First Name:
Last Name:

Student Number:

MATH 200 – Calculus III

FINAL EXAM – DECEMBER 12, [redacted]
TOTAL MARKS : 100

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1. (4 points) Find the angle between the planes $x + 2y + 3z = 12$ and $x - 2y + z = 7$.

$\vec{a} = \langle 1, 2, 3 \rangle$ and $\vec{b} = \langle 1, -2, 1 \rangle$ are normal vectors to the planes

Since $\vec{a} \cdot \vec{b} = 0$, the planes are perpendicular

2. (6 points) For the vector field

$$\vec{F}(x,y,z) = < 3x^2, 2xyz, 3x - z >$$

find

(a) $\nabla \times \vec{F}$ $= \left\langle \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & 3x-z \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ 3x-z & 3x^2 \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 3x^2 & 2xyz \end{vmatrix} \right\rangle$

(curl $\vec{F}$)

$$= \langle -2xy, -3, 2yz \rangle$$

(b) $\nabla \cdot \vec{F}$.

(div $\vec{F}$) $6x + 2xz + (-1)$

3. (11 points)

$$F(x, y, z) = x^3 + x^2yz + 2z - 3, \quad P = (1, -1, 2).$$

(a) Find ∇F at P

$$\nabla F = \langle 3x^2 + 2xyz,\ x^2z,\ x^2y + 2 \rangle$$

$$\nabla F(1, -1, 2) = \langle -1, 2, 1 \rangle$$

(b) Give a parametric equation for the line normal to the level surface $F(x, y, z) = 0$ at the point P.

$$\frac{x-1}{-1} = \frac{y+1}{2} = \frac{z-2}{1}$$

(c) What is the maximum rate of change of F at P?

$$|\nabla F(1, -1, 2)| = |\langle -1, 2, 1 \rangle| = \sqrt{1+4+1} = \sqrt{6}$$

(d) What is the rate of change of F at P in the direction of $\vec{v} =< 5, 0, 12 >$?

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 5,0,12 \rangle}{13} = \left\langle \frac{5}{13}, 0, \frac{12}{13} \right\rangle$$

The rate of change of F at P in direction $\vec{v}$ is

$$\nabla F(1,-1,2) \cdot \vec{u} = \langle -1,2,1 \rangle \left\langle \frac{5}{13}, 0, \frac{12}{13} \right\rangle = \frac{7}{13}$$

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4. (4 points) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^4 x^4}{x^{12} + y^6}$$

does not exist.

5. (7 points) For the vectors

$$\vec{a} = <1, 2, 2>, \quad \vec{b} = <0, 3, 4>$$

(a) Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$.

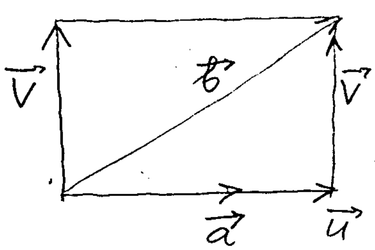
$\vec{a} = \langle 1, 2, 2 \rangle$

$\vec{b} = \langle 0, 3, 4 \rangle$

$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$

$\vec{a} \times \vec{b} = \langle 2, -4, 3 \rangle$

(b) Decompose $\vec{b}$ into a vector $\vec{u}$ parallel to $\vec{a}$ and a vector $\vec{v}$ orthogonal to $\vec{a}$.



$$\vec{u} = \text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \cdot \vec{a}$$

$$= \frac{14}{9} \cdot \vec{a} = \left\langle \frac{14}{9}, \frac{28}{9}, \frac{28}{9} \right\rangle$$

$$\vec{V} = \vec{b} - \vec{u} = \left\langle -\frac{14}{9}, -\frac{1}{9}, \frac{8}{9} \right\rangle$$

6. (5 points) Find all the critical points (a, b) of the function

$$f(x, y) = 6xy + 3y^2 - x^3 + 1.$$

Use the second derivative test to decide whether these correspond to local maxima, minima or saddle points.

$$f_x = 6y - 3x^2 = 3(2y - x^2)$$
$$f_y = 6x + 6y = 6(x+y)$$

$$\begin{cases} 2y - x^2 = 0 \\ x + y = 0 \end{cases} \Rightarrow y = -x \Rightarrow -2x - x^2 = 0 \Rightarrow x = 0 \text{ or } x = -2$$

