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Inequalities Involving Moments of a Continuous Random Variable Defined Over a Finite Interval

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Abstract—Some results based on the Korkine's identity and integral inequalities of Hölder and Grüss are obtained for the moments of a continuous random variable whose probability distribution is a convex function on the interval of real numbers. Applications of these results are considered in deriving the inequalities involving higher moments and special means and also in evaluating moments of a beta random variable.

Keywords—Korkine identity, Grüss integral inequality, Hölder inequality, Special means, Beta distribution.

1. INTRODUCTION

Distribution functions and density functions provide complete descriptions of the distribution of probability for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions is useful in making comparisons. Knowing the probability function, we can determine the moments. There are, however, applications wherein the exact forms of probability distributions are not known or are mathematically intractable so that the moments cannot be calculated—as an example, an application in insurance in connection with the insurer's payout on a given contract or group of contracts that follows a mixture or compound probability distribution. It is this problem that motivates researchers to obtain alternative estimations for the moments of a probability distribution. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied [1–6]. In this paper, we further develop some estimations for the moments of a continuous random variable taking its values on a finite interval.

Set X to denote a *continuous random variable* (referred to as *random variable* in what follows now) whose probability density function $f : I \subseteq \mathbb{R} \to \mathbb{R}_+$ is a convex function on the interval of real numbers I and $a, b \in I$ $(a < b)$.

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Denote by M_r the r^{th} moment of X, $r \geq 0$, defined as

$$M_r = \int_a^b t^r f(t)\,dt. \tag{1.1}$$

The mean and variance of X are

$$\mu = M_1 = \int_a^b t f(t)\,dt, \tag{1.2}$$

$$\sigma^2 = M_2 - M_1^2 = \int_a^b (t-\mu)^2 f(t)\,dt. \tag{1.3}$$

In what follows now, when reference is made to the r^{th} moment of a particular distribution, we assume that the appropriate integral converges for that distribution.

2. PRELIMINARIES

For the integrable mappings $m, g, h : [a,b] \to \mathbb{R}$, the following identity, inequalities, and results hold and are presented for ready reference.

The Korkine's identity [7]

$$\begin{aligned}\int_a^b m(t)\,dt \int_a^b m(t)g(t)h(t)\,dt - \int_a^b m(t)g(t)\,dt \int_a^b m(t)h(t)\,dt \\ = \frac{1}{2}\int_a^b \int_a^b m(t)m(s)[g(t)-g(s)][h(t)-h(s)]\,dt\,ds\end{aligned} \tag{2.1}$$

holds provided all integrals involved in (2.1) exist and are finite.

The Hölder's integral inequality for double integrals [7]

$$\int_a^b \int_a^b g(t)g(s)\,dt\,ds \leq \left(\int_a^b \int_a^b g^p(t)g^p(s)\,dt\,ds\right)^{1/p} \left(\int_a^b \int_a^b g^q(t)g^q(s)\,dt\,ds\right)^{1/q}, \tag{2.2}$$

where $p > 1$ and $1/p + 1/q = 1$.

The Grüss integral inequality [8]

$$|T(g,h)| \leq \frac{(\Phi-\phi)(\Gamma-\gamma)}{4}, \tag{2.3}$$

where

$$T(g,h) = \frac{1}{b-a}\int_a^b g(t)h(t)\,dt - \frac{1}{b-a}\int_a^b g(t)\,dt \cdot \frac{1}{b-a}\int_a^b h(t)\,dt, \tag{2.4}$$

ϕ, Φ, γ, and Γ are real numbers such that $\phi \leq g(t) \leq \Phi$ and $\gamma \leq h(t) \leq \Gamma$ a.e. on $[a,b]$.

A premature Grüss inequality that provides a sharper bound than the above Grüss inequality [8]

$$|T(g,h)| \leq \frac{(\Phi-\phi)}{2}|T(h,h)|^{1/2}. \tag{2.5}$$

The Grüss type inequality [9]

$$0 \leq \frac{\int_a^b g(t)h^2(t)\,dt}{\int_a^b g(t)\,dt} - \left(\frac{\int_a^b g(t)h(t)\,dt}{\int_a^b g(t)\,dt}\right)^2 \leq \frac{(M-m)^2}{4}, \tag{2.6}$$

provided all integrals exist and are finite, $\int_a^b g(t)\,dt > 0$, and $m \leq g(t) \leq M$ a.e. on $[a,b]$.

It may be noted that inequalities (2.3) and (2.6) are sharp in the sense that the constant 1/4 cannot be replaced by a smaller one.

3. INEQUALITIES INVOLVING MOMENTS

The following results for the moments of the random variable X hold.

THEOREM 3.1. *For a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$, and $r \geq 0$,*

$$M_r - \mu M_{r-1} \leq \begin{cases} \dfrac{(b-a)\left(b^{r-1} - a^{r-1}\right)}{2}, \\ (b-a)\left(\dfrac{b^{r+1} - a^{r+1}}{r+1} - \dfrac{(a+b)\left(b^r - a^r\right)}{4}\right) \|f\|_\infty^2, \end{cases} \tag{3.1}$$

provided $f \in L_\infty[a,b]$.

