First Name: **Student Number:**
Last Name:

MATH 200 – Calculus III

MIDTERM I – OCT 7

TOTAL MARKS : 50

Show all working. You may use non-programmable, non-graphing calculators

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1. (18 points) Find the following quantities, when

$$\vec{a} = <2, 3, -1>, \quad \vec{b} = <1, -2, 2>.$$

(i). $\|\vec{b}\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$

(ii). $3\vec{a} - 2\vec{b}$

$3\vec{a} = \langle 6, 9, -3 \rangle$

$-2\vec{b} = \langle -2, 4, -4 \rangle$

$3\vec{a} - 2\vec{b} = \langle 4, 13, -7 \rangle$

(iii). $\vec{a} \cdot \vec{b}$

$= 2(1) + 3(-2) + (-1)(2) = 2 - 6 - 2 = -6$

(iv). $\vec{a} \times \vec{b}$

$\vec{a} = \langle 2, 3, -1 \rangle$

$\vec{b} = \langle 1, -2, 2 \rangle$

$$\vec{a} \times \vec{b} = \left\langle \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \right\rangle$$

$$= \langle 4, -5, -7 \rangle$$

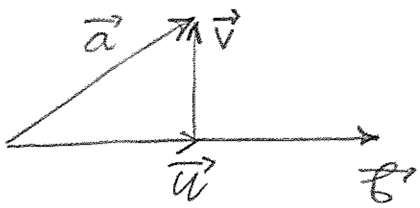
(v). A unit vector in the direction of $\vec{b}$. $\|\vec{b}\| = 3$ from (i)

$$\frac{1}{\|\vec{b}\|}\vec{b} = \frac{1}{3}\langle 1,-2,2\rangle = \left\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right\rangle$$

Vector $\vec{a} = < 2, 3, -1 >$ is decomposed into a vector $\vec{u}$ parallel to and a vector $\vec{v}$ orthogonal to $\vec{b} = < 1, -2, 2 >$.

(vi). Find $\vec{u}$

$$\vec{a} = \vec{u} + \vec{v}$$



$$\vec{u} = \overrightarrow{\text{proj}_{\vec{b}}}\,\vec{a} = \frac{\vec{b}\cdot\vec{a}}{\|\vec{b}\|^2}\cdot\vec{b}$$

$\vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b} = -6$ (from (iii))

$\|\vec{b}\| = 3$, so $\vec{u} = -\frac{2}{3}\vec{b} = \left\langle -\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right\rangle$

(vii). Find $\vec{v}$

$$\vec{v} = \vec{a} - \vec{u} = \langle 2,3,-1\rangle - \left\langle -\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}\right\rangle$$

$$= \left\langle \frac{8}{3}, \frac{5}{3}, \frac{1}{3}\right\rangle = \frac{1}{3}\langle 8,5,1\rangle$$

(viii). How could you check that $\vec{v}$ really is orthogonal to $\vec{b}$?

Check that $\vec{v}\cdot\vec{b} = 0$:

$$\vec{v}\cdot\vec{b} = \frac{1}{3}\langle 8,5,1\rangle\cdot\langle 1,-2,2\rangle$$

$$= \frac{1}{3}(8 - 10 + 2) = 0$$

2. (6 points) Give a parametric equation for the line through the points $A(1,2,3)$ and $B(-1,-2,5)$. Where does this line cross the xy-plane?

The line passes through $A(1,2,3)$ and it is parallel to $\overrightarrow{AB} = \langle -2,-4,2\rangle$

$= -2\langle 1,2,-1\rangle$

(vector-)
Parametric equation is

$$\vec{r} = \overrightarrow{OA} + t\langle 1,2,-1\rangle$$

or

$$x = 1 + t$$
$$y = 2 + 2t$$
$$z = 3 - t$$

The xy-plane has equation $z=0$; this gives $3-t=0$, so $t=3$.
The point of intersection is

$$x = 1+3$$
$$y = 2+2(3)$$
$$z = 0$$

or $(4,8,0)$

3. (3 points) What is the volume of the parallelepiped with corners $O(0,0,0)$, $P(1,0,-1)$, $Q(0,3,5)$ and $R(2,-1,2)$ and adjacent edges OP, OQ and OR?

$\overrightarrow{OP} = \langle 1, 0, -1 \rangle$

$\overrightarrow{OQ} = \langle 0, 3, 5 \rangle$

$\overrightarrow{OR} = \langle 2, -1, 2 \rangle$

The volume is the absolute value of the determinant

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 3 & 5 \\ 2 & -1 & 2 \end{vmatrix} = 17$$

Volume $= 17$.

4. (4 points) Find the angle between the planes

$$\underbrace{x - y + \sqrt{2}z = 2}_{\alpha}, \quad \underbrace{x + y + \sqrt{2}z = 4}_{\beta}.$$

θ – angle between the planes α and β

$\vec{n}_\alpha = \langle 1, -1, \sqrt{2} \rangle$

$\vec{n}_\beta = \langle 1, 1, \sqrt{2} \rangle$

$\|\vec{n}_\alpha\| = \|\vec{n}_\beta\| = \sqrt{1+1+2} = 2$

$\vec{n}_\alpha \cdot \vec{n}_\beta = 2$

$$\Rightarrow \cos\theta = \frac{\vec{n}_\alpha \cdot \vec{n}_\beta}{\|\vec{n}_\alpha\| \cdot \|\vec{n}_\beta\|} = \frac{2}{2(2)} = \frac{1}{2}$$

$\Rightarrow \theta = 60^\circ$

5. (6 points) The position of a particle after t seconds is given by

$$\vec{r}(t) = < 3\sin t, 3\cos t, 4t >$$

(a) Find the velocity of the particle after t seconds.

$\vec{v}(t) = \vec{r}\,'(t) = \langle 3\cos t, 3(-\sin t), 4\rangle$

(b) What is the speed of the particle after t seconds?

$\|\vec{r}\,'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$

$= \sqrt{9+16} = 5$

(c) What was the distance travelled by the particle along its path during the first π seconds?

$\int_0^{\pi} \|\vec{r}\,'(t)\|\, dt = \int_0^{\pi} 5\, dt = 5[t]_0^{\pi} = 5\pi$

6. (6 points) Give the equation of the plane containing the three points $A(1, 0, -1)$, $B(3, 0, 5)$ and $C(2, -1, 3)$.

$$\overrightarrow{AB} = \langle 2, 0, 6 \rangle = 2\langle 1, 0, 3 \rangle$$

$$\overrightarrow{AC} = \langle 1, -1, 4 \rangle$$

A normal vector for the plane is

$$\frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, 0, 3 \rangle \times \langle 1, -1, 4 \rangle$$

$$= \left\langle \begin{vmatrix} 0 & 3 \\ -1 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \right\rangle$$

$$= \langle 3, -1, -1 \rangle$$

The plane passes through $A(1, 0, -1)$ and has a normal vector $\langle 3, -1, -1 \rangle$, so the equation is

$$3(x-1) + (-1)(y-0) + (-1)(z-(-1)) = 0$$

$$3x - y - z - 4 = 0$$