Hence critical pts are $(0,0)$ and $(-2,2)$

$$f_{xx} = -6x, \quad f_{xy} = 6$$
$$f_{yy} = 6, \quad f_{yx} = 6$$

$$D(x,y) = \begin{vmatrix} -6x & 6 \\ 6 & 6 \end{vmatrix} = 36 \begin{vmatrix} -x & 1 \\ 1 & 1 \end{vmatrix} = -36(x+1)$$

$D(0,0) = -36$, so $(0,0)$ gives a saddle point

$D(-2,2) = 36$ and $f_{xx}(-2,2) = 12 > 0$, so $(-2,2)$ produces a local minimum

7. (6 points) The vector field

$$\vec{F}(x,y,z) = < yz + y + z + 2x, xz + 4y + x, xy + x - 1 >$$

is conservative.

(a) Find a potential function f for $\vec{F}$.

$f_x = yz + y + z + 2x \qquad (1)$

$f_y = xz + 4y + x \qquad (2)$

$f_z = xy + x - 1 \qquad (3)$

From (3) $\Rightarrow$ $f = xyz + xz - z + g(x,y)$

$\cancel{xz} + 4y + x = f_y = \cancel{xz} + g_y(x,y)$

$\Rightarrow g(x,y) = 2y^2 + xy + h(x)$

$\Rightarrow f = xyz + xz - z + 2y^2 + xy + h(x)$

$\cancel{yz} + \cancel{y} + \cancel{z} + 2x = f_x = \cancel{yz} + \cancel{z} + \cancel{y} + h'(x)$

$\therefore h(x) = x^2 + C$

$\Rightarrow f = xyz + xz - z + 2y^2 + xy + x^2 + C$

(b) Hence (or otherwise) evaluate

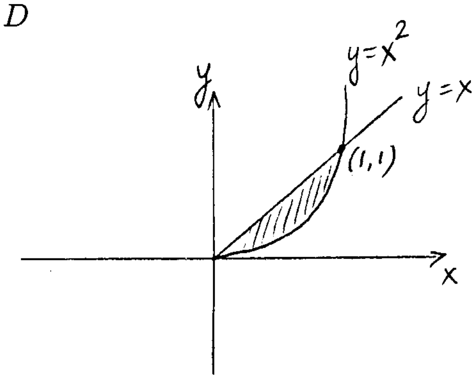
We found f s.t. $\nabla f = \vec{F}$ $\qquad \int_C \vec{F} \cdot d\vec{r}$

where C is the line segment from $(0,0,0)$ to $(1,1,1)$. By FTLI

$$\int_C \vec{F} \cdot d\vec{r} = f(1,1,1) - f(0,0,0) = 5$$

8. (8 points) D is the region bounded by the lines $y = x^2$, $y = x$.

(a) Sketch D



(b) Write D as both a Type I and Type II region.

$$I: \{(x,y) \mid 0 \le x \le 1,\ \ x^2 \le y \le x\}$$

$$II: \{(x,y) \mid 0 \le y \le 1\ \ y \le x \le \sqrt{y}\}$$

(c) If C is the boundary of D (positively oriented) use Green's Theorem to evaluate

$$\int_C (x\, dx + 2xy\, dy).$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y, \quad \text{so}$$

$$\int_C x\,dx + 2xy\,dy = \iint_D 2y\, dA = \int_0^1 \int_{x^2}^{x} 2y\, dy\, dx$$

$$= \int_0^1 \left[y^2\right]_{x^2}^{x} dx = \int_0^1 (x^2 - x^4)\, dx = \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{2}{15}.$$

9. (10 points)

(a) Evaluate the (scalar) line integral

$$\int_C z\, ds$$

where C is given parametrically by

$$\vec{r}(t) = < 3\cos t, 3\sin t, t >, \quad 0 \le t \le \pi.$$

$$\int_C z\, ds = \int_0^{\pi} t\sqrt{9+1}\, dt = \sqrt{10}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{\pi^2\sqrt{10}}{2}$$