PROOF. We choose the mappings $m(t) = f(t)$, $g(t) = (t - \mu)$, and $h(t) = t^{r-1}$ in Korkine's identity (2.1). The left-hand side of (2.1) provides

$$\begin{aligned} &\int_a^b f(t)\,dt \int_a^b t^{r-1}(t-\mu)f(t)\,dt - \int_a^b (t-\mu)f(t)\,dt \int_a^b t^{r-1}f(t)\,dt \\ &= \int_a^b t^{r-1}(t-\mu)f(t)\,dt \qquad \left(\text{since} \int_a^b f(t)\,dt = 1 \text{ and } \int_a^b (t-\mu)f(t)\,dt = 0\right) \\ &= \int_a^b t^r f(t)\,dt - \mu \int_a^b t^{r-1} f(t)\,dt = M_r - \mu M_{r-1}, \end{aligned} \tag{3.2}$$

and the right-hand side of (2.1)

$$\frac{1}{2}\int_a^b \int_a^b (t-s)\left(t^{r-1} - s^{r-1}\right) f(t)f(s)\,dt\,ds. \tag{3.3}$$

Observe that

$$\begin{aligned} &\int_a^b \int_a^b (t-s)\left(t^{r-1} - s^{r-1}\right) f(t)f(s)\,dt\,ds \\ &\leq \sup_{(t,s)\in[a,b]^2} \left|(t-s)\left(t^{r-1} - s^{r-1}\right)\right| \int_a^b \int_a^b f(t)f(s)\,dt\,ds \\ &= (b-a)\left(b^{r-1} - a^{r-1}\right) \qquad \left(\text{since} \int_a^b \int_a^b f(t)f(s)\,dt\,ds = 1\right), \end{aligned}$$

hence, the first part of the moment inequality (3.1).

The second part of (3.1) follows as

$$\begin{aligned} &\int_a^b \int_a^b (t-s)\left(t^{r-1} - s^{r-1}\right) f(t)f(s)\,dt\,ds \\ &\leq \sup_{(t,s)\in[a,b]^2} |f(t)f(s)| \int_a^b \int_a^b (t-s)\left(t^{r-1} - s^{r-1}\right)\,dt\,ds \\ &= \sup_{(t,s)\in[a,b]^2} |f(t)f(s)| \left[2(b-a)\left(\frac{b^{r+1} - a^{r+1}}{r+1} - \frac{(a+b)\left(b^r - a^r\right)}{4}\right)\right] \\ &= \|f\|_\infty^2 \left[2(b-a)\left(\frac{b^{r+1} - a^{r+1}}{r+1} - \frac{(a+b)\left(b^r - a^r\right)}{4}\right)\right]. \end{aligned}$$

Using the Grüss type inequality (2.6), we prove the following theorem.

THEOREM 3.2. *For a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$, and $r \geq 0$,*

$$M_{2r} - M_r^2 \leq \frac{1}{4}\left(b^r - a^r\right)^2. \tag{3.4}$$

PROOF. We choose in the Grüss type inequality (2.6): $g(t) = f(t)$ and $h(t) = t^r$, $t \in [a,b]$. Thus, $m = a^r$ and $M = b^r$, and

$$0 \leq \frac{\int_a^b t^{2r} f(t)\,dt}{\int_a^b f(t)\,dt} - \left(\frac{\int_a^b t^r f(t)\,dt}{\int_a^b f(t)\,dt}\right)^2 \leq \frac{(b^r - a^r)^2}{4}, \qquad \text{or}$$

$$M_{2r} - M_r^2 \leq \frac{1}{4}\left(b^r - a^r\right)^2 \qquad \left(\text{since } \int_a^b f(t)\,dt = 1\right).$$

The following results hold also.

THEOREM 3.3. *For a random variable X with probability density function $f : [a,b] \subset R \to R^+$, for any $x \in [a,b]$ and $r \geq 0$,*

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i$$
$$\leq \begin{cases} \|f\|_\infty \left[\dfrac{(x-a)^{r+1} - (x-b)^{r+1}}{r+1}\right], & \text{provided } f \in L_\infty[a,b], \\ \|f\|_p \left[\dfrac{(x-a)^{rq+1} - (x-b)^{rq+1}}{rq+1}\right]^{1/q}, & \text{provided } f \in L_p[a,b],\ p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1, \\ \left[\dfrac{b-a}{2} + \left|x - \dfrac{a+b}{2}\right|\right]^r. \end{cases} \tag{3.5}$$

PROOF. Applying the binomial expansion

$$(x-t)^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^i t^i x^{r-i},$$

we have

$$\int_a^b (x-t)^r f(t)\,dt = \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i. \tag{3.6}$$

Further observe that

$$\begin{aligned} \int_a^b (x-t)^r f(t)\,dt &\leq \operatorname*{ess\,sup}_{t \in [a,b]} |f(t)| \int_a^b (x-t)^r\,dt \\ &= \|f\|_\infty \left[\frac{(x-a)^{r+1} - (x-b)^{r+1}}{r+1}\right], \qquad \text{provided } f \in L_\infty[a,b], \end{aligned}$$

and thus, the first inequality in (3.5).

For proving the second inequality in (3.5), we have from the Hölder's integral inequality (2.2),

$$\begin{aligned} \int_a^b (x-t)^r f(t)\,dt &\leq \left(\int_a^b f^p(t)\,dt\right)^{1/p} \left(\int_a^b (x-t)^{rq}\,dt\right)^{1/q} \\ &= \|f\|_p \left[\frac{(x-a)^{rq+1} - (x-b)^{rq+1}}{rq+1}\right]^{1/q}, \end{aligned} \tag{3.7}$$

provided $f \in L_p[a,b]$, $p > 1$, and $1/p + 1/q = 1$.

Now observing that

$$\begin{aligned}\int_a^b (x-t)^r f(t)\,dt &\le \sup_{t\in[a,b]} |(x-t)^r| \int_a^b f(t)\,dt\\ &= [\max(x-a, b-x)]^r\\ &= \left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^r,\end{aligned}$$

we prove the third inequality in (3.5).

