10-4+1)^2 = 64 + 49 = 113 > 10^2. \qquad \square$$

(ii) In the four-element set $M = \{(0,0),(1,1),(-3,0),(2,-1)\}$ it is allowed to replace any pair (a,b) by the pair $(a+2c, b+2d)$, if (c,d) also belongs to M. Decide whether it is possible to obtain the four-element set $M' = \{(-1,2),(2,-1),(4,0),(1,1)\}$ through an appropriate sequence of such operations.

SOLUTION. We note that $3 \mid (x-y)$ for each pair (x,y) in the original set M. This property is an invariant, since the number

$$(a+2c) - (b+2d) = (a-b) + 2(c-d)$$

is a multiple of 3 if both numbers $a-b$ and $c-d$ are. Therefore, no sequence of operations will lead to the new pair $(4,0)$. □

(iii) Suppose that several ones, twos, and threes are written on a blackboard. It is allowed to erase any two different digits and adjoin the remaining third digit (thus the number of digits on the blackboard is decreased by 1). Show that if after a number of such operations one single digit remains on the board, then this digit is determined by the original situation, that is, it does not depend on the particular sequence of the allowable operations.

SOLUTION. The situation in which there are exactly p ones, q twos, and r threes written on the blackboard will be denoted by the triple (p,q,r). An allowable operation is then the change of (p,q,r) to one of the triples $(p-1,q-1,r+1)$, $(p-1,q+1,r-1)$, $(p+1,q-1,r-1)$. We note that in each of these operations the parities of all three numbers p,q,r change. Therefore, the parity of each of the sums

$$s_1 = p+q, \quad s_2 = p+r, \quad \text{and} \quad s_3 = q+r$$

is invariant. Let us write down the values s_i for the states where one digit remains on the blackboard:

(p,q,r)	s_1	s_2	s_3
(1,0,0)	1	1	0
(0,1,0)	1	0	1
(0,0,1)	0	1	1

Any two of these three states differ in the parity of two of the numbers s_i, and can therefore not be the result of the same starting position. □

(iv) Along the circumference of a circular park there are n linden trees, on each of which there is one lark. From time to time two of the larks fly simultaneously to a neighboring tree, but in opposite directions. Decide whether it is possible that at some time all the larks end up in one tree.

SOLUTION. We number the trees consecutively, in one direction, by $1,2,\dots,n$. If n is odd, $n = 2k+1$, then the desired situation can occur, for instance, when successively the pairs from trees 2 and $2k+1$, 3 and $2k$, 4 and $2k-1$, …, $k+1$ and $k+2$ all fly to linden tree 1.

We use the method of invariants to show that for even n such a situation can never occur. Suppose that at some moment exactly p_j larks sit on tree j $(1 \le j \le n)$. Then we consider the sum

$$S = 1\cdot p_1 + 2\cdot p_2 + \dots + n\cdot p_n.$$

When a lark flies to the neighboring tree in the direction of the numbering, then the value of S either increases by 1 or decreases by $n-1$; upon flying in the opposite direction, the value of S either decreases by 1 or increases by $n-1$. Therefore, when a pair of larks fly to their new trees in an allowable fashion, then the change of S is equal to 0, n, or $-n$. Therefore, the remainder of the number S upon division by n is an invariant. In the starting position the value

$$S = 1\cdot 1 + 2\cdot 1 + \dots + n\cdot 1 = n\cdot \frac{n+1}{2}$$

is not divisible by n (the number $\frac{n+1}{2}$ is not an integer, since n is even); on the other hand, in the position where all larks sit on linden tree j we have $S = n\cdot j$. This completes the proof. □

(v) In the sequence $1,0,1,0,1,0,3,5,0,\dots$, each term (beginning with the seventh) is equal to the last digit of the sum of the preceding six terms. Show that in this infinite sequence the numbers $0,1,0,1,0,1$ will never occur in this order.

SOLUTION. Consider the mapping that associates to the sextuple $(x_1, x_2, \dots, x_6)$ the sextuple $(x_2, x_3, \dots, x_6, x_7)$, where x_7 is the final digit of the sum $x_1 + x_2 + \dots + x_6$. An invariant of this mapping is the final digit of the expression

$$I(x_1, x_2, \dots, x_6) = 2x_1 + 4x_2 + 6x_3 + 8x_4 + 10x_5 + 12x_6,$$

since the difference

$$I(x_2,x_3,\dots,x_7) - I(x_1,x_2,\dots,x_6) = 10x_7 + 2(x_7 - (x_1 + x_2 + \cdots + x_6))$$

is, by definition of the digit x_7, divisible by 10. It remains to add that

$$I(1,0,1,0,1,0) = 18 \quad \text{and} \quad I(0,1,0,1,0,1) = 24;$$

the first sextuple is the starting one, and the second is the one whose nonattainability we wanted to show. □

5.6 *Exercises*

(i) On a magic tree there are 25 lemons and 30 oranges. The gardener picks two fruits every day, but the following night one new fruit grows on the tree: An orange (respectively a lemon) if the fruits picked during the day were the same (respectively different). What fruit is the last one to grow on the tree?

(ii) Peter tears a sheet of paper into 10 pieces, then he tears some of the pieces into 10 smaller pieces, etc. Is it possible to get exactly 1991 pieces in this way?

(iii) In the decimal representation of the number 2^{1991} we remove the first digit (on the left) and add it to the remaining number. We keep repeating this operation until we obtain a ten-digit number A. Show that at least two of the digits of A are the same.

(iv) A generalization of 5.5.(iv). We consider the more general initial situation where $p_1, p_2, \dots, p_n$ larks, in this order, sit in the linden trees, where $p_1+p_2+\cdots+p_n = N > 1$. Show that if the same rules as before apply to the pairs of larks, then the state $(p_1,p_2,\dots,p_n)$ can be changed to $(p_1',p_2',\dots,p_n')$ if and only if $p_1'+p_2'+\cdots+p_n' = N$ and $p_1 + 2p_2 + \cdots + np_n \equiv p_1' + 2p_2' + \cdots + np_n' \pmod n$.

(v) We return to Example 5.5.(iii), using the notation from its solution. Show that if we can carry out at least one operation on the original position (p,q,r), then upon repetition we can always get exactly one of the four positions $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(2,0,0)$.

*(vi) Suppose that the numbers $1,2,\dots,n$ are written on a blackboard, each number exactly once. We may choose any two of these numbers a,b, erase them, and write the number $|a-b|$ on the blackboard. If we repeat this operation $(n-1)$ times, a single number will remain on the blackboard. What can its value be?

*(vii) Given $k,m \geq 1$, on the set M_m^k of all k-tuples $(x_1,x_2,\dots,x_k)$ of numbers chosen from $\{0,1,\dots,m-1\}$ we consider the transformation $(x_1,x_2,\dots,x_k) \mapsto (x_2,x_3,\dots,x_k,x_{k+1})$, where $x_{k+1} \equiv x_1 + x_2 + \cdots + x_k \pmod m$. Show that an invariant of the form

$$I(x_1,x_2,\dots,x_k) \equiv c_1x_1 + c_2x_2 + \cdots + c_kx_k \pmod m$$

with constants $c_1,c_2,\dots,c_k \in \{0,1,\dots,m-1\}$ exists if and only if the numbers $k-1$ and m are not relatively prime. (Recall that every invariant is a nonconstant mapping, and therefore $c_i \neq 0$ for some $i \in \{1,2,\dots,k\}$.)

5.7 *The Method of Valuations*

For studying iterative sequences of a relation Ω on X we will see that in addition to invariants (see 5.3) a more general mapping $J\colon X \to K$ will be useful, where the values upon iterating the elements of X change monotonically. Thus we call the mapping $J\colon X \to K$ a *nonincreasing* (respectively *decreasing*) *valuation* of the relation Ω if for any pair $(x,y) \in \Omega$, $x \neq y$, we have $J(x) \geq J(y)$ (respectively $J(x) > J(y)$). Similarly, we define a *nondecreasing*, respectively *increasing*, valuation; in all cases K is one of the sets of numbers $\mathbf{R}, \mathbf{N}_0, \mathbf{Z}$ with the usual ordering.

If we notice, for instance, that a certain relation has a nonincreasing valuation J, then for any iterative sequence $x_1, x_2, \dots$ we have

$$J(x_1) \geq J(x_2) \geq \cdots,$$

which often makes it possible to solve the periodicity problem (the condition $J(x_1) = J(x_2)$ usually leads to a description of all possible elements x_1). If furthermore J is a decreasing valuation with values in $\mathbf{N}_0$, then in each infinite iterative sequence $x_1, x_2, \dots$ there exists an index n such that $J(x_n) = J(x_{n+1}) = \cdots$, that is, $x_n = x_{n+1} = \cdots$, since each nonempty subset of $\mathbf{N}_0$ contains its smallest element; this would solve the finiteness problem, respectively the stabilization problem (see the introduction to Section 5).

The problems solved here with the method of monotonic valuations will be divided between the two subsections 5.7 and 5.9.

(i) From the quadruple of real numbers (a,b,c,d) we form the new quadruple

$$(a-b, b-c, c-d, d-a),$$

and keep repeating this transformation. Show that, as long as the original quadruple does not satisfy $a=b=c=d$, after a certain number of steps we obtain a quadruple that contains at least one number larger than 10^6.

