First Name: **Student Number:**
Last Name:

MATH 200 – Calculus III

FINAL EXAM – DECEMBER 12, [redacted]

TOTAL MARKS : 100

Show all working. You may use non-programmable, non-graphing calculators

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1. (4 points) Find the angle between the planes $x + 2y + 3z = 12$ and $x - 2y + z = 7$.

2. (6 points) For the vector field

$$\vec{F}(x, y, z) =< 3x^2, 2xyz, 3x - z >$$

find

(a) $\nabla \times \vec{F}$

(b) $\nabla \cdot \vec{F}$.

3. (11 points)

$$F(x, y, z) = x^3 + x^2yz + 2z - 3, \quad P = (1, -1, 2).$$

(a) Find ∇F at P

(b) Give a parametric equation for the line normal to the level surface $F(x, y, z) = 0$ at the point P.

(c) What is the maximum rate of change of F at P?

(d) What is the rate of change of F at P in the direction of $\vec{v} =< 5, 0, 12 >$?

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4. (4 points) Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{y^4x^4}{x^{12}+y^6}$$

does not exist.

5. (7 points) For the vectors

$$\vec{a} =< 1, 2, 2 >, \quad \vec{b} =< 0, 3, 4 >$$

(a) Find a unit vector orthogonal to both $\vec{a}$ and $\vec{b}$.

(b) Decompose $\vec{b}$ into a vector $\vec{u}$ parallel to $\vec{a}$ and a vector $\vec{v}$ orthogonal to $\vec{a}$.

6. (5 points) Find all the critical points (a, b) of the function

$$f(x, y) = 6xy + 3y^2 - x^3 + 1.$$

Use the second derivative test to decide whether these correspond to local maxima, minima or saddle points.

7. (6 points) The vector field

$$\vec{F}(x, y, z) =< yz + y + z + 2x, xz + 4y + x, xy + x - 1 >$$

is conservative.

(a) Find a potential function f for $\vec{F}$.

(b) Hence (or otherwise) evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is the line segment from $(0, 0, 0)$ to $(1, 1, 1)$.

8. (8 points) D is the region bounded by the lines $y = x^2$, $y = x$.

(a) Sketch D

(b) Write D as both a Type I and Type II region.

(c) If C is the boundary of D (positively oriented) use Green's Theorem to evaluate

$$\int_C (x\ dx + 2xy\ dy).$$

9. (10 points)

(a) Evaluate the (scalar) line integral

$$\int_C z\ ds$$

where C is given parametrically by

$$\vec{r}(t) =< 3\cos t, 3\sin t, t >, \quad 0 \leq t \leq \pi.$$

(b) Parametrize the straight line L from $(0, -1, 1)$ to $(1, 0, 3)$.

(c) Evaluate the (vector) line integral over this straight line

$$\int_L \vec{F} \cdot d\vec{r}, \qquad \vec{F} = < 2y, x, z > .$$

10. (6 points) B is the region below the paraboloid $z = 9 - (x^2 + y^2)$ and above the xy-plane ($z = 0$).

(a) Sketch B and describe B in terms of cylindrical coordinates.

(b) Evaluate the triple integral

$$\int\int\int_B 5(x^2 + y^2)^{1/2}\ dV.$$

11. (10 points) The parallelogram P has corners $(0,0)$, (2,5), (3,4) and $(1,-1)$

(a) Write P as the image of a rectangle R under a transformation

$$x = g(u,v), \quad y = h(u,v).$$

(b) Find the Jacobian of this transformation.

(c) By a change of variables or otherwise evaluate

$$\int\int_P (x+y)\,dA.$$

12. (6 points) Using the Divergence Theorem or otherwise evaluate

$$\int\int_S \vec{F} \cdot \, d\vec{S}$$

where

$$\vec{F} =< 2xy, e^z, x^3 >$$

and S is the surface of the cube $[0,1] \times [0,1] \times [0,1]$ with positive (outward) orientation.

13. (12 points) S is the surface

$$\vec{r}(u,v) =< u\cos v, u\sin v, u >, \quad 0 \le u \le 1, \quad 0 \le v \le 2\pi.$$

(i.e. S is the portion of the cone $z = \sqrt{x^2+y^2}$ below $z = 1$).

(a) Evaluate the normal vector $\vec{r}_u \times \vec{r}_v$

(b) Give the equation of the plane tangent to the surface S at the point $(-1/2, 0, 1/2)$ (corresponding to $u = 1/2$, $v = \pi$).

(c) Evaluate the (scalar) surface integral

$$\int\int_S z \, dS.$$

(d) Evaluate the (vector) surface integral

$$\int\int_S \vec{F} \cdot d\vec{S}, \quad \vec{F} =< y, -x, z >$$

(with the normal pointing upwards).

14. (5 points) Sketch the quadric surface $x^2 + y^2 = 1 + z^2$.