(b) Parametrize the straight line L from $(0, -1, 1)$ to $(1, 0, 3)$.

$$\vec{r}(t) = (1-t)\langle 0,-1,1\rangle + t\langle 1,0,3\rangle$$
$$= \langle t, t-1, 1+2t\rangle$$

$$\begin{cases} x = 0 + t \\ y = -1 + t \\ z = 1 + 2t \end{cases} \qquad 0 \le t \le 1$$

(c) Evaluate the (vector) line integral over this straight line

$$\int_L \vec{F} \cdot d\vec{r}, \quad \vec{F} =< 2y, x, z > .$$

$$\vec{F}(\vec{r}(t)) = \vec{F}\langle 2t-2, t, 2t+1\rangle$$

$$d\vec{r} = \langle dx, dy, dz\rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \langle 1, 1, 2\rangle dt$$

$$\Rightarrow \int_L \vec{F}\, d\vec{r} = \int_0^1 \langle 2t-2, t, 2t+1\rangle \langle 1, 1, 2\rangle dt$$

$$= \int_0^1 (2t-2+t+4t+2)\, dt$$

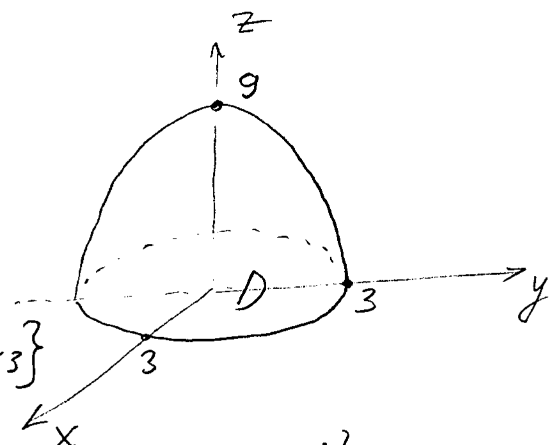
$$= \int_0^1 7t\, dt = 7\left[\frac{t^2}{2}\right]_0^1 = \frac{7}{2}$$

10. (6 points) B is the region below the paraboloid $z = 9 - (x^2 + y^2)$ and above the xy-plane ($z = 0$).

(a) Sketch B and describe B in terms of cylindrical coordinates.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$z = 9 - r^2$



$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 3\}$

$B = \{(r,\theta,z) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 9 - r^2\}$

(b) Evaluate the triple integral

$$I = \iiint_B 5(x^2+y^2)^{1/2}\,dV.$$

$$\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} 5r^2\,dz\,dr\,d\theta = 5\int_0^{2\pi} d\theta \int_0^3 r^2 [z]_0^{9-r^2}\,dr$$

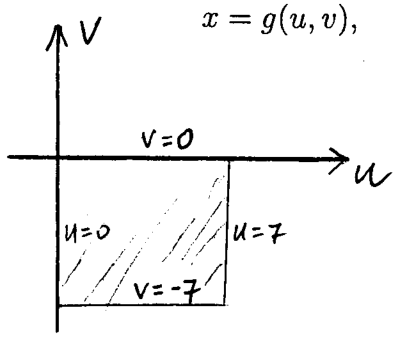
$$= 10\pi \int_0^3 (9r^2 - r^4)\,dr = 10\pi \left[3r^3 - \frac{r^5}{5}\right]_0^3$$

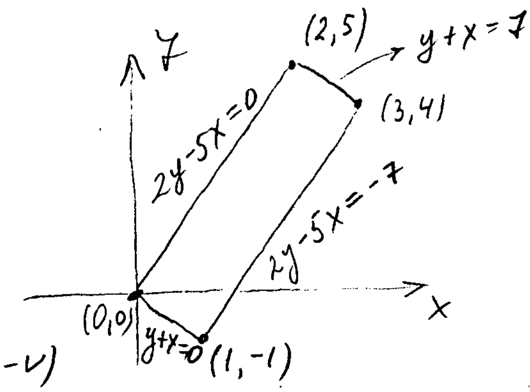
$$= 10\pi \left(81 - \frac{243}{5}\right) = 324\pi$$

11. (10 points) The parallelogram P has corners $(0,0)$, (2,5), (3,4) and $(1,-1)$