Solution. We consider the nonnegative valuation

$$J(a,b,c,d) = a^2 + b^2 + c^2 + d^2$$

and denote by (a_n,b_n,c_n,d_n) the quadruple that is obtained from the original quadruple after n steps. It is easy to see that $a_n + b_n + c_n + d_n = 0$ for all $n \geq 1$, so for each such n we can write

$$\begin{aligned} J(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) &= (a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - d_n)^2 + (d_n - a_n)^2 \\ &= 2J(a_n, b_n, c_n, d_n) - 2(a_n b_n + b_n c_n + c_n d_n + d_n a_n) \\ &= 2J(a_n, b_n, c_n, d_n) - 2(a_n + c_n)(b_n + d_n) \\ &= 2J(a_n, b_n, c_n, d_n) + 2(a_n + c_n)^2 \geq 2J(a_n, b_n, c_n, d_n). \end{aligned}$$

This gives the estimate $J(a_n, b_n, c_n, d_n) \geq 2^{n-1} J(a_1, b_1, c_1, d_1)$ for all $n \geq 1$. Therefore, unless $J(a_1, b_1, c_1, d_1) = 0$ (which occurs only when $a = b = c = d$), it means that $J(a_n, b_n, c_n, d_n) = a_n^2 + b_n^2 + c_n^2 + d_n^2 > 36 \cdot 10^{12}$ for sufficiently large n; but then at least one of the numbers $|a_n|, |b_n|, |c_n|, |d_n|$ must be greater than $3 \cdot 10^6$. However, since $a_n + b_n + c_n + d_n = 0$, we would obtain from the assumption $\max\{a_n, b_n, c_n, d_n\} \leq 10^6$ the bound $\min\{a_n, b_n, c_n, d_n\} \geq -3 \cdot 10^6$, and thus $\max\{|a_n|, |b_n|, |c_n|, |d_n|\} \leq 3 \cdot 10^6$, which is a contradiction.

It is worth remarking that the defining transformation is a linear operator $\mathbb{R}^4 \to \mathbb{R}^4$, so the problem could also be solved in a standard way by finding the eigenvalues and eigenvectors of the corresponding 4×4 matrix. □

(ii) Let a sequence of integers $x_1, x_2, \dots, x_n$ be given. If i, j are arbitrary indices such that $x_i - x_j = 1$, then we may replace the terms x_i, x_j by the numbers $x_i + 1$, $x_j - 1$ (in this order). Show that only a finite number of repetitions of this transformation is possible.

SOLUTION. The integer valuation $J = x_1^2 + x_2^2 + \cdots + x_n^2$ increases by 4 with each transformation, since $(x_i+1)^2 + (x_j-1)^2 - (x_i^2 + x_j^2) = 2(x_i - x_j) + 2 = 4$. If we set $m = \min\{x_1, x_2, \dots, x_n\}$ and $M = \max\{x_1, x_2, \dots, x_n\}$, it suffices to show that for any attainable n-tuple $(y_1, \dots, y_n)$ we have the bounds $m - 3n < y_i < M + 3n$ for each $i = 1, 2, \dots, n$. (These bounds imply that the valuation J is bounded on any iterative sequence.) Our approach is based on the following observation. For each $k \in \mathbb{Z}$ we have that if $\{x_1, \dots, x_n\} \cap \{k-1, k, k+1\} \neq \emptyset$, then also $\{y_1, \dots, y_n\} \cap \{k-1, k, k+1\} \neq \emptyset$ for every n-tuple $(y_1, \dots, y_n)$ that is attainable from $(x_1, \dots, x_n)$. Therefore, if we suppose that the n-tuple $(y_1, \dots, y_n)$, which is attainable from $(x_1, \dots, x_n)$, satisfies $y_i \geq M + 3n$ for some $i \in \{1, 2, \dots, n\}$, then every integer a, $M \leq a \leq y_i$, has to be an element of some n-tuple occurring in the iteration. Hence the set $\{y_1, \dots, y_n\}$ must have a nonempty intersection with each of the $(n+1)$ disjoint sets $\{M-1, M, M+1\}$, $\{M+2, M+3, M+4\}, \dots, \{M+3n-1, M+3n, M+3n+1\}$, which is a contradiction. Similarly, one also excludes the case $y_i \leq m - 3n$ for some $i \in \{1, 2, \dots, n\}$. This completes the proof. □

(iii) Suppose that n real numbers, $n \geq 4$, are written on the circumference of a circle. If four adjacent numbers a, b, c, d satisfy $(a-d)(b-c) < 0$, then we may interchange the places of the neighboring numbers b, c. Show that only a finite number of such transformations can be carried out.

SOLUTION. We first remark that even though there are only finitely many arrangements of n numbers along the circumference of a circle, it is not so clear why a sequence of the transformations described cannot be infinite (e.g., periodic). We note that we have

$$(a-d)(b-c) = (ab + cd) - (ac + bd),$$

where within the first parentheses on the right we have the products of the pairs situated at both ends of the quadruple (a, b, c, d), and within the second pair of parentheses we have the same products for the quadruple (a, c, b, d), which is obtained from the original quadruple by switching the places of the numbers b, c; furthermore, this center pair is the same in both quadruples (up to order). It is therefore convenient to denote the numbers, consecutively in one direction, by $x_1, x_2, \dots, x_n$ and to consider the valuation

$$J = x_1 x_2 + x_2 x_3 + \cdots + x_{n-1} x_n + x_n x_1,$$

which is increasing for the allowable operations. Since for a given n-tuple there can be only finitely many values of J, after a certain number of steps J will attain a maximal value, and further transformations are not possible. □

(iv) Given a triple a, b, c of integers, we form a new triple

$$|a-b|, \quad |b-c|, \quad |c-a|,$$

and keep repeating this transformation. Show that after a certain number of steps we reach a triple containing the number 0. Does this assertion also hold in the case where the initial numbers a, b, c are real?

SOLUTION. We assume that there is an initial triple of numbers $a, b, c \in \mathbb{Z}$, from which after any number of steps we always obtain triples of nonzero (and thus positive) integers. We may clearly assume that the integers a, b, c are also positive (otherwise, we delete the first triple). Let us set

$$J = \max\{a, b, c\} \quad \text{and} \quad J' = \max\{|a-b|, |b-c|, |c-a|\}.$$

For any positive numbers x, y we have

$$|x - y| = \max\{x, y\} - \min\{x, y\} < \max\{x, y\},$$

so we immediately obtain $J' < J$, that is, $J' \leq J - 1$, since the values of J lie in $\mathbb{N}_0$. Hence after n steps we obtain from the triple $[a, b, c]$ the triple of positive numbers $[a_n, b_n, c_n]$ for which

$$0 < \max\{a_n, b_n, c_n\} \leq \max\{a, b, c\} - n$$

holds, but this is a contradiction for $n > \max\{a, b, c\}$. This proves the assertion in the case of integers a, b, c.

In the case of real numbers a, b, c we base the construction of a counterexample on the following consideration. If there exists a triple of positive numbers $[a, b, c]$ that is transformed into the triple $[pa, pb, pc]$ for some $p > 0$, then after an arbitrary number n of transformations this triple turns into the triple of positive numbers $[p^n a, p^n b, p^n c]$. Hence we look for numbers $0 < a < b < c$ such that

$$\frac{b-a}{a} = \frac{c-b}{b} = \frac{c-a}{c} = p > 0.$$

The reader should verify that these equalities hold if and only if $b = (p+1)a$, $c = (p+1)^2 a$, where p satisfies the equation $p^2 + p - 1 = 0$. This equation has the positive root $p = \frac{-1+\sqrt{5}}{2}$; if we choose $a = 1$, then we obtain a triple of the form $\left[1, \frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right]$. Now verify by direct computation of the iterations that this triple has the desired property: No number of steps will give a triple that contains a 0. □

5.8 *Exercises*

(i) Suppose that n positive integers are written on the circumference of a circle. Between all pairs of neighboring integers we write their greatest common divisor, then we delete the original numbers and repeat the process with the new n-tuple of integers. Show that after a certain number of steps we obtain an n-tuple of identical integers.

(ii) Suppose that 100 numbers are written on a blackboard: 10 zeros, 10 ones, ..., 10 nines. We may choose any two of these numbers and replace both of them with their arithmetic mean (for example, the pair $[3, 6]$ may be changed to $[\frac{9}{2}, \frac{9}{2}]$). Determine the smallest positive number that can appear on the blackboard after a sequence of such operations.

(iii) A sequence $x_1, x_2, \dots, x_n$ is formed with nonzero real numbers. We are allowed to choose any two of its terms x_i, x_j $(i \neq j)$ and replace them by $x_i + \frac{x_j}{2}, x_j - \frac{x_i}{2}$, in this order. Show that by iterating this transformation we can never obtain either the original sequence or a sequence that differs from the original one only by the order of its terms.

(iv) From a quadruple of real numbers (a, b, c, d) we form the quadruple $(a+b, b+c, c+d, d+a)$, and keep repeating this process. Show that if at two different times we obtain the same quadruple of numbers (possibly in a different order), then the initial quadruple was of the form $(a, -a, a, -a)$ for an appropriate $a \in \mathbf{R}$.

(v) A generalization of (iv). From an n-tuple of real numbers $a_1, a_2, \dots, a_n$ we form the n-tuple $(a_1 + a_2, a_2 + a_3, \dots, a_n + a_1)$. Does a conclusion similar to that in (iv) hold for every even $n \geq 4$?

*(vi) A generalization of 5.7.(i). For a fixed $n \geq 6$ we consider iterations of n-tuples from $\mathbf{R}^n$ under the transformation

$$(x_1, x_2, \dots, x_n) \mapsto (x_1 - x_2, x_2 - x_3, \dots, x_n - x_1).$$

We assume that the initial n-tuple of real numbers does not satisfy $x_1 = x_2 = \cdots = x_n$. Do we always reach, after a certain number of steps, an n-tuple containing at least one number exceeding 10^6?

(vii) Find all triples of real numbers that have the property that after a certain number of the transformations described in 5.7.(iv) the initial triple is obtained again.

*(viii) For a fixed $m \geq 3$ we consider the mapping

$$(a_1, a_2, \dots, a_m) \mapsto (|a_2 - a_1|, |a_3 - a_2|, \dots, |a_m - a_{m-1}|, |a_1 - a_m|)$$

defined on the set $\mathbf{R}^m$. Show that an initial m-tuple can be chosen in such a way that in the sequence of its iterations no m-tuple contains the number 0.