REMARK 3.4. Choosing $r = 2$ in (3.5) results in the inequality established in Theorem 2.4 by Barnett *et al.* [2].

COROLLARY 3.5. *The best inequality from (3.5) may be seen for $x = (a+b)/2$. For $r \ge 0$,*

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i \le \begin{cases} \|f\|_\infty \left[\dfrac{(b-a)^{r+1} - (a-b)^{r+1}}{2^{r+1}(r+1)}\right], & \text{provided } f \in L_\infty[a,b],\\ \|f\|_p \left[\dfrac{(b-a)^{rq+1} - (a-b)^{rq+1}}{2^{rq+1}(rq+1)}\right]^{1/q}, & \text{provided } f \in L_p[a,b],\ p>1,\ \dfrac{1}{p}+\dfrac{1}{q}=1,\\ \left[\dfrac{b-a}{2}\right]^r. \end{cases} \tag{3.8}$$

An interesting case follows from (3.5) when $p = q = 2$.

COROLLARY 3.6. *For a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$, $p = q = 2$, and $r \ge 0$,*

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i \le \|f\|_2 \left[\frac{(x-a)^{2r+1} - (x-b)^{2r+1}}{2r+1}\right]^{1/2}, \qquad \text{provided } f \in L_2[a,b]. \tag{3.9}$$

From (3.8), we can evaluate an upper bound for the variance of X as follows.

COROLLARY 3.7. *For a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$, $p = q = r = 2$,*

$$\sigma^2 \le \mu[(a+b) - \mu] + \|f\|_2 \frac{(b-a)^{5/2}}{4\sqrt{5}} - \left(\frac{a+b}{2}\right)^2, \qquad \text{provided } f \in L_2[a,b]. \tag{3.10}$$

4. PERTURBED RESULTS FOR MOMENTS

We apply the Grüss type inequalities (2.3) to (2.5) to prove results involving the moments.

THEOREM 4.1. *For a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$, $m \le f \le M$, and $r \ge 0$,*

$$\left|M_r - \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}\right| \le \frac{(b-a)(M-m)}{2} \sqrt{|T(h,h)|}, \tag{4.1}$$

where

$$T(h,h) = \frac{b^{2r+1} - a^{2r+1}}{(b-a)(2r+1)} - \left(\frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}\right)^2. \tag{4.2}$$

PROOF. Let $g(t) = f(t)$ and $h(t) = t^r$ in the Grüss integral inequality (2.4). Then,

$$T(g,h) = \frac{1}{b-a}\int_a^b t^r f(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b t^r\,dt$$
$$= \frac{1}{b-a}M_r - \frac{b^{r+1}-a^{r+1}}{(b-a)^2(r+1)},$$

which is the left-hand side of (4.1), and

$$T(h,h) = \frac{1}{b-a}\int_a^b t^r t^r\,dt - \frac{1}{b-a}\int_a^b t^r\,dt \cdot \frac{1}{b-a}\int_a^b t^r\,dt$$
$$= \frac{b^{2r+1}-a^{2r+1}}{(b-a)(2r+1)} - \left(\frac{b^{r+1}-a^{r+1}}{(b-a)(r+1)}\right)^2.$$

Applying inequality (2.5), we prove the theorem.

COROLLARY 4.2. *A reverse inequality from (4.1) provides the moment estimation for a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$, $m \le f \le M$, and $r \ge 0$,*

$$M_r \le \frac{b^{r+1}-a^{r+1}}{(b-a)(r+1)} + \frac{(b-a)(M-m)}{2}\sqrt{|T(h,h)|}. \tag{4.3}$$

In what follows now, we have a theorem that provides an inequality involving the r^{th} moment ($r \ge 0$) of X about any arbitrary constant $c \in [a,b]$, defined as

$$M_r(c) = \int_a^b (t-c)^r f(t)\,dt.$$

THEOREM 4.3. *For a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x, c \in [a,b]$, $m \le f \le M$, and $r \ge 0$,*

$$\left| M_r(c) - \frac{(b-c)^{r+1}-(a-c)^{r+1}}{(b-a)(r+1)} \right| \le \frac{(b-a)(M-m)}{2}\sqrt{|T(h,h)|}, \tag{4.4}$$

where

$$T(h,h) = \frac{(b-c)^{2r+1}-(a-c)^{2r+1}}{(b-a)(2r+1)} - \left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{(b-a)(2r+1)}\right)^2. \tag{4.5}$$

The proof is similar to that of Theorem 4.1 by letting $g(t) = f(t)$ and $h(t) = (t-c)^r$.

COROLLARY 4.4. *A reverse inequality from (4.4) provides the estimation for $M_r(c)$ of a random variable X with probability density function $f : [a,b] \subset R \to R^+$, $x, c \in [a,b]$, $m \le f \le M$, and $r \ge 0$,*

$$M_r \le \frac{b^{r+1}-a^{r+1}}{(b-a)(r+1)} + \frac{(b-a)(M-m)}{2}\sqrt{|T(h,h)|}, \tag{4.6}$$

where $T(h,h)$ is given by (4.5).

REMARK 4.5. The best inequality from (4.5) is attained at $c = (a+b)/2$ as

$$\left| M_r\left(\frac{a+b}{2}\right) - \frac{(b-a)^{r+1}-(a-b)^{r+1}}{2^{r+1}(b-a)(r+1)} \right| \le \frac{(b-a)(M-m)}{2}\sqrt{|T(h,h)|}, \tag{4.7}$$

where

$$T(h,h) = \frac{(b-a)^{2r+1}-(a-b)^{2r+1}}{2^{2r+1}(b-a)(2r+1)} - \left(\frac{(b-a)^{r+1}-(a-b)^{r+1}}{2^{r+1}(b-a)(r+1)}\right)^2. \tag{4.8}$$

Below we obtain some results for the probability density functions $f(x)$ that are differentiable, i.e., for absolutely continuous probability functions $f(x)$.

