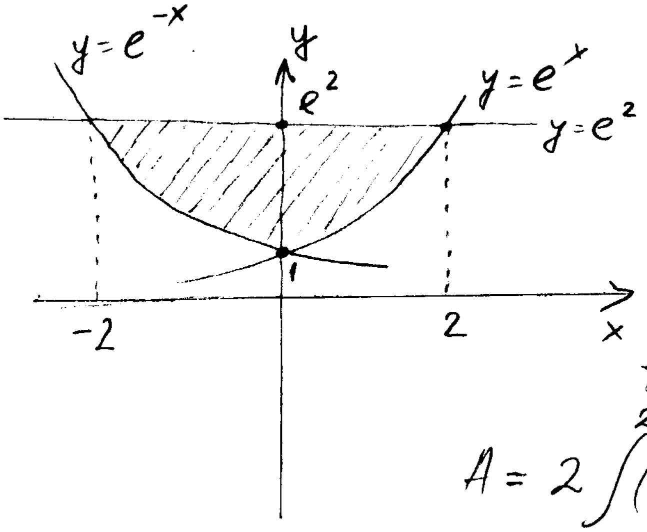
Sample Final Solutions

1. Sketch the region bounded by the curves $y = e^x$, $y = e^{-x}$ and $y = e^2$, and find the area of the region.



$e^x = e^2 \Rightarrow x = 2$

$e^{-x} = e^2 \rightarrow x = -2$

Because of symmetry, the required area is

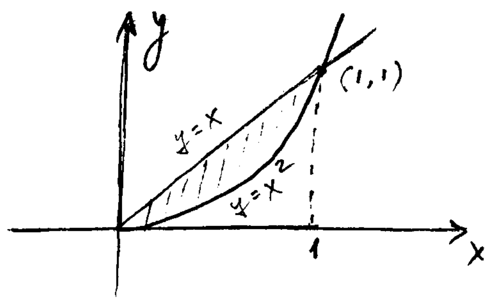
$$A = 2\int_0^2 (e^2 - e^x)\,dx$$

$$= 2\left[e^2 x - e^x\right]_0^2$$

$$= 2\left[(2e^2 - e^2) - (0-1)\right]$$

$$= 2(e^2+1).$$

2. Find the volume of the solid obtained by rotating about the x-axis the region bounded by the curves $y = x$ and $y = x^2$.

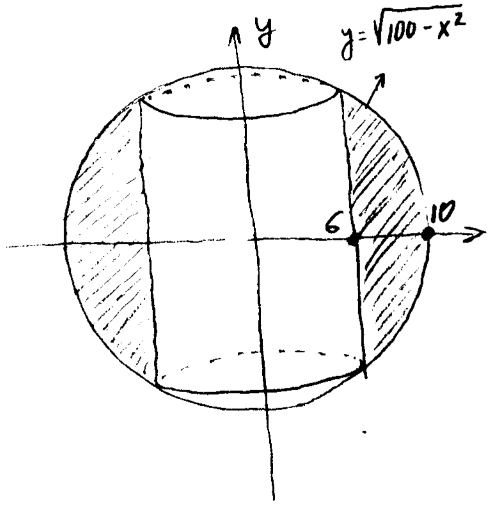


$$V = \pi \int_0^1 [(x)^2 - (x^2)^2]\,dx$$

$$= \pi \left[\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\right]_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}.$$

3. A cylindrical hole of radius 6cm has been drilled straight through the center of a sphere of radius 10cm. Use the method of cylindrical shells to find the volume of the remaining solid.



The sphere has been obtained by rotating the circle $x^2+y^2=10^2$ about the y-axis.

$$V = 2\int_6^{10} 2\pi x \sqrt{100-x^2}\, dx$$

$$= -2\pi \int_6^{10} \sqrt{100-x^2}\, d(100-x^2)$$

$$= -2\pi \left[\frac{(100-x^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_6^{10} = -2\pi \left(0 - \frac{(100-36)^{\frac{3}{2}}}{\frac{3}{2}} \right) = \frac{2048}{3}\pi$$

4. **Evaluate the integral**

$$\int_0^1 x \tan^{-1} x \, dx.$$

$$\int_0^1 \tan^{-1}x \; d\left(\frac{x^2}{2}\right) = \left[\frac{x^2}{2}\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x^2}{2}\frac{1}{1+x^2}\,dx$$

$$= \left[\frac{x^2}{2}\tan^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$$

$$= \left[\frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left(x - \tan^{-1}x\right)\right]_0^1$$

$$= \frac{1}{2}\frac{\pi}{4} - \frac{1}{2} + \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}.$$