(a) Write P as the image of a rectangle R under a transformation

$$x = g(u,v), \quad y = h(u,v).$$





Hence $\begin{cases} u = y+x \\ v = 2y-5x \end{cases} \rightarrow \begin{cases} x = \frac{1}{7}(2u-v) \\ y = \frac{1}{7}(5u+v) \end{cases}$

(b) Find the Jacobian of this transformation.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{5}{7} & \frac{1}{7} \end{vmatrix} = \frac{1}{7}$$

(c) By a change of variables or otherwise evaluate

$$I = \iint_P (x+y)\, dA.$$

$$I = \int_{-7}^{0}\int_{0}^{7} \frac{1}{7} u \, du \, dv = \frac{1}{7}\int_{-7}^{0}\left[\frac{u^2}{2}\right]_0^7 dv = \frac{1}{7}\cdot\frac{49}{2}\left[v\right]_{-7}^{0} = \frac{49}{2}$$

12. (6 points) Using the Divergence Theorem or otherwise evaluate

$$\int\int_S \vec{F} \cdot d\vec{S}$$

where

$$\vec{F} = < 2xy, e^z, x^3 >$$

and S is the surface of the cube $[0,1] \times [0,1] \times [0,1]$ with positive (outward) orientation.

$$\operatorname{div} \vec{F} = 2y + 0 + 0 = 2y$$

$$\iint_S \vec{F}\, d\vec{S} = \iiint_E 2y\, dV = \int_0^1\int_0^1\int_0^1 2y\, dy\, dx\, dz$$

$$= \int_0^1\int_0^1 \left[y^2\right]_0^1 dx\, dz = \int_0^1 dx \int_0^1 dz = 1$$

13. (12 points) S is the surface

$$\vec{r}(u,v) = < u\cos v, u\sin v, u >, \quad 0 \le u \le 1, \quad 0 \le v \le 2\pi.$$

(i.e. S is the portion of the cone $z = \sqrt{x^2+y^2}$ below $z = 1$).

(a) Evaluate the normal vector $\vec{r}_u \times \vec{r}_v$

$$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$$
$$\vec{r}_v = \langle -u\sin v, u\cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -u\cos v, -u\sin v, u \rangle$$

(b) Give the equation of the plane tangent to the surface S at the point $(-1/2, 0, 1/2)$ (corresponding to $u = 1/2$, $v = \pi$).

$\vec{n}(u,v) = \vec{r}_u \times \vec{r}_v$ is a normal vector for the tangent plane

$\vec{n}(\frac{1}{2}, \pi) = \langle +\frac{1}{2}, 0, \frac{1}{2} \rangle$)

hence the tangent plane is

$$+\frac{1}{2}(x + \frac{1}{2}) + \frac{1}{2}(z - \frac{1}{2}) = 0$$

$$+\frac{1}{2}x + \frac{1}{4} + \frac{1}{2}z - \frac{1}{4} = 0$$

$$z = -x$$

(c) Evaluate the (scalar) surface integral

$$\int\int_S z\, dS.$$

$$|\vec{r_u} \times \vec{r_v}| = \sqrt{2}\, u$$

$$\iint_S z\, dS = \iint_D u \sqrt{2} u \, dA, \text{ where } D = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 2\pi\}$$

$$= \sqrt{2} \int_0^{2\pi} \int_0^{1} u^2 \, du\, dv = \sqrt{2} \int_0^{2\pi} \left[\frac{u^3}{3}\right]_0^1 dv$$

$$= \frac{\sqrt{2}}{3} 2\pi .$$

(d) Evaluate the (vector) surface integral

$$\int\int_S \vec{F} \cdot d\vec{S}, \quad \vec{F} = < y, -x, z >$$

(with the normal pointing upwards).

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r_u} \times \vec{r_v})\, dA$$

$$= \iint_D \langle u \sin v, -u \cos v, u \rangle \cdot \langle -u \cos v, -u \sin v, u \rangle \, dA$$

$$= \iint_D u^2 \, dA = \int_0^{2\pi} \int_0^{1} u^2 \, du\, dv = \frac{2}{3}\pi ,$$

by the result in c).