THEOREM 4.6. *Let a random variable X have probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$. Suppose that f is differentiable and is such that $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then, for $r \geq 0$,*

$$\left| M_r - \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} \right| \leq \frac{(b-a)}{\sqrt{12}} \|f'\|_\infty \left(\frac{(b-a)\left(b^{2r+1} - a^{2r+1}\right)}{(2r+1)} - \left(\frac{b^{r+1} - a^{r+1}}{(r+1)} \right)^2 \right)^{1/2}. \quad (4.9)$$

PROOF. Let $g, h : [a,b] \to \mathbb{R}$ be absolutely continuous and h', g' be bounded. Then, from Chebyshev's inequality [10],

$$T(g,h) \leq \frac{(b-a)^2}{12} \sup_{t\in[a,b]} |g'(t) \cdot h'(t)|.$$

Matic, Pečaric and Ujevic [8] have shown that

$$|T(g,h)| \leq \frac{(b-a)}{\sqrt{12}} \sup_{t\in[a,b]} |g'(t)| \cdot \sqrt{T(h,h)}. \quad (4.10)$$

Let $g(t) = f(t)$ and $h(t) = t^r$. Then,

$$\sup_{t\in[a,b]} |g'(t)| = \|f'\|_\infty,$$

and from (4.1), (4.2), and (4.10), we get

$$\left| \frac{1}{b-a} M_r - \frac{b^{r+1} - a^{r+1}}{(b-a)^2(r+1)} \right| \leq \frac{(b-a)}{\sqrt{12}} \left(\frac{b^{2r+1} - a^{2r+1}}{(b-a)(2r+1)} - \left(\frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} \right)^2 \right)^{1/2}.$$

COROLLARY 4.7. *Let a random variable X have probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$. Suppose that f is differentiable. Then, from (4.9), the reverse inequality for $r \geq 0$ provides*

$$M_r \leq \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} + \frac{(b-a)}{\sqrt{12}} \|f'\|_\infty \left(\frac{(b-a)\left(b^{2r+1} - a^{2r+1}\right)}{(2r+1)} - \left(\frac{b^{r+1} - a^{r+1}}{(r+1)} \right)^2 \right)^{1/2}. \quad (4.11)$$

We apply the results from Lupas [11] and Matic, Pečaric and Ujevic [8] to prove the following theorem.

THEOREM 4.8. *Let a random variable X have probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$. Suppose that f is locally absolutely continuous on (a,b) and $f' \in L_2(a,b)$. Then, for $r \geq 0$,*

$$\left| M_r - \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} \right| \leq \frac{(b-a)}{\pi} \|f'\|_2 \sqrt{\frac{(b-a)\left(b^{2r+1} - a^{2r+1}\right)}{(2r+1)} - \left(\frac{b^{r+1} - a^{r+1}}{(r+1)} \right)^2}. \quad (4.12)$$

PROOF. For $g, h : (a,b) \to \mathbb{R}$ locally absolutely continuous on (a,b), and $g', h' \in L_2(a,b)$, Lupas [11] established

$$|T(g,h)| \leq \frac{(b-a)^2}{\pi^2} \|g''\|_2^\dagger \|h''\|_2^\dagger,$$

where

$$\|g''\|_2^\dagger := \left(\frac{1}{b-a} \int_a^b |k(t)|^2 \, dt \right)^{1/2}, \qquad \text{for } k \in L_2(a,b).$$

Further, Matic, Pečaric and Ujevic [8] have shown that

$$|T(g,h)| \leq \frac{(b-a)}{\pi} \|g''\|_2^\dagger \cdot \sqrt{T(h,h)}. \quad (4.13)$$

Letting $g(t) = f(t)$ and $h(t) = t^r$ in (4.13), we prove the theorem.

COROLLARY 4.9. *Let a random variable X have probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$. Suppose that f is locally absolutely continuous on (a,b) and $f' \in L_2(a,b)$. Then, from (4.12), the reverse inequality for $r \geq 0$ provides*

$$M_r \leq \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} + \frac{(b-a)}{\pi}\|f'\|_2\sqrt{\frac{(b-a)\left(b^{2r+1} - a^{2r+1}\right)}{(2r+1)} - \left(\frac{b^{r+1} - a^{r+1}}{(r+1)}\right)^2}. \tag{4.14}$$

In what follows now, we apply the results from the Grüss type inequalities to develop estimations for the central moments of X. Let

$$\mathcal{S}(h(x)) = h(x) - \mathcal{M}(h), \tag{4.15}$$

where

$$\mathcal{M}(h) = \frac{1}{b-a}\int_a^b h(u)\,du. \tag{4.16}$$

From (2.5),

$$T(g,h) = \mathcal{M}(gh) - \mathcal{M}(g)\mathcal{M}(h).$$

Dragomir and McAndrew [9] established the identity

$$T(g,h) = T(\mathcal{S}(g), \mathcal{S}(h)). \tag{4.17}$$

We now apply (4.15) through (4.17) to obtain the following results.

