First Name:
Last Name:

Student Number:

MATH 200 – Calculus III

MIDTERM II – NOVEMBER 18

TOTAL MARKS : 50

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1. (12 points)

$$F(x, y, z) = 2x^3 + xy - z, \quad P = (1, 0, 2)$$

(a) Find ∇F at P.

$\nabla F = \langle 6x^2 + y, x, -1 \rangle$

$\nabla F(1,0,2) = \langle 6, 1, -1 \rangle$

(b) Find the equation of the plane tangent to the surface $F(x, y, z) = 0$ at the point P.

$6(x-1) + 1(y-0) - 1(z-2) = 0$

$6x + y - z - 4 = 0$

(c) Find the rate of change of F at P in the direction of the vector

$$\vec{v} = < 3, 4, 12 > .$$

$$|\vec{v}| = \sqrt{3^2+4^2+12^2} = 13$$ $\Rightarrow$ unit vector in the direction of $\vec{v}$ is

$$\frac{1}{|\vec{v}|}\vec{v} = \frac{1}{13}<3,4,12>$$

Then $D_{\vec{v}} F(1,0,2)$ is the rate of change of F at P in the direction of $\vec{v}$:

$$D_{\vec{v}} F(1,0,2) = \nabla F(1,0,2) \cdot \frac{1}{|\vec{v}|} \vec{v}$$

$$= <6,1,-1> \cdot \frac{1}{13} <3,4,12>$$

$$= \frac{1}{13}(18+4-12) = \frac{10}{13} .$$

(d) What is the maximum rate of change of F at P?

It is $|\nabla F(1,0,2)| = \sqrt{6^2+1^2+(-1)^2} = \sqrt{38}$.

2. (7 points) Use the method of Lagrange Multipliers to find the maximum value of

$$f(x, y) = 2x + y$$

subject to the constraint

$$x^2 + 2y^2 = 18.$$

$g(x,y) = x^2 + 2y^2 - 18$

(No points if you don't use Lagrange Multipliers).

$\nabla f = \langle 2, 1 \rangle$

$\nabla g = \langle 2x, 4y \rangle$

$\nabla f = \lambda \nabla g \implies \begin{cases} 2 = \lambda(2x) \\ 1 = \lambda(4y) \end{cases} \implies \begin{cases} 2(\lambda x - 1) = 0 \\ 4\lambda y - 1 = 0 \end{cases}$

$\implies \begin{cases} x = \frac{1}{\lambda} \\ y = \frac{1}{4\lambda} \end{cases}$ Using the constraint,

$\left(\frac{1}{\lambda}\right)^2 + 2\left(\frac{1}{4\lambda}\right)^2 = 18$

$\implies \frac{9}{8} \frac{1}{\lambda^2} = 18 \implies \lambda^2 = \frac{1}{16} \implies \lambda^2 = \pm \frac{1}{4}$

If $\lambda = \frac{1}{4}$, then $x = 4$, $y = 1$

If $\lambda = -\frac{1}{4}$, then $x = -4$, $y = -1$

$f(4,1) = 9$ is the maximum

($f(-4,-1) = -9$ is the minimum)

3. (7 points) Find all the critical points (a, b) of the function

$$f(x, y) = x^3 - 12x + y^3 - 3y + 1.$$

Use the second derivative test to decide whether these correspond to local maxima, minima or saddle points.

$f_x = 3x^2 - 12$, $f_y = 3y^2 - 3$

Critical points from

$\begin{cases} 3x^2 - 12 = 0 \\ 3y^2 - 3 = 0 \end{cases}$ $\quad x = \pm 2$, $y = \pm 1$

This gives four critical points:

$(2,1)$, $(2,-1)$, $(-2,1)$, $(-2,-1)$

$f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = f_{yx} = 0$

$D(x,y) = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$

$D(2,1) > 0$, $f_{xx}(2,1) > 0$ $\Rightarrow$ local minimum at $(2,1)$

$D(2,-1) < 0$ $\Rightarrow$ saddle point at $(2,-1)$

$D(-2,1) < 0$ $\Rightarrow$ saddle point at $(-2,1)$

$D(-2,-1) > 0$, $f_{xx}(-2,-1) < 0$ $\Rightarrow$ local maximum at $(-2,-1)$

4. (4 points) Give the value of the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^4x^4}{x^8+y^8}$$

or say why the limit does not exist.

Let $y = kx$.

We have $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^4x^4}{x^8+y^8} = \lim\limits_{x\to 0} \dfrac{k^4x^4x^4}{x^8+k^8x^8}$

$= \dfrac{k^4}{1+k^8}$

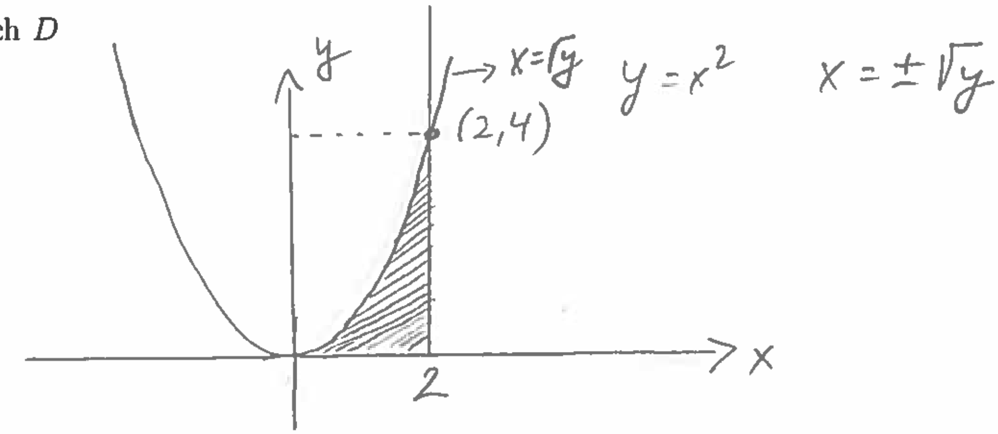
If $k=1$, then the limit is $\frac{1}{2}$

If $k=2$, the limit is $\frac{16}{257}$

Therefore, the limit of $\dfrac{y^4x^4}{x^8+y^8}$ does not exist when $(x,y)\to(0,0)$.

5. (8 points) D is the region bounded by the lines $y = x^2$, $y = 0$ and $x = 2$.

(a) Sketch D



(b) Write D as both a Type I and Type II region.

TYPE I : $\{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$

TYPE II : $\{(x,y) \mid 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}$

(c) Evaluate $\int\int_D 3x\ dA$.

TYPE I

$$\int_0^2 \int_0^{x^2} 3x\, dy\, dx = \int_0^2 3x \int_0^{x^2} dy\, dx$$

$$= \int_0^2 3x\, [y]_0^{x^2}\, dx = \int_0^2 3x^3\, dx = 3\left[\frac{x^4}{4}\right]_0^2$$

$$= 3(4-0) = 12.$$