*(ix) Suppose that in the initial m-tuple from (viii) all the numbers $a_1, a_2, \dots, a_m$ are integers. Show that if m is a power of 2, then the iterative sequence contains an m-tuple consisting only of zeros.

(x) Embedded into a horizontal straight line there is a finite number of arrows (as, for example, in Figure 14), some of which point to the left, and the remaining ones to the right. We choose any two neighboring arrows that point to each other (in Figure 14, for example, the fourth and the fifth from the left) and change the directions of both. Show that this transformation can be repeated only several times, and that the final state (where further changes are no longer possible) is determined by the initial situation; that is, it does not depend on the order in which the changes were carried out. Furthermore, explain why the total number of changes also does not depend on the order of carrying them out.

Figure 14

*(xi) Suppose that n piles with $a_1, a_2, \dots, a_n$ pebbles lie on a table. We may choose any two piles and from one of them move as many pebbles to the other as there were already in this second pile (thus the number of its pebbles is doubled). Find a necessary and sufficient condition on the numbers $a_1, a_2, \dots, a_n$ that makes it possible to get all pebbles onto one pile after a finite number of steps. (Hint: Consider the change of the greatest common divisor d of all numbers $a_1, a_2, \dots, a_n$.)

*(xii) Suppose that an arrangement $a_1, a_2, \ldots, a_n$ of the numbers $1, 2, \ldots, n$ is such that $a_k \neq k$ for some $k \in \{1, 2, \ldots, n\}$. We choose any such k and move the number a_k to the a_kth position in the arrangement $a_1, a_2, \ldots, a_n$. To be more exact: If $a_k = c$, then the new arrangement in the case $c < k$ has the form

$$a_1, a_2, \ldots, a_{c-1}, c, a_c, a_{c+1}, \ldots, a_{k-1}, a_{k+1}, a_{k+2}, \ldots, a_n, \quad (32)$$

while in the case $c > k$ it has the form

$$a_1, a_2, \ldots, a_{k-1}, a_{k+1}, a_{k+2}, \ldots, a_c, c, a_{c+1}, a_{c+2}, \ldots, a_n. \quad (33)$$

Show that after a finite sequence of such changes we obtain the arrangement $1, 2, \ldots, n$ (when no further changes are possible), no matter how the numbers a_k to be moved are chosen.

5.9 The Method of Valuations – Continuation

We now consider three further, and somewhat more difficult, problems that can successfully be solved with the method of monotonic valuations, as introduced in 5.7.

(i) Suppose that a sequence of $2n+1$ integers has the following property: If we remove any of its terms, then the remaining terms can be divided into two n-element sets such that the sums of the numbers in both of them are the same. Show that every such sequence is formed of $2n+1$ identical numbers.

SOLUTION. Each term of the sequence under consideration differs from the sum of all $2n+1$ terms by an even number. This means that the whole sequence consists either exclusively of even or exclusively of odd numbers. Thus we have either a sequence of the form $2a_1, 2a_2, \ldots, 2a_{2n+1}$, or a sequence of the form $2a_1 - 1, 2a_2 - 1, \ldots, 2a_{2n+1} - 1$; in both cases we may instead consider the sequence of integers $a_1, a_2, \ldots, a_{2n+1}$. This "reduction" preserves the original property of the sequence, since for any two n-element sets of indices $I, J \subseteq \{1, 2, \ldots, 2n+1\}$ we have the identities

$$\sum_{i \in I} 2a_i = \sum_{j \in J} 2a_j, \qquad \sum_{i \in I} (2a_i - 1) = \sum_{j \in J} (2a_j - 1),$$

if and only if

$$\sum_{i \in I} a_i = \sum_{j \in J} a_j.$$

Since the inequalities $|x| \le |2x|$ and $|x| \le |2x-1|$ hold for all $x \in \mathbb{Z}$, the above reduction does not increase the sum S of absolute values of all $2n+1$ terms of the sequence. Furthermore, the value S is a nonnegative integer, and therefore we obtain after a certain number of steps a sequence of integers $b_1, b_2, \ldots, b_{2n+1}$, such that the sum $S = |b_1| + |b_2| + \cdots + |b_{2n+1}|$ does not change upon further reductions. Then for all $i \in \{1, 2, \ldots, 2n+1\}$ we have either $|b_i| = |2b_i|$ (that is, $b_i = 0$) or $|b_i| = |2b_i - 1|$ (that is, $b_i = 1$). However, the numbers $b_1, b_2, \ldots, b_{2n+1}$ have the same parity, as we have already seen, and therefore we have either $b_i = 0$ $(1 \le i \le 2n+1)$ or $b_i = 1$ $(1 \le i \le 2n+1)$. Now it is easy to verify by induction that all the preceding $(2n+1)$-tuples also consist of identical numbers. □

(ii) From an arbitrary n-tuple of integers $a_1, a_2, \ldots, a_n$ we form a new n-tuple

$$\left(\frac{a_1+a_2}{2}, \frac{a_2+a_3}{2}, \ldots, \frac{a_{n-1}+a_n}{2}, \frac{a_n+a_1}{2}\right),$$

and keep repeating this transformation. Find all possible initial n-tuples if you know that all n-tuples that occur successively are formed exclusively of integers.

SOLUTION. If $a_1 = a_2 = \cdots = a_n$, then clearly, after each transformation we obtain the same n-tuple of identical integers. However, let us not jump to the premature conclusion that these are all the desired n-tuples. For instance, in the case where n is even, $(c, d, c, d, \ldots, c, d)$ also has the desired property, where c and d are any pair of integers with the same parity. Let now $(a_1, a_2, \ldots, a_n)$ be any n-tuple of integers with the desired property, and let

$$M = \max\{a_1, a_2, \ldots, a_n\} \quad \text{and} \quad m = \min\{a_1, a_2, \ldots, a_n\}.$$

Since M (respectively m) is a nonincreasing (respectively nondecreasing) integer valuation of the given transformation and since $M \ge m$, this means that the values M and m on our infinite iterative sequence do not change from a certain place on (say, starting with the iteration $(b_1, b_2, \ldots, b_n)$). But we have an equality

$$\max\left\{\frac{b_1+b_2}{2}, \frac{b_2+b_3}{2}, \ldots, \frac{b_n+b_1}{2}\right\} = \max\{b_1, b_2, \ldots, b_n\} \ (= M')$$

if and only if in the sequence $b_1, b_2, \ldots, b_n, b_1$ the number M' occurs in some two neighboring places. If we repeat the same argument for the following iterations, we obtain by induction on $k \ge 1$ the following assertion: In the infinite periodic sequence

$$b_1, b_2, \ldots, b_n, b_1, b_2, \ldots, b_n, \ldots$$

the number M' occurs in some $k+1$ neighboring places. This is possible for all $k \ge 1$ only if $b_1 = b_2 = \cdots = b_n$. If the n-tuple $(b_1, b_2, \ldots, b_n)$ is not the initial one, then the preceding iteration $(c_1, c_2, \ldots, c_n)$ must satisfy

$$\frac{c_1+c_2}{2} = \frac{c_2+c_3}{2} = \cdots = \frac{c_n+c_1}{2},$$

that is, $c_1 = c_3 = c_5 = \cdots$ and $c_2 = c_4 = \cdots$. From this it follows in the case where n is odd that $c_1 = c_2 = \cdots = c_n$ (and thus clearly the initial n-tuple consists of n identical numbers). For even n we then have

$$(c_1, c_2, \dots, c_n) = (c, d, c, d, \dots, c, d),$$

where c and d are integers of the same parity. Finally, we prove that in the case $c \neq d$ the n-tuple $(c_1, c_2, \dots, c_n)$ is the initial one; that is, there do not exist integers $e_1, e_2, \dots, e_n$ simultaneously satisfying the identities

$$\frac{e_1 + e_2}{2} = \frac{e_3 + c_4}{2} = \cdots = \frac{e_{n-1} + e_n}{2} = c$$

and

$$\frac{e_2 + e_3}{2} = \frac{e_4 + e_5}{2} = \cdots = \frac{e_n + e_1}{2} = d.$$

Indeed, these identities imply

$$nc = e_1 + e_2 + e_3 + e_4 + \cdots + e_{n-1} + e_n = nd,$$

and thus $c = d$, which is a contradiction. □

(iii) Suppose that the integers $x_1, x_2, \dots, x_n$ are successively written on the corners of a regular n-gon, where

$$S = x_1 + x_2 + \cdots + x_n > 0.$$

If for some $k \in \{1, 2, \dots, n\}$ we have $x_k < 0$, we may carry out the following transformation: The numbers x_{k-1}, x_k, x_{k+1} are changed to $x_{k-1} + x_k$, $-x_k$, $x_{k+1} + x_k$, in this order; here we set $x_0 = x_n$ and $x_{n+1} = x_1$. We may repeat this transformation with the new n-tuple, etc. Show that in the cases $n = 3$ and $n = 5$ only a finite number of transformations can be carried out, that is, after a certain number of steps we obtain an n-tuple of nonnegative integers. (The case of general n and real values $x_1, x_2, \dots, x_n$ is discussed in Section 5.10.)

SOLUTION. First we note that

$$(x_{k-1} + x_k) + (-x_k) + (x_{k+1} + x_k) = x_{k-1} + x_k + x_{k+1},$$

so the sum S is an invariant of the transformation in question. In the case $n = 3$ it is not difficult to come up with the appropriate valuation

$$J(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2; \tag{34}$$

in view of symmetry it suffices to consider a change of J for $k = 2$. Let therefore $x_2 < 0$; the transformation gives

$$\begin{aligned} J(x_1 + x_2, -x_2, x_3 + x_2) &= (x_1 + x_2)^2 + (-x_2)^2 + (x_3 + x_2)^2 \\ &= (x_1^2 + x_2^2 + x_3^2) + 2x_2(x_1 + x_2 + x_3) \\ &= J(x_1, x_2, x_3) + 2x_2 S < J(x_1, x_2, x_3). \end{aligned}$$

Hence J is a decreasing valuation; since by (34) the value of J is a nonnegative integer, this completes the proof of the finiteness assertion in the case $n = 3$.