THEOREM 4.10. *Let a random variable X have probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$. Then, for $r \geq 0$,*

$$\left|M_r - \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}\right| = \left|\int_a^b \mathcal{S}\left(t^r\right)\left(f(t) - \frac{1}{b-a}\right)dt\right|. \tag{4.18}$$

PROOF. Let $g(t) = f(t)$ and $h(t) = t^r$. Using identity (4.15), we have

$$\int_a^b t^r f(t)\,dt - \mathcal{M}(t^r) = \int_a^b \left[t^r - \mathcal{M}\left(t^r\right)\right]\left(f(t) - \frac{1}{b-a}\right)dt, \tag{4.19}$$

where

$$\mathcal{M}\left(t^r\right) = \frac{1}{b-a}\int_a^b t^r\,dt = \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} \tag{4.20}$$

and

$$\mathcal{S}(t^r) = t^r - \mathcal{M}\left(t^r\right). \tag{4.21}$$

From (1.1) and (4.19)–(4.21),

$$M_r - \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)} = \int_a^b \mathcal{S}\left(t^r\right)\left(f(t) - \frac{1}{b-a}\right)dt,$$

and taking the modulus, we prove the theorem.

COROLLARY 4.11. *Let a random variable X have probability density function $f : [a,b] \subset R \to R^+$, $x \in [a,b]$, and $f \in L_\infty[a,b]$. Then, for $r \geq 0$,*

$$\left|M_r - \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}\right| \leq \left\|\left(f(.) - \frac{1}{b-a}\right)\right\|_\infty \int_a^b \left|S\left(t^r\right)\right|dt. \tag{4.22}$$

REMARK 4.12. We can obtain other estimations for the moments from (4.18) for $f \in L_p[a,b]$, $1/p + 1/q = 1$, $p > 1$. However, they will involve calculation of

$$\left(\int_a^b \left|S\left(t^r\right)\right|^q dt\right)^{1/q}, \qquad \text{where } S\left(t^r\right) = t^r - \frac{b^{r+1} - a^{r+1}}{(b-a)(r+1)}.$$

5. ESTIMATIONS WHEN PROBABILITY DENSITY FUNCTION IS ABSOLUTELY CONTINUOUS

We start with the following lemma.

LEMMA 5.1. *Let a random variable X be such that its probability density function $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$. Then, for $r \ge 0$,*

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i = \frac{(x-a)^{r+1} - (x-b)^{r+1}}{(b-a)(r+1)} + \frac{1}{b-a} \int_a^b \int_a^b (x-t)^r p(t,s) f'(s)\, ds\, dt, \quad (5.1)$$

where $p : [a,b]^2 \to \mathbb{R}$ is

$$p(t,s) := \begin{cases} s-a, & \text{if } a \le s \le t \le b, \\ s-b, & \text{if } a \le t < s \le b, \end{cases}$$

for all $x \in [a,b]$.

PROOF. From (3.6), we have the identity

$$\int_a^b (x-t)^r f(t)\, dt = \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i, \quad (5.2)$$

for all $x \in [a,b]$.

Further, integrating by parts,

$$f(t) = \frac{1}{b-a} \int_a^b f(s)\, ds + \frac{1}{b-a} \int_a^b p(t,s) f'(s)\, ds, \quad (5.3)$$

for all $t \in [a,b]$.

On substituting (5.3) in (5.2), we prove the lemma.

The following theorem holds for the probability density functions which are absolutely continuous and have essentially bounded derivatives.

THEOREM 5.2. *Let a random variable X be such that its probability density function $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$ and $f' \in L_\infty[a,b]$, i.e., $\|f'\|_\infty := \operatorname{ess\,sup}_{t\in[a,b]} |f'(t)| < \infty$. Then, for $r \ge 0$,*

$$\begin{aligned} &\left| \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i - \frac{(x-a)^{r+1} - (x-b)^{r+1}}{(b-a)(r+1)} \right| \\ &\quad \le \frac{\|f'\|_\infty}{2(b-a)} \int_a^b |(x-t)^r| \left[(t-a)^2 + (b-t)^2 \right] dt, \end{aligned} \quad (5.4)$$

for all $x \in [a,b]$.

PROOF. Applying identity (5.1) from the lemma, we have

$$\begin{aligned} &\left| \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i - \frac{(x-a)^{r+1} - (x-b)^{r+1}}{(b-a)(r+1)} \right| \\ &\quad = \frac{1}{b-a} \left| \int_a^b \int_a^b (x-t)^r p(t,s) f'(s)\, ds\, dt \right| \\ &\quad \le \frac{1}{b-a} \int_a^b \int_a^b |(x-t)^r p(t,s)|\, |f'(s)|\, ds\, dt \\ &\quad \le \frac{\|f'\|_\infty}{b-a} \int_a^b \int_a^b |(x-t)^r p(t,s)|\, ds\, dt. \end{aligned}$$

Further,

$$\int_a^b \int_a^b |(x-t)^r p(t,s)|\, ds\, dt$$
$$\leq \int_a^b |(x-t)^r| \left[\int_a^b (s-a)\, ds + \int_a^b (b-s)\, ds \right] dt$$
$$= \int_a^b |(x-t)^r| \left[\frac{(t-a)^2 + (b-t)^2}{2} \right] dt,$$

and hence, the theorem.

