

## MATH 101 Lab 5

In this lab we look at some of the tools that can be used for solving problems on sequences and series. We know that a huge set of functions can be created by using the elementary functions  $x^r$ , where  $r$  is a real constant, all of the polynomial and rational functions, the trigonometric, logarithmic, exponential, hyperbolic functions and all of their inverses. The set would include all of the combinations of these functions using finitely many compositions and finitely many algebraic operations. We may need Maple's help, but we can evaluate them, graph them, manipulate them and study their behavior.

The collection is large, and we can certainly use these functions to create a variety of complex mathematical models. Yet, one of the truly fascinating aspects of mathematics is that this supply of accessible functions occupies only a small corner in a much larger warehouse of nice and well-behaved (continuous) functions that play a significant role in mathematics and its applications. Most of the functions in this warehouse live in a world that can be explored by the use of series.

Examples come from a variety of mathematical situations. We have already mentioned that antiderivatives of continuous functions are frequently beyond our reach, even though the Fundamental Theorem of Calculus says that they always exist. The same can be said about solutions (of the form  $y = f(x)$ ) to some fairly simple equations in  $x$  and  $y$ . The same can be said about inverses to some functions, even simple functions like  $f(x) = x + \sin x$ , for example. In all of these situations, and others as well, answers to very practical problems can turn out to exist, yet to be beyond the reach of familiar functions. In this lab we look at some of the tools that can be used to study such functions. It turns out that functions in this list can be described by expressions that look and behave very much like the simplest functions, namely polynomials

$$p(x) = \sum_{i=0}^n a_i x^i, \text{ or } p(x) = \sum_{i=0}^n a_i (x - a)^i.$$

The only difference is that we must allow the degree of the polynomial to go to infinity. In the process, something that we can call "long polynomials",

$$p(x) = \sum_{i=0}^{\infty} a_i x^i, \text{ or } p(x) = \sum_{i=0}^{\infty} a_i (x - a)^i,$$

actually, called **power series** or **Taylor series**, are created. Maple has a command

```
> taylor();
```

for producing a Taylor series. First we look at sequences.

Mathematically, a sequence is just a function whose domain is the set of all positive integers, or a subset of that set. It follows that the standard methods for defining functions in Maple can be used to define sequences just as well. For example, we can find terms of the sequence  $\left\{ \frac{(-1)^n}{n^2} \right\}$  by

```
> a:=n->(-1)^n/n^2;
```

```
> a(3),a(10);
```

$$\frac{-1}{9}, \frac{1}{100}$$

We can also use the special command

```
> seq(a(n),n=N..M);
```

to list consecutive terms of the sequence. The following command adds the terms  $a_N, a_{N+1}, \dots, a_M$ .

```
> sum(a(n),n=N..M);
```

Note that, in the second sum,  $M$  can be **infinity**. Since we will be more interested in **infinite sequences**, they will usually be defined as Maple functions; the `seq()` command can still be used to evaluate and list some of the terms of a sequence:

**Example** . Show that the sequence  $\{a_n\}$  defined by  $a_n = \frac{n^{100}}{100^n}$  is eventually a decreasing sequence which converges to 0. At what point in the sequence does it become decreasing? Determine an  $N$  such that

$0 < a_n < 10^{-8}$  for all  $n > N$ .

We begin by evaluating, for the sake of interest, the first few terms of the sequence:

```
> a:=n->n^100/100^n;  
.....  
> seq(evalf(a(n),2),n=1..10);
```

.010, .13 10<sup>27</sup>, .52 10<sup>42</sup>, .16 10<sup>53</sup>, .79 10<sup>60</sup>, .65 10<sup>66</sup>, .32 10<sup>71</sup>, .20 10<sup>75</sup>, .27 10<sup>78</sup>, .10 10<sup>81</sup>

Does this sequence diverge to infinity?

```
> limit(a(n),n=infinity);
```

0

So the sequence converges to 0, and merely listing the first 10 terms, and witnessing the phenomenal growth, did not suggest the long term pattern of values. Since the sequence is defined as a Maple function, we can study the sequence by plotting the function. We do this next. This seems to show that the sequence does eventually decrease. By placing the pointer at the top of the graph and clicking, we can see that the sequence appears to be decreasing starting with the 22nd term. Certainly this is not very rigorous approach. Our eyes cannot really establish with any certainty that the graph continues to decrease forever. Thus, to show the decreasing nature of this sequence more rigorously, we can take a look at the derivative of the function representing the sequence.

```
> plot(a(n),n=1..40);  
.....  
> ap:=diff(a(n),n);
```

$$ap := 100 \frac{n^{99}}{100^n} - \frac{n^{100} \ln(100)}{100^n}$$

```
> ap:=normal(ap);
```

$$ap := -\frac{n^{99}(-100 + n \ln(100))}{100^n}$$

```
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$$ap := 100 \frac{n^{99}}{100^n} - \frac{n^{100} \ln(100)}{100^n}$$

```
> ap:=normal(ap);
```

$$ap := -\frac{n^{99}(-100 + n \ln(100))}{100^n}$$

```
> solve(-100+n*ln(100)=0,n);
```

$$\frac{100}{\ln(100)}$$

```
> evalf("");
```

21.71472410

It follows that the derivative  $a_n$  is negative for  $n > 22$ , and so the sequence decreases for  $n > 22$ .

Maple might have some difficulties with the last part of the problem. Even if Maple cannot solve the inequality, this is not a major problem. Since we already know that the sequence is decreasing, we only need to get a decimal solution for  $n$  in the equation  $a_n = 10^{-8}$ . Once we have this number, we can say that  $a_n$  is smaller yet for all larger values of  $n$ .