For $n = 5$ the situation is more complicated; we can convince ourselves that a direct analogue with (34) would not work. Guessing an appropriate valuation $J(x_1, x_2, \dots, x_5)$ would require considerable imagination; however, if we assume that J will be a quadratic form, it is possible to use the method of undetermined coefficients, namely, to search for a J of the form

$$\begin{aligned} J(x_1, x_2, x_3, x_4, x_5) = p\,&(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2) \\ &+ q(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1) \\ &+ r(x_1x_3 + x_2x_4 + x_3x_5 + x_4x_1 + x_5x_2) \end{aligned} \tag{35}$$

with unknown constants p, q, r. The coefficients in (35) are chosen such that J is an invariant with respect to those permutations of the quintuple $x_1, x_2, \dots, x_5$ that do not change the situation of the problem; furthermore, this reduces the study of the change to J again to the case $k = 2$. A routine calculation now gives

$$\begin{aligned} &J(x_1 + x_2, -x_2, x_3 + x_2, x_4, x_5) - J(x_1, x_2, x_3, x_4, x_5) \\ &\quad = 2p\,(x_1x_2 + x_2^2 + x_2x_3) + q\,(-2x_1x_2 - 2x_2^2 - 2x_2x_3 + x_2x_4 + x_2x_5) \\ &\qquad + r\,(x_1x_2 + x_2^2 + x_2x_3 - x_2x_4 - x_2x_5) \\ &\quad = x_2[(2p - 2q + r)(x_1 + x_2 + x_3) + (q - r)(x_4 + x_5)]. \end{aligned}$$

It is clearly convenient to require that $2p - 2q + r = q - r = c > 0$; the term in brackets will then be equal to

$$c(x_1 + x_2 + x_3 + x_4 + x_5) = cS > 0,$$

which together with the condition $x_2 < 0$ shows that J is a decreasing valuation. The assumptions on the numbers p, q, r can be rewritten as

$$p = \frac{r + 3c}{2} \quad \text{and} \quad q = r + c,$$

where the numbers r and c are still arbitrary. By substituting into (35) and rearranging, we get

$$\begin{aligned} J(x_1, x_2, x_3, x_4, x_5) &= \frac{r + c}{2}(x_1 + x_2 + x_3 + x_4 + x_5)^2 \\ &+ \frac{c}{2}\left[(x_1 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_1)^2 + (x_5 - x_2)^2\right]. \end{aligned}$$

From this we see that the value J is a nonnegative integer if both numbers are even, $c > 0$, and $r \geq -c$. The easiest valuation is obtained for $c = 2$ and $r = -2$:

$$J = (x_1 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_1)^2 + (x_5 - x_2)^2.$$

This completes the proof for the case $n = 5$. □

5.10 Exercises

(i) We now weaken the property of the sequence in 5.9.(i); namely, we no longer require the two sets with equal sums of elements to have the same number of elements. Does the conclusion still hold that all such sequences consist of identical numbers?

The following exercises continue the study of the transformation from 5.9.(iii) under the initial condition $S > 0$.

(ii) Find a decreasing valuation with values in $\mathbb{N}_0$ in the case $n = 4$.

*(iii) Show that if we disallow the transformation of the triple x_{k-1}, x_k, x_{k+1} for a fixed k, for instance $k = 1$, then only a finite number of transformations can be carried out, even if the initial numbers $x_1, x_2, \dots, x_n$ are real (and not necessarily integers).

*(iv) Find a decreasing valuation with values in $\mathbb{N}_0$ in the case $n \geq 6$.

*(v) With the help of (iii) and the valuation from the solution to (iv), prove the assertion concerning finiteness of the number of transformations in the general case $n \geq 6$, $x_1, x_2, \dots, x_n \in \mathbb{R}$.

(vi) From the initial numbers $x_1, x_2, \dots, x_n \in \mathbb{R}$ set up a formula for some constant K with the following properties: The absolute value of each of the n numbers that occur in the corners of the n-gon after an arbitrary number of transformations does not exceed the number K.

5.11 Problems on Piles of Pebbles

To finish this section, we return to a problem that we used in 5.1 to introduce the general idea of iterations. We will now consider the general initial situation where n pebbles, divided into several piles, lie on a table. Let us recall the individual transformations: From every pile we take one pebble each, and form a new pile with these pebbles.

It is important to note that in view of the finite number of all possible distributions of n pebbles into piles, after a certain number of steps the situation will necessarily become periodic. However, it is not clear how long the period will be, or whether any initial state will eventually lead to the same "universal" state. We will now proceed to solve these problems.

Let the positive integers $a_1 \geq a_2 \geq \cdots \geq a_m$ denote the numbers of pebbles in the individual piles, so clearly, $1 \leq m \leq n$ and $a_1 + a_2 + \cdots + a_m = n$. We note that the number m of piles in general changes in the course of the transformations. Nevertheless, we consider the valuation

$$J = \sum_{k=1}^{m} \left(a_k^2 + c_k a_k\right), \tag{36}$$

where $c_1, c_2, \dots$ are constants that we will choose in a moment. After carrying out one transformation, the distribution $(a_1, a_2, \dots, a_m)$ changes to $(m,\ a_1 - 1,\ a_2 - 1,\ \dots,\ a_m - 1)$, and if we substitute the new numbers into J in exactly this order, we obtain the new value

$$J' = m^2 + c_1 m + \sum_{k=1}^{m} \left[(a_k - 1)^2 + c_{k+1}(a_k - 1)\right].$$

We note that if $a_k = 1$ for some $k \in \{1, 2, \dots, m\}$, then the corresponding pile disappears; but this clearly has no influence on the calculation of the value of J'. We choose the numbers c_k in such a way that the equality $J = J'$ holds identically in the variables $a_1, a_2, \dots, a_m$. We can rewrite this as

$$\sum_{k=1}^{m} (c_k + 2 - c_{k+1}) a_k = m^2 + m - \sum_{k=1}^{m} (c_{k+1} - c_1), \tag{37}$$

where the left-hand side does not depend on $a_1, a_2, \dots, a_m$ if $c_k + 2 - c_{k+1} = 0$, that is, if $c_k = c_1 + 2(k-1)$ for all $k \geq 1$. But then the right-hand side of (37) is equal to zero, as the reader should verify, which then guarantees that $J = J'$. If, for instance, we choose $c_1 = 2$, then we obtain $c_k = 2k$ for all k, and the definition (36) takes the form

$$J = \sum_{k=1}^{m} \left(a_k^2 + 2k a_k\right). \tag{38}$$

We have agreed that for the computation of J the numbers of pebbles are substituted into J in their nonincreasing order. However, the implication

$$a_1 \geq a_2 \geq \cdots \geq a_m \implies m \geq a_1 - 1 \geq \cdots \geq a_m - 1$$

does not hold in general. If we have to change the arrangement of $m, a_1 - 1, \dots, a_m - 1$ into the nonincreasing one, then the value of J decreases because of the choice of the coefficients c_k. Therefore, J is not an invariant, but only a nonincreasing valuation. However, each value of J is a positive integer, and therefore only a finite number of changes is possible. Hence after a certain number of transformations the numbers $a_1, a_2, \dots, a_m$ of pebbles in the piles must all satisfy the inequalities $m \geq a_k - 1$ $(1 \leq k \leq m)$. We have thus proved the following assertion:

After a certain number of transformations the number of pebbles in any pile is always at most one more than the number of piles at a given moment.

As an illustration we show in Figure 15 an infinite iterative sequence in the case $n = 8$ with initial numbers of pebbles $[5, 1, 1, 1]$. If we compare Figure 15 with Figure 11, we see that in the case $n = 8$ the situation is different from $n = 6$: There is no "final" state that remains unchanged by further transformations and is reached after a certain number of steps from any initial situation.

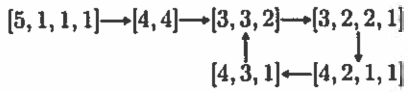


Figure 15

In closing, we show that such a final state exists not only for $n = 6$, but also for every n of the form $n = \frac{d(d+1)}{2}$, where $d \in \mathbf{N}$, and that this state has the form $[1, 2, \dots, d]$. (An analysis of the situations for the remaining values of n is described in Section 5.12.)