COROLLARY 5.3. *Let the probability density function $f : [a,b] \to \mathbb{R}_+$ be absolutely continuous on $[a,b]$ and $f' \in L_\infty[a,b]$. Then, for all $x \in [a,b]$, $1/p + 1/q = 1$, and even integers $r \geq 2$,*

$$\left| \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i - \frac{(x-a)^{r+1} - (x-b)^{r+1}}{(b-a)(r+1)} \right|$$
$$\leq \frac{(-1)^r \|f'\|_\infty}{2(b-a)} \left\{ \begin{array}{l} (x-a)^{r+3} \tilde{B}\left(\dfrac{b-a}{x-a}, r+1, 3\right) \\ +(b-x)^{r+3} \tilde{B}\left(\dfrac{b-a}{b-x}, r+1, 3\right) \end{array} \right\}, \tag{5.5}$$

where $\tilde{B}(.,.,.)$ is the quasi-incomplete Euler's Beta mapping

$$\tilde{B}(.,.,.) = \int_0^z (u-1)^{\alpha-1} u^{\beta-1}\, du, \qquad \alpha, \beta > 0, \quad z \geq 1.$$

PROOF. From (5.4) for even integers $r \geq 2$,

$$\int_a^b |(x-t)^r| \left[(t-a)^2 + (b-t)^2 \right] dt$$
$$= \int_a^b (x-t)^r \left[(t-a)^2 + (b-t)^2 \right] dt \tag{5.6}$$
$$= (-1)^r \left(\int_a^b (t-x)^r (t-a)^2\, dt + \int_a^b (t-x)^r (b-t)^2\, dt \right).$$

We evaluate the integrals

$$I_1 := \int_a^b (t-x)^r (t-a)^2\, dt$$
$$= (x-a)^{r+3} \int_0^{(b-a)/(x-a)} (u-1)^r u^2\, du \tag{5.7}$$
$$= (x-a)^{r+3} \tilde{B}\left(\frac{b-a}{x-a}, r+1, 3\right),$$

by changing variable to $t = (1-u)a + ux$, and

$$I_2 := \int_a^b (t-x)^r (b-t)^2\, dt$$
$$= (b-x)^{r+3} \int_0^{(b-a)/(b-x)} (v-1)^r v^2\, dv \tag{5.8}$$
$$= (x-a)^{r+3} \tilde{B}\left(\frac{b-a}{b-x}, r+1, 3\right),$$

by changing variable to $t = (1-v)a + vx$.

Substituting from (5.7) and (5.8) in (5.6), we get (5.5), and hence, the corollary.

COROLLARY 5.4. *Let the probability density function $f : [a,b] \to \mathbb{R}_+$ be absolutely continuous on $[a,b]$ and $f' \in L_\infty[a,b]$. Then, for $x = (a+b)/2$, $1/p + 1/q = 1$, and even integers $r \geq 2$,*

$$\begin{aligned} &\left| \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i - \frac{(b-a)^r \left(1 - (-1)^{r+1}\right)}{(r+1)} \right| \\ &\leq \frac{(-1)^r (b-a)^{r+2} \|f'\|_\infty}{2^{r+3}} [B(r+1,3) + \Psi(r+1,3)], \end{aligned} \tag{5.9}$$

where $B(.,.)$ is the Euler's Beta mapping and

$$\Psi(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1+u)^{\beta-1}\, du, \qquad \alpha,\beta > 0, \quad z \geq 1.$$

PROOF. In (5.5), set $x = (a+b)/2$. The left side of (5.9) is then obvious. For the right side,

$$\begin{aligned} \tilde{B}\left(\frac{b-a}{x-a}, r+1, 3\right) &= \tilde{B}(2, r+1, 3) \\ &= \int_0^2 u^2 (u-1)^r\, du \\ &= \int_0^1 u^2(u-1)^r\, du + \int_1^2 u^2(u-1)^r\, du \\ &= B(r+1,3) + \Psi(r+1,3). \end{aligned}$$

The right side of (5.9) becomes

$$\begin{aligned} &\frac{(-1)^r \|f'\|_\infty}{2(b-a)} \left\{ \begin{array}{l} (x-a)^{r+3} \tilde{B}\left(\dfrac{b-a}{x-a}, r+1, 3\right) \\ +(b-x)^{r+3} \tilde{B}\left(\dfrac{b-a}{b-x}, r+1, 3\right) \end{array} \right\} \\ &\qquad = \frac{(-1)^r (b-a)^{r+2} \|f'\|_\infty}{2^{r+3}} \left[B(r+1,3) + \Psi(r+1,3)\right], \end{aligned}$$

and hence, the corollary.

We now obtain results where f' is a Lebesgue $p-$ integrable mapping, $p \in (1,\infty)$.

THEOREM 5.5. *Let the probability density function $f : [a,b] \to \mathbb{R}_+$ be absolutely continuous on $[a,b]$ and $f' \in L_p[a,b]$, i.e.,*

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p\, dt \right)^{1/p} < \infty, \qquad p \in (1,\infty).$$

Then, for $r \geq 0$,

$$\begin{aligned} &\left| \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i - \frac{(x-a)^{r+1} - (x-b)^{r+1}}{(b-a)(r+1)} \right| \\ &\leq \frac{\|f'\|_p}{(b-a)^{1/q}} \left(\int_a^b |(x-t)^r|^q \left[\frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1} \right] dt \right)^{1/q}, \end{aligned} \tag{5.10}$$

for all $x \in [a,b]$, $p > 1$, and $1/p + 1/q = 1$.

PROOF. Applying Lemma 5.1, we have

$$\begin{aligned} &\left| \sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i - \frac{(x-a)^{r+1} - (x-b)^{r+1}}{(b-a)(r+1)} \right| \\ &\quad = \frac{1}{b-a} \left| \int_a^b \int_a^b (x-t)^r p(t,s) f'(s)\, ds\, dt \right| \\ &\quad \leq \frac{1}{b-a} \int_a^b \int_a^b |(x-t)^r p(t,s)|\, |f'(s)|\, ds\, dt. \end{aligned} \tag{5.11}$$

Substituting from (5.15) and (5.16) in (5.14), we prove the corollary.