```
> solve(a(n)<10^(-8),n);  
(lengthy run of Maple)  
> fsolve(a(n)=10^(-8),n);
```

.8655891064

Now, this obviously is not the result we are looking for (look at the graph). Unsuccessful attempts at using the `fsolve()` command can be resolved by telling Maple where to look for a solution. In the present case, we can note that  $a_{100} = 1$ . By restricting our search for a solution to values of  $n > 100$ , we are able to get a decimal solution.

```
> fsolve(a(n)=10^(-8),n,n=100..infinity);
```

105.0749621

This result shows that  $0 < a_n < 10^{-8}$  for all  $n > 106$ .

## Series

Maple has two commands for forming sums. These are the `sum()` and `add()` commands. We start with a discussion on their similarities and differences. For finite sums, there is little difference between the two commands. The command `sum(a(n),n=1..m)` evaluates the exact sum of the terms  $a(n)$  only if  $m - l$  is less than 1000. If  $m - l > 1000$ , then Maple resorts to more theoretical means to evaluate the sum. In contrast, `add(a(n),n=1..m)` *always* evaluates the exact sum.

Another difference is that Maple can often evaluate `sum(a(n),n=1..infinity)`, and it can often find a formula for `sum(a(n),n=1..m)` when  $l$  and  $m$  are not specified. The `add()` command can never be used in either of these situations. The arithmetic and symbolic computations can quickly get out of hand. If fractions are involved, Maple will, for example, have to compute a common denominator. Therefore, it is advisable, especially for  $m - l < 1000$ , to enclose the `sum()` and `add()` commands inside the `evalf()` command.

According to the definition, a series is a sequence – a sequence  $\{s_n\}$  of its partial sums. The series is convergent or divergent if the sequence of partial sums is convergent or divergent, correspondingly. So we can focus on the sequence  $\{s_n\}$ . Does it converge or diverge? If it converges, can we compute its limit? Let us consider some examples. Maple might be able to establish divergence to  $\infty$ .

```
> S:=sum(1/n,n=1..infinity);
```

$S := \infty$

Maple can find the sums of some infinite series in “closed form”, that is, an exact expression. Quite often the output will be in the form of some unfamiliar special function of mathematics. Using the `evalf()` command can give a good approximation to the sum.

```
> S:=sum(1/n^2,n=1..infinity);
```

$$S := \frac{1}{6} \pi^2$$

> `S:=sum(1/n^3,n=1..infinity);`

$$S := \zeta(3)$$

> `evalf(S);`

1.202056903

If Maple cannot find a closed form expression, it just returns the series.

> `S:=sum(n^2/(2^n+1),n=1..infinity);`

$$S := \sum_{n=1}^{\infty} \frac{n^2}{2^n + 1}$$

The `evalf()` command can be used to find an approximation to the sum.

> `evalf(S);`

5.412796795

There are cases for which the `evalf()` command does not come up with an answer either.

> `S:=sum(2^n/(2^n+n+1),n=1..infinity);`

$$S := \sum_{n=1}^{\infty} \frac{2^n}{2^n + n + 1}$$

> `evalf(S);`

$$\sum_{n=1}^{\infty} \frac{2^n}{2^n + n + 1}$$

Perhaps we were a bit unfair to Maple here: this series diverges. Sometimes Maple detects divergence to  $\infty$ , but not in this case. But there are also convergent series for which `evalf()` fails. For example,

> `evalf(sum(ln(n+1)/n^3,n=1..infinity));`

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^3}$$

Though Maple cannot deal directly with some series, it can be helpful in establishing convergence and approximations to a broad spectrum of series. One possible approach is to apply some of the tests for convergence and use Maple for the necessary computations. The students are strongly encouraged to try this approach in the remaining labs.

Please submit commented sheets for the following questions.

## Question 1

A sequence  $\{a_n\}$  is defined by

$$a_n = \frac{[\ln(n+1)]^{\ln(n+1)}}{n^3}.$$

- List the first 50 terms as decimal numbers. By looking at the list, does the sequence appear to converge or diverge?
- If it appears to converge, try to predict its limit.
- Compute  $a(400)$  as decimal. Does this confirm or contradict your previous conclusions?
- Show that the sequence is eventually an increasing sequence. Find the term (its number would suffice) in the sequence where it changes from a decreasing to an increasing sequence.

## Question 2

Some of the mystery surrounding the notion of a series can be removed by appreciating that a series is just a sequence – the sequence of partial sums. Viewing a series in this way may seem unnecessary in calculations, but to fully understand what a series is, as a concept, it helps to pay some attention to the definition of a series. With this in mind, use Maple to

- set up the sequence of partial sums of the series  $\sum_{n=0}^{\infty} \frac{1}{n!}$ . Use the `evalf()` command when you define the sequence.
- List the first 20 terms of this sequence. (The output should be a sequence of decimal numbers.) Increase the number of digits of accuracy to 16, and repeat these calculations. (This can be done by entering 16 as an optional second argument in the `evalf()` command.)
- You will soon know, if you do not already know, that this series converges to  $e$ . Compare the 20 terms in this sequence to a decimal expansion of  $e$ .