So, let $n = 1 + 2 + \cdots + d$. We choose an arbitrary initial situation and let $m_0, m_1, m_2, \dots$ denote the numbers of piles that successively appear on the table; we then write the numbers of pebbles in the piles, always in a nonincreasing order:

$$\begin{gathered} a_0(1) \geq a_0(2) \geq \cdots \geq a_0(m_0), \\ a_1(1) \geq a_1(2) \geq \cdots \geq a_1(m_1), \\ a_2(1) \geq a_2(2) \geq \cdots \geq a_2(m_2), \\ \vdots \end{gathered} \tag{39}$$

According to the preceding result we may ignore the first few iterations and therefore assume that we have

$$a_{j+1}(1) = m_j \geq a_j(1) - 1, \; a_{j+1}(k) = a_j(k-1) - 1, \tag{40}$$

where $2 \leq k \leq m_{j+1}$ and $j \in \mathbf{N}_0$. Here the number m_j is determined for each $j \geq 1$ by the relation

$$m_j = 1 + \max\{k \in \{1, 2, \dots, m_{j-1}\} : a_{j-1}(k) > 1\}.$$

Since for a fixed n there is only a finite number of distributions of n pebbles, in the system (39) there is only a finite number of different chains that are repeated from a certain row on. We may therefore assume that this repetition begins with the first row and that the number m_0 is the largest of the numbers $m_0, m_1, m_2, \dots$. By (40) we have $a_1(1) = m_0$, $a_2(2) = m_0 - 1$, $a_3(3) = m_0 - 2$, $\dots$, $a_{m_0}(m_0) = 1$, which implies $a_{m_0+1}(1) = m_0$. By induction we easily obtain

$$a_{pm_0+1}(1) = m_0 \quad (p \in \mathbf{N}_0). \tag{41}$$

The right-hand part of the estimate

$$m_0 - 1 \leq a_j(1) \leq m_0 \quad (j \in \mathbf{N}_0) \tag{42}$$

follows from (40) and from the fact that $m_j \leq m_0$ for all $j \in \mathbf{N}_0$. Let us now prove the left-hand part of (42) by contradiction. If we assume that

for some $j > 1$ we have $a_j(1) < m_0 - 1$, then for every $q \in \mathbf{N}_0$ we must also have

$$a_{q(m_0-1)+j}(1) < m_0 - 1. \tag{43}$$

Indeed, let (43) be false for some $q \in \mathbf{N}$; we take the smallest such q. Then from $a_{q(m_0-1)+j}(1) \geq m_0 - 1$ and (40) we get successively

$$\begin{gathered} a_{q(m_0-1)+j-1}(m_0 - 1) \geq 1, \; a_{q(m_0-1)+j-2}(m_0 - 2) \geq 2, \; \dots \\ \dots, \; a_{(q-1)(m_0-1)+j}(1) \geq m_0 - 1, \end{gathered}$$

and this is a contradiction to the choice of the number q. Hence (43) holds for all $q \in \mathbf{N}_0$. However, the relations (41) and (43) with $p = q = j - 1$ lead to a contradiction, since then $pm_0 + 1 = q(m_0 - 1) + j$. This proves the left-hand part of (42).

From the estimates (42) and the relations (40) we now easily derive for each $j \in \mathbf{N}_0$,

$$m_0 - 2 \leq a_{j+1}(2) \leq m_0 - 1, \quad m_0 - 3 \leq a_{j+2}(3) \leq m_0 - 2, \quad \text{etc.},$$

and since the sequence of chains in (39) repeats from the beginning, this implies

$$m_0 - k \leq a_j(k) \leq m_0 - k + 1 \quad (1 \leq k \leq m_j, \; j \in \mathbf{N}_0). \tag{44}$$

If we add these inequalities for $j = 0$ and $k = 1, 2, \dots, m_0$, we get

$$\frac{(m_0 - 1)m_0}{2} < a_0(1) + a_0(2) + \cdots + a_0(m_0) \leq \frac{m_0(m_0 + 1)}{2}.$$

(The left inequality is strict, since $a_0(m_0) > 0$.) In view of the fact that

$$a_0(1) + a_0(2) + \cdots + a_0(m_0) = n = \frac{d(d+1)}{2},$$

we obtain from this $m_0 = d$ and $a_0(k) = d - k + 1$ $(1 \leq k \leq d)$. The proof is now complete.

5.12 *Exercises*

(Continuation of the problem on piles of pebbles from 5.11.)

(i) For each $n \geq 1$ find all "final" distributions of pebbles that do not change under the transformation.

*(ii) Show that in the case $\frac{(d-1)d}{2} < n < \frac{d(d+1)}{2}$ $(d \in \mathbf{N})$ after a certain number of transformations the iteration becomes periodic with a period $p > 1$ that is a divisor of d such that the number $\frac{d}{p}$ divides the difference $n - \frac{(d-1)d}{2}$. Describe those distributions that repeat periodically.

*(iii) In the situation of (ii), find the number of distributions of the pebbles that are periodically repeated in the iterations with a given period p.

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Combinatorial Geometry

The development of geometry, as inspired by the deep results of Bernhard Riemann in the second half of the nineteenth century, has meant that scientific work in this field moved quite far from the "naive" or elementary geometry practiced by the Greek mathematicians of around the beginning of our era, and their numerous successors in later times. Classically, the main focus of geometry has been on the *proofs* or *constructions* connected with properties of basic geometrical objects (points, straight lines, circles, triangles, half-planes, tetrahedra, etc.), that is, problems that can be visualized. On the other hand, a paradoxical characteristic of contemporary scientific works in "pure" geometry is the fact that the majority of them are completely devoid of pictures; or the fact that "geometrical intuition" is more often required by specialists in mathematical analysis or algebra than by mathematicians who consider themselves geometers. The apparent dissatisfaction of a number of mathematicians with this situation has led to new directions of research in geometry; this is mainly in response to modern problems in *optimization*. As examples of such geometrical optimization problems one can consider the problem of *filling* a plane or a space (or some of their parts) with some system of geometrical objects, or *covering* parts of a plane or a space with the smallest possible number of copies of a geometrical object. These and several other reasons have led, especially in the past half century, to the rapid development of several nontraditional branches of geometry (or areas closely related to geometry), among them *combinatorial geometry*.

It is rather difficult to define the contents of combinatorial geometry precisely, since this discipline is closely connected with a number of re-

(iv) Show that for the valuation

$$J(x_1,x_2,\ldots,x_n)=\sum_{k=0}^{n-1}p_k(x_1x_{1+k}+x_2x_{2+k}+\cdots+x_nx_{n+k}) \qquad (55)$$

(where $x_{n+j}=x_j$ for $j=1,2,\ldots,n-1$) with constants $p_k=p_{n-k}$ you have $J(x_1+x_2,-x_2,x_3+x_2,x_4,\ldots,x_n)-J(x_1,x_2,\ldots,x_n)=2x_2[(p_0-2p_1+p_2)(x_1+x_2+x_3)+(p_3-2p_2+p_1)x_4+(p_4-2p_3+p_2)x_5+\cdots+(p_{n-1}-2p_{n-2}+p_{n-3})x_n]=2x_2c\cdot S$, if $c=p_2-2p_1+p_0=p_3-2p_2+p_1=\cdots=p_{n-1}-2p_{n-2}+p_{n-3}$. (In view of symmetry it is not necessary to consider other transformations.) The last identities hold if and only if the sequence $p_1-p_0,p_2-p_1,\ldots,p_{n-1}-p_{n-2}$ is arithmetic with difference c, that is, $p_k-p_{k-1}=p_1-p_0+(k-1)c$ $(1\le k\le n-1)$, from which by summing you obtain $p_k=p_0-k(p_0-p_1)+\frac{k(k-1)}{2}c$ $(0\le k\le n-1)$. By substituting into $p_1=p_{n-1}$ you will obtain $p_0-p_1=\frac{n-1}{2}\cdot c$, and thus

$$p_k=p_0-\frac{k(n-k)}{2}c \quad (0\le k\le n-1). \qquad (56)$$

Verify that p_k defined by (56) possesses all the desired properties. If you choose any $p_0\in\mathbb{R}$ and $c\in\mathbb{R}^+$, then by substituting p_k from (56) into (55) you obtain a decreasing valuation J. Next you must consider whether J is a nonnegative function for some pair p_0,c. If you set, for instance, $p_0=\frac{n^2+n}{2}$ and $c=2$, then $p_k=\binom{n-k+1}{2}+\binom{k+1}{2}$ for all $k\in\{1,2,\ldots,n-1\}$, and $J=S_1+S_2+\cdots+S_n$, with $S_j=\sum_{i=1}^{n}(x_i+x_{i+1}+\cdots+x_{i+j-1})^2\ge 0$ for $1\le j\le n$.

(v) Assume that there is an infinite iterative sequence. From the properties described in (iv) it follows that the corresponding infinite sequence of valuations of the n-tuples of our iterative sequence is decreasing and bounded, so convergent to a number L. Therefore, you can choose an iterative term with a valuation J_0 satisfying $L<J_0<L+\frac{4S^2}{n}$ and leave out all the previous terms. So, without loss of generality, you can assume that these inequalities are satisfied already by the initial n-tuple $(x_1,\ldots,x_n)$. Then the infinite sequence consisting of the negative central terms $x_k,x'_k,x''_k,\ldots$ of all transformed triples satisfies

$$(-x_k)+(-x'_k)+(-x''_k)+\cdots=\frac{1}{4S}(J_0-L)<\frac{S}{n}. \qquad (57)$$

The largest of the numbers $x_1,\ldots,x_n$ in the initial n-tuple is at least $\frac{S}{n}$; let, for instance, $x_1\ge\frac{S}{n}$. From (57) it then follows that in the place of x_1 after an arbitrary number of steps there will be a nonnegative number. Hence the index $k=1$ is disallowed in the sense of (iii), which means a finite number of transformations, which is a contradiction.

(vi) A suitable constant K is, for instance, of the form

$$K=\max_{1\le k\le n}\left\{\max_{0\le i\le n-1}|x_k+x_{k+1}+\cdots+x_{k+i}|\right\}.$$

Show that the expression on the right is a nonincreasing valuation; to do this, use inequalities of the type $x_2<x_1+2x_2+x_3+\cdots+x_n<x_1+x_2+\cdots+x_n$ (if $x_2<0$).

5.12 (i) A final distribution exists only when $n=1+2+\cdots+d$ for some $d\in\mathbb{N}$, and it is of the form $[d,d-1,\ldots,1]$. Proof: Comparing the distribution $[a_1,a_2,\ldots,a_m]$ with $[m,a_1-1,a_2-1,\ldots,a_m-1]$ under the assumption $a_1\ge a_2\ge\cdots\ge a_m$ you obtain $a_m=1$. If $m>1$, then $a_{m-1}-1=a_m$, that is, $a_{m-1}=2$, etc.