COROLLARY 5.7. *Let the probability density function $f : [a,b] \to \mathbb{R}_+$ be absolutely continuous on $[a,b]$ and $f' \in L_p[a,b]$. Then, for $x = (a+b)/2$, $1/p + 1/q = 1$, $p > 1$, and even integers $r \geq 2$,*

$$\begin{aligned}&\left|\sum_{i=0}^{r}\binom{r}{i}(-1)^i x^{r-i} M_i - \frac{(b-a)^r\left(1-(-1)^{r+1}\right)}{2^{r+1}(r+1)}\right| \\ &\quad \leq \frac{(-1)^r(b-a)^{r+1+1/q}\|f'\|_p}{2^{r+1+1/q}(q+1)}[B(rq+1,q+2)+\Psi(rq+1,q+2)]^{1/q}.\end{aligned} \tag{5.17}$$

PROOF. In (5.13), let $x = (a+b)/2$. The left side of (5.17) is obvious. For the right side, we consider

$$\begin{aligned}\tilde{B}\left(\frac{b-a}{x-a}, rq+1, q+2\right) &= \tilde{B}(2, rq+1, q+2) \\ &= \int_0^2 u^{q+1}(u-1)^{qr}\,du \\ &= \int_0^1 u^{q+1}(u-1)^{qr}\,du + \int_1^2 u^{q+1}(u-1)^{qr}\,du \\ &= B(rq+1, q+2) + \Psi(rq+1, q+2).\end{aligned}$$

The right side of (5.17) is

$$\begin{aligned}&\frac{\|f'\|_p}{(b-a)^{1/q}}\left(\frac{(-1)^{qr}}{q+1}\left\{\begin{array}{l}(x-a)^{qr+q+2}\tilde{B}\left(\dfrac{b-a}{x-a}, qr+1, q+2\right) \\ +(b-x)^{qr+q+2}\tilde{B}\left(\dfrac{b-a}{b-x}, qr+1, q+2\right)\end{array}\right\}\right)^{1/q} \\ &\quad = \frac{(-1)^r(b-a)^{r+1+1/q}\|f'\|_p}{2^{r+1+1/q}(q+1)}[B(rq+1,q+2)+\Psi(rq+1,q+2)],\end{aligned}$$

and hence, the corollary.

An interesting case of $p = q = r = 2$ from (5.17) results in the following upper bound for the variance of X.

COROLLARY 5.8. *Let the probability density function $f : [a,b] \to \mathbb{R}_+$ be absolutely continuous on $[a,b]$ and $f' \in L_2[a,b]$. Then, for $x = (a+b)/2$, $p = q = r = 2$,*

$$\sigma^2 \leq \mu[(a+b)-\mu] + 0.0833(b-a)^2 + 0.0330(b-a)^{7/2}\|f'\|_2. \tag{5.18}$$

Further, if f is absolutely continuous, $f' \in L_1[a,b]$, and $\|f'\|_1 = \int_a^b |f'(t)|\,dt$, then we have the following theorem.

THEOREM 5.9. *Letting the probability density function $f : [a,b] \to \mathbb{R}_+$ be absolutely continuous on $[a,b]$, then for $r \geq 0$,*

$$\begin{aligned}&\left|\sum_{i=0}^{r}\binom{r}{i}(-1)^i x^{r-i} M_i - \frac{(x-a)^{r+1}-(x-b)^{r+1}}{(b-a)(r+1)}\right| \\ &\quad \leq \|f'\|_1(b-a)\left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^r,\end{aligned} \tag{5.19}$$

for all $x \in [a,b]$.

PROOF. For $r \geq 0$,

$$\begin{aligned}\int_a^b \int_a^b &|(x-t)^r p(t,s)|\, |f'(s)|\, ds\, dt \\ &\leq \sup_{t,s\in[a,b]^2} [|(x-t)^r|\, |p(t,s)|] \int_a^b \int_a^b |f'(s)|\, ds\, dt \\ &= \|f'\|_1 I,\end{aligned}$$

where

$$\begin{aligned} I &= \sup_{t,s\in[a,b]^2} [|(x-t)^r|\, |p(t,s)|] \\ &\leq (b-a) \sup_{t\in[a,b]} |(x-t)^r| \\ &= (b-a)[\max(|x-a|, |b-x|)]^r \\ &= (b-a)\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right]^r, \end{aligned}$$

and hence, the theorem.

From (5.19), we get the following corollary when $x = (a+b)/2$.

COROLLARY 5.10. *Let the probability density function $f : [a,b] \to \mathbb{R}_+$ be absolutely continuous on $[a,b]$. Then, for $r \geq 0$,*

$$\begin{aligned}\left|\sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i - \frac{(b-a)^r \left(1-(-1)^{r+1}\right)}{2^{r+1}(r+1)}\right| \\ \leq \|f'\|_1 \frac{(b-a)^{r+1}}{2^r}.\end{aligned} \tag{5.20}$$

6. APPLICATIONS TO SPECIAL MEANS

We consider the following convex mappings that result in the special means.