5. **Evaluate the integral**

$$\int \cos^4 x \sin^3 x \, dx.$$

$$= -\int \cos^4 x \sin^2 x \; d(\cos x)$$

$$= -\int \cos^4 x \, (1-\cos^2 x) \; d(\cos x)$$

$$= -\int (\cos^4 x - \cos^6 x) \; d(\cos x)$$

$$= -\frac{(\cos x)^5}{5} + \frac{(\cos x)^7}{7} + C$$

6. **Evaluate the integral**

$$I = \int \frac{1}{x^2\sqrt{1-x^2}}\, dx. \qquad \text{Let } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Substitute $x = \sin\theta$. Then $dx = \cos\theta \, d\theta$

So $$I = \int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta \sqrt{\cos^2\theta}} \, d\theta = \int \frac{\cos\theta}{\sin^2\theta \, |\cos\theta|} \, d\theta$$

$$= \int \frac{d\theta}{\sin^2\theta} = -\cot\theta + C$$

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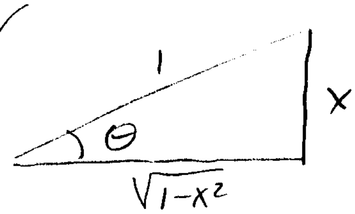

$$\sin\theta = x$$
$$\cot\theta = \frac{\sqrt{1-x^2}}{x}$$
)

$$= -\frac{\sqrt{1-x^2}}{x} + C.$$

7. **Evaluate the integral**

$$\int \underbrace{\ln^2 x}_{u}\, \underbrace{dx}_{v}.$$

$= x \ln^2 x - \int x \, d(\ln^2 x)$

$= x \ln^2 x - \int x \cdot 2 \ln x \cdot \frac{1}{x} \, dx$

$= x \ln^2 x - 2 \int \underbrace{\ln x}_{u} \, \underbrace{dx}_{v}$

$= x \ln^2 x - 2 \left(x \ln x - \int x \, d(\ln x) \right)$

$= x \ln^2 x - 2 \left(x \ln x - \int x \frac{1}{x} dx \right)$

$= x \ln^2 x - 2x \ln x + 2 \int dx$

$= x \ln^2 x - 2x \ln x + 2x + C.$

8. Find the length of the curve defined as $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 3$.

$$\frac{dx}{dt} = 3 - 3t^2 \qquad \frac{dy}{dt} = 6t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3\sqrt{(1-t^2)^2 + 4t^2} = 3\sqrt{(1+t^2)^2} = 3(1+t^2)$$

$$L = \int_0^3 3(1+t^2) = 3\left[\left(t + \frac{t^3}{3}\right)\right]_0^3 = 3\,(3+9) = 36\,.$$

9. **What is the connection between polar and Cartesian coordinates? Show that $r = 6\cos\theta + 8\sin\theta$ is the equation of a circle and find its center and radius.**

$x = r\cos\theta$
$y = r\sin\theta$ $\Rightarrow$ $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$
$x^2+y^2 = r^2$

So $r = 6\frac{x}{r} + 8\frac{y}{r}$

$r^2 = 6x+8y$

$x^2+y^2 = 6x+8y$

$x^2+y^2-6x-8y+9+16 = 9+16$

$(x-3)^2+(y-4)^2 = 5^2$ — this is a circle of radius 5 centered at $(3,4)$.

10. **Find**

$$\lim_{n\to\infty}(\sqrt{n^2+1}-n)\sqrt{3n^2+n}.$$

$$= \lim_{n\to\infty} \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}\sqrt{3n^2+n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{3n^2+n}}{\sqrt{n^2+1}+n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{3+\frac{1}{n}}}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{\sqrt{3}}{2}.$$

11. Find the values for which

$$\sum_{n=0}^{\infty}(2x)^n$$

converges. Find the sum of the series for those values.

This is a geometric series with $a=1$, $r=2x$

It converges when $|2x|<1$, that is, $-\frac{1}{2}<x<\frac{1}{2}$

The sum is $\frac{1}{1-2x}$.

12. **Test for convergence**

$$\sum_{n=1}^{\infty} \frac{2n+1}{3n^2-1}.$$

$$\lim \frac{\frac{2n+1}{3n^2-1}}{\frac{1}{n}} = \frac{2}{3}$$, so divergent by the Comparison Test.

13. **Test for convergence**

$$\sum_{n=1}^{\infty} \frac{(n+1)2^n}{n3^{n+1}}.$$

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \frac{(n+2)\,2^{n+1}}{(n+1)\ 3^{n+2}} \ \frac{n\ 3^{n+1}}{(n+1)\,2^n} = \frac{2}{3} \lim \frac{n(n+2)}{(n+1)^2} = \frac{2}{3},$$

so convergent by the Ratio test.

14. Test for convergence

$$\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}.$$

$$\int_3^{\infty} \frac{1}{x^2-4}\,dx = \frac{1}{4}\left[\ln \frac{x-2}{x+2}\right]_3^{\infty} = \frac{1}{4}\left(0 - \ln \frac{1}{5}\right)$$

So convergent by the Integral Test.