(ii) It suffices to modify the conclusion of Section 5.11 after the inequalities (44), from which in the case $\frac{(d-1)d}{2}<n<\frac{d(d+1)}{2}$ it follows by adding that $m_0=d$ and $a_j(k)=d-k+\varepsilon_j(k)$, with $\varepsilon_j(k)\in\{0,1\}$, $1\le k\le d$, $j\in\mathbb{N}_0$, where $a_j(d)=0$ when $m_j<d$, that is, when $m_j=d-1$. Show that $(\varepsilon_0(1),\varepsilon_0(2),\ldots,\varepsilon_0(d))$ can be any d-tuple $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_d)$ of the numbers 0 and 1 satisfying $\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_d=n-\frac{(d-1)d}{2}$, and that for each $j\ge 1$ you have $(\varepsilon_{j+1}(1),\varepsilon_{j+1}(2),\ldots,\varepsilon_{j+1}(d))=(\varepsilon_j(d),\varepsilon_j(1),\varepsilon_j(2),\ldots,\varepsilon_j(d-1))$, that is, $\varepsilon_j(k)=\varepsilon_{j+k}$ $(1\le k\le d,\ j\in\mathbb{N}_0)$, if you set $\varepsilon_{d+j}=\varepsilon_j$ for all $j\ge 1$. This implies a description of all periodic p-tuples of the iterations $a_j(k)=d-k+\varepsilon_{j+k}$ $(1\le k\le d,\ j\in\mathbb{N}_0)$ and the assertion about its length p being equal to the (smallest) period of the infinite sequence $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_d,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_d,\ldots$, since $n-\frac{(d-1)d}{2}$ copies of 1 in the d-tuple $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_d)$ are distributed into $\frac{d}{p}$ copies of the p-tuple $(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_p)$.

(iii) Set $m=n-\frac{(d-1)d}{2}$. From the analysis done in the solution of (ii) it follows that the repeating distributions are exactly the d-tuples $[d-1+\varepsilon_1,d-2+\varepsilon_2,\ldots,1+\varepsilon_{d-1},\varepsilon_d]$, where $\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_d\in\{0,1\}$ and $\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_d=m$. The period of the repetition is then equal to the smallest $k\in\mathbb{N}$ such that $\varepsilon_i=\varepsilon_{i+k}$ (here, obviously, put $\varepsilon_{d+j}=\varepsilon_j$ for all $j\in\mathbb{N}$). For the given m, d $(1\le m<d)$, and any positive integer p, let $f(p)$ denote the number of distributions with the period p, and $g(p)$ the number of distributions with an arbitrary period q, $q\mid p$. By (ii) you have $f(p)=0$ whenever $p\nmid d$ or $d\nmid pm$. If $p\mid d$ and $d\nmid pm$, then for any divisor $q\mid p$ we also have $d\nmid qm$, so $g(p)=0$. On the other hand, if $p\mid d$ and $d\mid pm$, then $g(p)$ is equal to the number of all ordered p-tuples of $\frac{pm}{d}$ ones and $\frac{p(d-m)}{d}$ zeros; that is, $g(p)=\binom{p}{pm/d}$. It follows directly from the definition that

$$g(p)=\sum_{q\mid p}f(q).$$

Then by the Möbius inversion formula (see Chapter 1, 6.13.(iv)), for $p\mid d$ and $d\mid pm$ we obtain (where $\mu(q)$ is the Möbius function)

$$f(p)=\sum_{q\mid p}\mu(q)g\left(\tfrac{p}{q}\right)=\sum_{q\mid p}\mu\left(\tfrac{p}{q}\right)g(q)=\sum_{\substack{q\mid p\\ d\mid qm}}\mu\left(\tfrac{p}{q}\right)\binom{q}{qm/d}.$$

some $n \geq 1$. Then clearly $a, b, c \in \mathbb{R}_0^+$. Next show, as in 5.7.(iv), that the numbers $A_k = \max\{a_k, b_k, c_k\}$ satisfy $A_0 \geq A_1 \geq \cdots \geq A_n$. Hence from $A_0 = A_n$ it follows that $A_0 = A_1$, which is possible only when $0 \in \{a, b, c\}$. Similarly, from $A_1 = A_2$ it follows that $0 \in \{a_1, b_1, c_1\}$ (which holds also when $n = 1$). Then two of the numbers a, b, c are identical. Therefore, $[a, b, c]$ is of the form $[a, 0, 0]$ or $[a, a, 0]$ for appropriate $a \geq 0$. Consider the iterations of both forms.

(viii) An appropriate m-tuple $(1, \alpha, \alpha^2, \ldots, \alpha^{m-1})$, where $\alpha > 1$, has a kth iteration of the form $(\beta^k, \beta^k\alpha, \beta^k\alpha^2, \ldots, \beta^k\alpha^{m-1})$ as long as

$$\frac{\alpha - 1}{1} = \frac{\alpha^2 - \alpha}{\alpha} = \cdots = \frac{\alpha^{m-1} - \alpha^{m-2}}{\alpha^{m-2}} = \frac{\alpha^{m-1} - 1}{\alpha^{m-1}} = \beta > 0.$$

This occurs if and only if $\beta = \alpha - 1$ and $\alpha > 1$ is a root of the equation $F_m(x) = 0$, where $F_m(x) = x^{m-1} - x^{m-2} - \cdots - x - 1$. The equation $F_m(x) = 0$ has a root in the interval $(1, 2)$ for every $m \geq 3$, since $F_m(1) < 0$ and $F_m(2) > 0$.

(ix) Independently of the initial m-tuple, you obtain after a certain number of steps an m-tuple of even numbers $(2b_1, 2b_2, \ldots, 2b_m)$; this follows from the result of 5.1.(iii) if you consider the numbers 1 and -1 as symbols for an even, respectively odd, number. Repeat the same argument for the m-tuple $(b_1, b_2, \ldots, b_m)$, etc.; hence for every $k \in \mathbb{N}$ you obtain after a certain number of steps the m-tuple $(2^k c_1, 2^k c_2, \ldots, 2^k c_m)$, where $c_i \in \mathbb{N}_0$, $1 \leq i \leq m$. On the other hand, the numbers in all m-tuples are bounded from above by twice the largest of the numbers $|a_1|, |a_2|, \ldots, |a_m|$. For sufficiently large k this means that $c_1 = c_2 = \cdots = c_m = 0$.

(x) Call an arrow pointing to the left (respectively to the right) a left (respectively right) arrow. The numbers L and R of left, respectively right, arrows on the line are invariants; since a transformation can be carried out if and only if some left arrow lies to the right of some right arrow, the final state is uniquely determined: Going from left to right, there are first L left and then R right arrows. Next, consider the valuation J defined by the number of (unordered) pairs of arrows with opposite directions and pointing to each other (they don't have to be neighbors). An allowable change of some pair of neighboring arrows is possible if and only if $J > 0$; after each such transformation the value of J decreases by 1. This means that the transformation can (independently of the particular sequence) be repeated exactly J_0 times, where J_0 is the initial value of J.

(xi) The desired condition has the form

$$a_1 + a_2 + \cdots + a_n = 2^m \cdot d, \quad m \in \mathbb{N}_0. \tag{54}$$

Indeed, the value of d either remains unchanged after a transformation or is doubled; this implies the necessity of (54). Its sufficiency can be proven by induction on the exponent m. If $m = 0$, then $n = 1$ and everything is trivial. If (54) holds for some $m \geq 1$, then under the assumption that $d = 1$ (this is no loss of generality, since the pebbles are shifted in integer multiples of d) the number of odd integers among $a_1, a_2, \ldots, a_n$ is even, say $2k$, where $k \in \mathbb{N}$. Therefore, the piles with odd numbers of pebbles can be "paired," and in each one of these k pairs you can carry out a transformation. After these k transformations you obtain $n'(\leq n)$ piles with numbers of pebbles $b_1, b_2, \ldots, b_{n'}$, all of which are even. Therefore, $2^m = a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_{n'} = 2^{m'} \cdot d'$ ($m' \in \mathbb{N}_0$), where d' is the greatest common divisor of the numbers $b_1, b_2, \ldots, b_{n'}$. Since $d' = 2^i$ for some $i \geq 1$, you have $m' \leq m - 1$, and therefore you can use the induction hypothesis.

(xii) Consider the two valuations

$$J_1(a_1, a_2, \ldots, a_n) = \sum_{i;\ a_i \leq i} 2^{a_i}, \quad J_2(a_1, a_2, \ldots, a_n) = \sum_{i;\ a_i \geq i} 2^{-a_i}.$$

If the indices c, k are as in the statement and if $|c - k| \geq 2$, then the inequalities $2^{k+1} + 2^{k+2} + \cdots + 2^{c-1} < 2^c$ (if $k < c$) or $2^{-c-1} + 2^{-c-2} + \cdots + 2^{-k+1} < 2^{-c}$ (if $k > c$) imply that after the transformation (32) the value of J_1 does not decrease and the value of J_2 increases, while after the transformation (33) the value of J_1 increases and that of J_2 does not decrease. Moreover, these properties of J_1 and J_2 are evident if $|c - k| = 1$. Therefore, $J = J_1 + J_2$ is an increasing valuation. Since there is a finite number of arrangements of the integers $1, 2, \ldots, n$, the value of J can increase only a finite number of times.

5.10 (i) The conclusion is not true for any $n > 1$. Consider a sequence of length $2n + 1$ formed with $n + 1$ copies of 1 and n copies of -1.

(ii) Show that $J = p(x_1^2 + x_2^2 + x_3^2 + x_4^2) + q(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1) + r(x_1x_3 + x_2x_4)$ is a suitable valuation if $2p - 2q + r = 2q - 2r = c > 0$, that is, if $q = r + \frac{c}{2}$ and $p = c + \frac{r}{2}$. By choosing $r = -2$ and $c = 4$ you obtain $J = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) + (x_1 - x_3)^2 + (x_2 - x_4)^2$.

