1. Trigonometric substitution.

$$I = \int \frac{1}{x^2\sqrt{x^2-1}}\,dx.$$

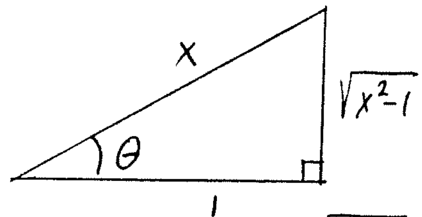
$$x = \sec\theta \implies dx = \sec\theta\tan\theta\,d\theta$$

$$I = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\,\tan\theta}$$

$$= \int \cos\theta\,d\theta$$

$$= \sin\theta + C$$

$$= \frac{\sqrt{x^2-1}}{x} + C$$



$$\implies \sin\theta = \frac{\sqrt{x^2-1}}{x}$$

2. Write out the general form of the partial fraction decomposition of the function

$$\frac{x^2-6x+9}{(x^2-6x+8)^2(x^2-6x+10)^2}.$$

DO NOT SOLVE for the coefficients.

$x^2-6x+8 = (x-2)(x-4)$

$x^2-6x+10$ is irreducible

$$\frac{x^2-6x+9}{(x^2-6x+8)^2(x^2-6x+10)^2}$$

$$= \frac{x^2-6x+9}{(x-2)^2(x-4)^2(x^2-6x+10)^2}$$

$$= \frac{A_1}{x-2} + \frac{A_2}{(x-2)^2} + \frac{B_1}{x-4} + \frac{B_2}{(x-4)^2} + \frac{M_1x+N_1}{x^2-6x+10} + \frac{M_2x+N_2}{(x^2-6x+10)^2}$$

3. Partial fractions.

$$I = \int \frac{1}{x^3 + 2x^2 + x}\, dx.$$

$x^3 + 2x^2 + x = x(x+1)^2$, so

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$
$$1 = A(x^2 + 2x + 1) + B(x^2 + x) + Cx$$
$$1 = (A+B)x^2 + (2A + B + C)x + A$$

$$\Longrightarrow \begin{cases} A + B = 0 \\ 2A + B + C = 0 \\ A = 1 \end{cases} \Rightarrow A = 1,\ B = -1,\ C = -1$$

Hence $\displaystyle \frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$, and

$$I = \int \frac{1}{x}\, dx - \int \frac{1}{x+1}\, dx - \int \frac{1}{(x+1)^2}\, dx$$
$$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C.$$
$$\left(= \ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + C \right)$$

4. **Rationalizing substitution.**

$$\int \frac{1}{1+\cos x}\,dx.$$

Universal substitution: $\tan\frac{x}{2} = t$. Then $dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$. Hence

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$$

$$= \int \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}}\,dt = \int dt = t + C$$

$$= \tan\frac{x}{2} + C.$$

Another solution:

$$\int \frac{1}{1+\cos x}\,dx$$

$$= \int \frac{1-\cos x}{1-\cos^2 x}\,dx = \int \frac{1-\cos x}{\sin^2 x}\,dx$$

$$= \int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x}\,dx$$

$$= -\cot x - \int (\sin x)^{-2}\,d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C.$$

$$\left(= \frac{1-\cos x}{\sin x} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2} \right)$$

5. **Find the length of the curve defined as $x = 3t - t^3$, $y = 3t^2$, $0 \le t \le 3$.**

$$\frac{dx}{dt} = 3 - 3t^2, \quad \frac{dy}{dt} = 6t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3\sqrt{(1-t^2)^2 + 4t^2} = 3\sqrt{(1+t^2)^2} = 3(1+t^2)$$

Then $$L = \int_0^3 3(1+t^2)\,dt = 3\left(t + \frac{t^3}{3}\right)\Big]_0^3 = 3(3+9) = 36.$$

6. What is the connection between polar and Cartesian coordinates? Show that $r = 6\cos\theta + 8\sin\theta$ is the equation of a circle and find its center and radius.

Connection: $x = r\cos\theta$
$y = r\sin\theta$

$\Rightarrow \cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

$x^2 + y^2 = r^2$

The equation becomes

$$r = 6\frac{x}{r} + 8\frac{y}{r}$$

or $r^2 = 6x + 8y$

or $x^2 + y^2 = 6x + 8y$

Complete the squares:

$$x^2 - 6x + 9 + y^2 - 8y + 16 = 25$$

$$(x-3)^2 + (y-4)^2 = 5^2$$

Therefore the curve is a circle of radius 5, centered at (3,4).