(iii) Suppose that the index $k = 1$ is disallowed. Note below how the numbers $S_i = x_1 + x_2 + \cdots + x_i$ $(1 \leq i \leq n)$ will change. When a transformation with the triple of numbers x_{k-1}, x_k, x_{k+1} is carried out, then the following holds: If $1 < k < n$, then the transformation is allowable only when $S_k < S_{k-1}$, and as a result, S_{k-1} and S_k are interchanged; if $k = n$, then the transformation is allowable only when $S_n < S_{n-1}$, in which case the n-tuple $(S_1, S_2, \ldots, S_n)$ is changed to $(S_1 + x_n,\ S_2 + x_n,\ \ldots,\ S_{n-2} + x_n,\ S_n + x_n,\ S_{n-1} + x_n)$. Each transformation of a triple x_{k-1}, x_k, x_{k+1}, where $1 < k \leq n$, then decreases the *number of inversions* in the n-tuple $(S_1, S_2, \ldots, S_n)$, that is, the number of pairs of indices (i, j) such that $1 \leq i < j \leq n$ and $S_i > S_j$. This proves the assertion on the finite number of possible transformations in the case where the index $k = 1$ is disallowed.

5.6 (i) The last one to grow is a lemon; their number is odd every morning.

(ii) The number of pieces of paper always satisfies $n \equiv 1 \pmod 9$, while $1991 \equiv 2 \pmod 9$.

(iii) The identity $(c \cdot 10^k + x) - (c + x) = c(10^k - 1)$ shows that the remainder upon division by 9 is an invariant of the transformation. Since 9 does not divide 2^{1991}, the digit sum of the number A cannot be equal to $0 + 1 + 2 + \cdots + 9 = 45$.

(iv) The equation $p_1' + p_2' + \cdots + p_n' = N$ is a preservation law for the number of larks; the necessity of the congruence condition was shown in 5.5.(iv). To prove sufficiency, begin by explaining how from any initial position of N larks you can, through a sequence of allowable transformations, achieve that at least $N-1$ larks sit in the tree with number n. The remaining Nth lark in this situation sits in the jth tree (the possibility $j = n$ is not excluded), where the index j is uniquely determined by the initial situation $(p_1, p_2, \dots, p_n)$ by way of the congruence $p_1 + 2p_2 + \cdots + np_n \equiv j + (N-1)n \pmod n$. Finally, use the fact that if the state p' is attainable from the state p, then also p is attainable from p'.

(v) For the position $(p, q, r) = (2, 0, 0)$, which was not tabulated in the solution of 5.5.(iii), all three sums s_i are even; therefore, from the given initial situation at most one of the four positions mentioned can be achieved. Now carry out the operations according to the following strategy: At each step adjoin the digit with the least number of occurrences (if this digit is not unique, then choose the smallest of them). Since the sum $p + q + r$ decreases by 1 after each step, you finally reach a position (p', q', r') where no further operation is possible. Clearly, p', q', r' are the numbers $n, 0, 0$ in some order $(n \geq 1)$. If $n = 1$, then everything is in order; in the case $n \geq 2$ the position (p', q', r') follows one consisting of the numbers $1, 1, n-1$, which in view of your strategy means that $n - 1 \leq 1$, that is, $n = 2$. This is then the position $(1, 1, 1)$, which is followed by the final position $(2, 0, 0)$.

(vi) Exactly those k, $0 \leq k \leq n$, for which $k \equiv 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \pmod 2$ holds can remain on the board. Indeed, on the one hand, the sum of all numbers on the board does not change parity after an operation, since $a + b \equiv |a - b| \pmod 2$ for any $a, b \in \mathbb{Z}$. On the other hand, for any such k you can carry out the operations in the following order (distinguish the cases (a) n even, k even; (b) n even, k odd; (c) n odd, k odd; (d) n odd, k even; write down all pairs (a, b) of consecutive erased numbers):

(a) $(2,3), (4,5), \dots, (k-2, k-1), (k+1, k+2), (k+3, k+4), \dots, (n-1, n)$;
(b) $(2,3), (4,5), \dots, (k-1, k), (k+2, k+3), (k+4, k+5), \dots, (n-1, n)$, $(k+1, 1)$;
(c) $(1,2), (3,4), \dots, (k-2, k-1), (k+1, k+2), (k+3, k+4), \dots, (n-1, n)$;
(d) $(1,2), (3,4), \dots, (k-1, k), (k+2, k+3), (k+4, k+5), \dots, (n-1, n)$, $(k+1, 1)$.

In all cases you then obtain on the board the number k and an even number of the integer 1, and it is clear how one has to proceed further so that only the number k remains. (The assertion about the parity of the number of ones follows from considering the invariant mentioned above.)

(vii) All congruences are modulo m. If you substitute $x_{k+1} \equiv x_1 + x_2 + \cdots + x_k$ into the congruence $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \equiv \alpha_1 x_2 + \alpha_2 x_3 + \cdots + \alpha_k x_{k+1}$, you obtain the condition $(\alpha_k - \alpha_1)x_1 + (\alpha_k + \alpha_1 - \alpha_2)x_2 + \cdots + (\alpha_k + \alpha_{k-1} - \alpha_k)x_k \equiv 0$, which holds for any $x_1, x_2, \dots, x_k$ if and only if $0 \equiv \alpha_k - \alpha_1 \equiv \alpha_k + \alpha_1 - \alpha_2 \equiv \cdots \equiv \alpha_k + \alpha_{k-1} - \alpha_k$, which is equivalent to the system $\alpha_i \equiv i\alpha_1$, $(1 \leq i \leq k-1)$, $\alpha_{k-1} \equiv 0$, and $\alpha_k \equiv \alpha_1$. Therefore, $\alpha_1 \not\equiv 0$ and $(k-1)\alpha_1 \equiv 0$ have to hold, which is possible if and only if the integers $k - 1$ and m are not relatively prime.

5.8 (i) The sum of all n numbers is a nonincreasing valuation.

(ii) The desired number is 2^{-10}. Note that the quotient $\frac{a}{2^n}$, where a is the smallest positive number and n the number of zeros on the board, is a nondecreasing valuation.

(iii) The valuation $J = x_1^2 + x_2^2 + \cdots + x_n^2$ is nondecreasing; after the first transformation the value of J increases.

(iv) Denote the nth iteration by (a_n, b_n, c_n, d_n) and assume cyclicity. The numbers $A_n = a_n + b_n + c_n + d_n$ satisfy $A_n = 2^n A_0$; since there are two identical numbers in the sequence $A_0, A_1, \dots$, we have $A_0 = a+b+c+d = 0$. The numbers $B_n = a_n^2 + b_n^2 + c_n^2 + d_n^2$ satisfy $B_{n+1} = 2B_n + 2(a_n + c_n)(b_n + d_n) = 2B_n + 2A_{n-1}^2 = 2B_n$ for all $n \geq 1$, so $B_n = 2^{n-1}B_1$ $(n \geq 2)$, which implies $B_1 = 0$. But this means that $a_1 = b_1 = c_1 = d_1 = 0$, which leads to $a = -b = c = -d$.

(v) The assertion does not hold for any $n = 3k$: Show that the initial n-tuple $(1, -1, 0, 1, -1, 0, \dots, 1, -1, 0)$ is transformed to itself after 6 iterations.

We add without proof that for all $n \neq 3k$ the following holds: If in some sequence of iterations two elements (that is, two ordered n-tuples) are identical, then the initial n-tuple is of the form $(a, -a, a, -a, \dots, a, -a)$ or $(0, 0, \dots, 0)$ according to whether n is even or odd.

(vi) For each $n \geq 6$ the answer is negative. A suitable counterexample is easier to construct in the field of complex numbers. Consider the n-tuple $(1, \varepsilon, \dots, \varepsilon^{n-1})$, where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Since $\varepsilon^n = 1$, it is not difficult to verify that the kth iteration is of the form $((1-\varepsilon)^k, (1-\varepsilon)^k\varepsilon, \dots, (1-\varepsilon)^k\varepsilon^{n-1})$ for all $k \geq 1$. Each number in this n-tuple has absolute value equal to

$$\left|(1-\varepsilon)^k\varepsilon^j\right| = |1-\varepsilon|^k = \left(\left(1 - \cos\tfrac{2\pi}{n}\right)^2 + \sin^2\tfrac{2\pi}{n}\right)^{\frac{k}{2}} = \left(2\sin\tfrac{\pi}{n}\right)^k,$$

which does not exceed 1 for any $k \geq 1$ if $n \geq 6$. A suitable example in real numbers is obtained by taking the real parts of the above complex numbers: $\left(1, \cos\frac{2\pi}{n}, \cos\frac{4\pi}{n}, \dots, \cos\frac{2(n-1)\pi}{n}\right)$.

(vii) Exactly the triples $[a, a, 0]$, where $a \geq 0$. Let $[a_k, b_k, c_k]$ be the kth iteration of the triple $[a, b, c]$, and suppose that $[a_n, b_n, c_n] = [a, b, c]$ for that they satisfy the bounds $a_i \geq 2^k$. Next assume that $a_1 = 2^{\alpha_1} \cdot b_1 < 2^k$. Then $b_1 \neq 3^\beta$, $b_1 \geq 5$, $k - \alpha_1 \geq 3$, $3^{\alpha_1+1} \cdot b_1 < 3^{\alpha_1+3} \cdot 2^{k-\alpha_1-3} < 3^k < 2n$, and therefore $L^* = \{3b_1, 3^2b_1, \ldots, 3^{\alpha_1+1} \cdot b_1\}$ is a subset of L. Define $I = \{i;\ b_i \in L^*\}$. If $\alpha_i \geq \alpha_1$ for all $i \in I$, then $a_1 \mid a_i$, which is a contradiction. Hence necessarily $0 \leq \alpha_i < \alpha_1$ $(i \in I)$, and by the pigeonhole principle among the $\alpha_1 + 1$ numbers α_i $(i \in I)$ some two are identical, $\alpha_i = \alpha_j$ $(i, j \in I,\ i \neq j)$, which means that either $a_i \mid a_j$ (if $b_i < b_j$), or $a_j \mid a_i$ (if $b_i > b_j$), but this is a contradiction.

(xii) For the numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_N$ under consideration, set

$$\sigma_k = \sum_{\substack{\{(i_1,i_2,\ldots,i_k):\\ 1\leq i_1<i_2<\cdots<i_k\leq n\}}} x_{i_1}x_{i_2}\cdots x_{i_k}, \quad s_k = \sum_{i=1}^{n} x_i^k, \quad \text{and} \quad t_k = \sum_{i=1}^{N} y_i^k,$$

where $1 \leq k \leq n$. From the theory of polynomials it is known that the collection of numbers $x_1, x_2, \ldots, x_n$ is uniquely determined by the values $\sigma_1, \sigma_2, \ldots, \sigma_n$ of the elementary symmetric polynomials. By Newton's theorem (see, e.g., [8], page 323), $s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \cdots + (-1)^{k-1}s_1\sigma_{k-1} = (-1)^{k-1}k\sigma_k$, where $1 \leq k \leq n$, and furthermore it follows that the numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ are uniquely determined by the power sums $s_1, s_2, \ldots, s_n$. Therefore, it suffices to show (by induction on k) that the number s_k is uniquely determined by the values of the sums $t_1, t_2, \ldots, t_k$, $1 \leq k \leq n$. Indeed, you have $s_1 = \frac{t_1}{n-1}$, and in the case $2^k \neq 2n$,

$$s_k = \frac{1}{2n - 2^k}\left[2t_k - \sum_{j=1}^{k-1}\binom{k}{j}s_js_{k-j}\right].$$

This last formula follows from the identity

$$2t_k + 2^ks_k = \sum_{i=1}^{n}\sum_{j=1}^{n}(x_i + x_j)^k$$

by using the binomial theorem for the powers on the right. Finally, for the values $n = 2^k$ you can recursively construct a counterexample of two disjoint n-element sets $A_n = \{x_1, \ldots, x_n\}$ and $B_n = \{y_1, \ldots, y_n\}$ for which the sums $x_i + x_j$, respectively $y_i + y_j$, form two identical $\binom{n}{2}$-element collections: Set $A_2 = \{0, 3\}$, $B_2 = \{1, 2\}$, $A_{2n} = A_n \cup (c + B_n)$, and $B_{2n} = B_n \cup (c + A_n)$, where the number c (depending on n) is chosen large enough so that the sets with index $2n$ have $2n$ elements and are disjoint.

5.2 (i) Assume to the contrary that after n steps all $2k + 1$ numbers were even, and take the *smallest* such n. Then in the preceding step all numbers would be odd, and in the step before that any two neighboring numbers would have different parities. This is not possible, since $2k + 1$ is an odd number.

(ii) Since after each step the sum of all 25 numbers doubles, after 100 steps it will be equal to -2^{100}. It remains to verify that $2^{100} > 25 \cdot 10^{28}$, or $2^{72} > 5^{30}$. You even have $2^{70} > 5^{30}$, since $2^7 = 128 > 125 = 5^3$.

(iii) If $k = 0$, then n is odd; if then the n-tuple $(1, 1, \ldots, 1)$ is not the initial one, then its first occurrence comes after the n-tuple $(-1, -1, \ldots, -1)$; this already is the initial n-tuple, since $(x_1x_2)(x_2x_3)\cdots(x_nx_1) = x_1^2x_2^2\cdots x_n^2 = 1 \neq (-1)^n$. Then base the induction step from $n = 2^k\ell$ to $n = 2^{k+1}\ell$ on considering the two sequences (29), as in the solution of 5.1.(iii).

5.4 (i) If you change the signs of the numbers ± 1, then the product of the numbers in any 2×2 subarray is an invariant. If you compare the 2×2 subarrays in the lower right corner, you find that the arrays in Figure 13 are mutually nonattainable.

(ii) The assertion is true. Use the invariant $I = (a_1a_4a_7a_{10})(a_3a_6a_9a_{12})$.

(iii) The assertion is true. Use the invariant $I = a_2a_3a_5a_6a_8a_9a_{11}a_{12}$.

(iv) Show that the sum $I(M) = \sum 2^{-x-y+2}$ is an invariant, where the summation is extended over all pairs $(x, y) \in M$. For the initial set M you have $I(M) = 1$. Explain why in any attainable set M' there is at most one pair $(1, y)$ and at most one pair $(x, 1)$. If you therefore assume that M' has the property required in the statement of the problem, then you obtain

$$I(M') \leq \frac{1}{8} + \frac{1}{8} + \sum_{(x,y)\in D\cap M'} 2^{-x-y+2},$$

where $D = \{(x, y) \in \mathbf{N} \times \mathbf{N}:\ x \geq 2 \wedge y \geq 2 \wedge x + y \geq 5\}$. Then you easily obtain the infinite series

$$\sum_{(x,y)\in D} 2^{-x-y+2} = \frac{3}{4}$$

which implies $I(M') < \frac{2}{8} + \frac{3}{4} = 1$, but this is a contradiction.

(v) Let $p = qd$. After one transformation the sign of exactly q factors in the product defining each of the numbers s_k changes; that is, the value of s_k changes its sign exactly when q is odd. Hence after any number of steps you have either $s'_k = s_k$ $(1 \leq k \leq d)$ or (only in the case of odd q and an odd number of steps) $s'_k = -s_k$ $(1 \leq k \leq d)$. Next show that by an appropriate iteration (with an even number of steps, so that the numbers s_k do not change) one can achieve changes in the signs of exactly two of the numbers a_i, a_j $(i \neq j)$ in the cases $i - j = p$, $p \mid (i - j)$ and even $d \mid (i - j)$. Therefore, if you have $s'_k = s_k$ $(1 \leq k \leq d)$, then you can transform the n-tuple $(a_1, a_2, \ldots, a_n)$ into the n-tuple $(a''_1, a''_2, \ldots, a''_n)$ such that $s''_k = s_k$ $(1 \leq k \leq d)$ and $a''_k = a'_k$ $(d < k \leq n)$. But then from the identity $a''_ka''_{k+d}\cdots a''_{k+n-d} = s''_k = s_k = s'_k = a'_ka'_{k+d}\cdots a'_{k+n-d}$ it also follows that $a''_k = a'_k$ $(1 \leq k \leq d)$. The case where $s'_k = -s_k$ $(1 \leq k \leq d)$ holds for odd q can be reduced to the preceding case by one transformation (in an arbitrary way).

the desk and there are 15! ways to arrange all the girls each in a gap between two neighboring boys.

We have $x = x_1 + x_2$, where $x_2 = 14! \times 15!$ and

$$x_1 + 2x_2 = 14! \times 15! \times 2^{15}.$$

Therefore $x = 14! \cdot 15!(2^{15} - 1)$.

50. [Baltic Way 1999] Two squares on an 8×8 chessboard are called *touching* if they have at least one common vertex. Determine if it is possible for a king to begin in some square and visit all the squares exactly once in such a way that all moves except the first are made into squares touching an even number of squares already visited.

Solution: It is not possible for the king to visit all the squares. Assume for a contradiction that there exists a path such that all moves except the first are made into squares touching an even number of squares already visited. Clearly, the first move is made into a square touching exactly one square already visited, namely the starting square. Summing the number of touching squares previously visited over all the moves, we therefore obtain an odd number. On the other hand, every pair of touching squares is counted exactly once in this sum, by the member of the pair that was visited second. Thus, the sum is equal to the total number of touching pairs. But this number is even, since the numbers of touching pairs oriented north-south and east-west are equal, as are the numbers of touching pairs oriented northeast-southwest and northwest-southeast. Thus we have a contradiction, and no path exists.

51. [St. Petersburg 1988] A total of 119 residents live in a building with 120 apartments. We call an apartment *overpopulated* if there are at least 15 people living there. Every day the inhabitants of an overpopulated apartment have a quarrel and each goes off to a different apartment in the building (so they can avoid each other☺). Is it true that this process will necessarily be completed someday?

Solution: Let $p_1, p_2, \ldots, p_{120}$ denote the 120 apartments, and let a_i denote the number of residents in apartment p_i. We consider the quantity

$$S = \frac{a_1(a_1 - 1)}{2} + \frac{a_2(a_2 - 1)}{2} + \cdots + \frac{a_{120}(a_{120} - 1)}{2}.$$

(Assume that all the residents in an apartment shake hand with each other at the beginning of the day, then quantity S denotes the number of the handshakes in that day.) If all $a_i < 15$, then the process is completed and we are done. If not, without loss of generality, we assume that $a_1 \geq 15$ and that the inhabitants in p_1 go off to different apartments in the building. Assume that they go to apartments $p_{i_1}, p_{i_2}, \ldots, p_{i_{a_1}}$. On the next day, the quantity is changed by an amount of

$$a_{i_1} + a_{i_2} + \cdots + a_{i_{a_1}} - \frac{a_1(a_1-1)}{2},$$

which is positive as

$$a_{i_1} + a_{i_2} + \cdots + a_{i_{a_1}} \leq 119 - a_1 \leq 119 - 15 = 104$$

and

$$\frac{a_1(a_1-1)}{2} \geq \frac{15 \times 14}{2} = 105.$$

Hence the quantity is decreasing during this process. On the other hand, S starts as a certain finite number and S is nonnegative. Therefore this process has to be completed someday.

4

Solutions to Ad

1. [AIME 1985] In a tournan each of the other players. I loser got 0 points, and eac was a tie. After the compl half of the points earned I ten players with the least lowest scoring players ean the ten). What was the tot

Solution: Assume that a t We will obtain two expres of points gathered by all o of points gathered by the gathered by the winners (o desired expressions, we wi another, then they played of $k(k-1)/2$ points to be tion, the n players gathere Similarly, the losers had 1(since this accounts for half points. In games among th