6.1. Mapping: $f(x) = x^p$, $p > 1$, $x > 0$, $a, b \in R$, $0 < a < b$

We have arithmetic mean $A(a^p, b^p) = (a^p + b^p)/2$, $a, b > 0$ and

$$f\left(\frac{a+b}{2}\right) = A^p(a,b), \qquad \frac{f(a)+f(b)}{2} = A\left(a^p, b^p\right), \qquad \frac{1}{b-a}\int_a^b f(x)\, dx = L_p^p(a,b). \tag{6.1}$$

PROPOSITION 6.1.1. *Let $p > 1$, $q = p/(p-1)$, and $0 < a < b$. Then, for $r \geq 0$,*

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i \leq \left(\frac{(x-a)^{pr+1} - (x-b)^{pr+1}}{(pr+1)}\right)^{1/p} (b-a)^{1/q} L_{pq}^p(a,b). \tag{6.2}$$

PROOF. For the convex mapping $f(x) = x^p$, we apply Hölder's integral inequality to get

$$\begin{aligned}\int_a^b (x-t)^r f(t)\, dt &\leq \left(\int_a^b f^q(t)\, dt\right)^{1/q} \left(\int_a^b (x-t)^{pr}\, dt\right)^{1/p} \\ &= \left(\int_a^b x^{pq}\, dx\right)^{1/q} \left(\frac{(x-a)^{pr+1} - (x-b)^{pr+1}}{(pr+1)}\right)^{1/p}.\end{aligned}$$

Using (3.6) and since

$$\int_a^b x^{pq}\, dx = (b-a) L_{pq}^{pq}(a,b),$$

we prove the required inequality (6.2).

An upper bound for the variance of X follows from (6.2).

COROLLARY 6.1.2. *For $r = 2$ and $x = (a+b)/2$ in (6.2),*

$$\sigma^2 \le \mu[(a+b) - \mu] + \left(\frac{(b-a)^{2p+1} - (a-b)^{2p+1}}{2^{2p+1}(2p+1)} - \left(\frac{a+b}{2}\right)^2\right)(b-a)^{1/q} L_{pq}^{p}(a,b). \tag{6.3}$$

6.2. Mapping: $f(x) = 1/x$, $x > 0$, $a, b \in R$, $0 < a < b$

We have

$$\text{logarithmic mean } L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & \text{if } a \neq b,\ a,b > 0, \\ a, & \text{if } a = b,\ a,b > 0, \end{cases}$$

$$\text{harmonic mean } H(a,b) = \frac{2}{1/a + 1/b}, \qquad a,b > 0,$$

and

$$\begin{gathered} -\frac{1}{a^2} \le f'(x) = -\frac{1}{x^2} \le -\frac{1}{b^2}, \qquad \text{for all } x \in [a,b], \\ f\left(\frac{a+b}{2}\right) = A^{-1}(a,b), \quad \frac{f(a)+f(b)}{2} = H^{-1}(a,b), \quad \frac{1}{b-a}\int_a^b f(x)\,dx = L^{-1}(a,b). \end{gathered} \tag{6.4}$$

PROPOSITION 6.2.1. *Let $p > 1$, $q = p/(p-1)$, and $0 < a < b$. Then, for $r \ge 0$,*

$$\sum_{i=0}^{r} \binom{r}{i} (-1)^i x^{r-i} M_i \le \left(\frac{(x-a)^{pr+1} - (x-b)^{pr+1}}{(pr+1)}\right)^{1/p} (b-a)^{1/q} L_{-pq}^{-p}(a,b). \tag{6.5}$$

We prove (6.5) by choosing the mapping $f(x) = 1/x$, and following the proof of Proposition 6.1. An estimation for the variance of X is obtained in the following corollary.

COROLLARY 6.2.2. *For $r = 2$ and $x = (a+b)/2$ in (6.5),*

$$\sigma^2 \le \mu[(a+b) - \mu] + \left(\frac{(b-a)^{2p+1} - (a-b)^{2p+1}}{2^{2p+1}(2p+1)} - \left(\frac{a+b}{2}\right)^2\right)(b-a)^{1/q} L_{-pq}^{-p}(a,b). \tag{6.6}$$

7. APPLICATIONS TO BETA DISTRIBUTIONS

A random variable X with parameters α and β have beta probability density function

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \qquad \alpha,\beta > 0, \quad 0 < x < 1, \tag{7.1}$$

where $B(.,.)$ denotes the beta function that is defined by $B(\alpha,\beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\,dz$.

Denoting by M the value of X for which $f(x)$ is maximum, we have

$$M = \frac{\alpha-1}{\alpha+\beta-2}, \qquad \alpha,\beta > 1.$$

From (4.2),

$$T(h,h) = \frac{r^2}{(2r+1)(r+1)^2}, \tag{7.2}$$

and from (4.3), for $r \ge 0$, $\alpha,\beta > 1$,

$$M_r \le \frac{1}{r+1}\left(1 + \frac{r(\alpha-1)}{2(\alpha+\beta-2)}\sqrt{\frac{1}{2r+1}}\right). \tag{7.3}$$

Further from (4.2) and (4.11), we have for $r \geq 0$, $\alpha, \beta > 1$,

$$M_r \leq \frac{1}{r+1} + \frac{1}{\pi}\sqrt{\frac{r^2}{(2r+1)(r+1)^2}}\|f'\|_2, \tag{7.4}$$

where

$$\|f'\|_2 = \big((\alpha-1)^2 B(2\alpha-3, 2\beta-1) + (\beta-1)^2 B(2\alpha-1, 2\beta-3) \\ -2(\alpha-1)(\beta-1)B(2\alpha-2, 2\beta-2)\big)^{1/2}.$$

For the beta random variables with parameters $\alpha = \beta$, $r \geq 0$, from (7.3),

$$M_r \leq \frac{1}{r+1}\left(1 + \frac{r}{4}\sqrt{\frac{1}{2r+1}}\right), \tag{7.5}$$

and from (7.4),

$$M_r \leq \frac{1}{r+1} + \frac{1}{\pi}\sqrt{\frac{r^2}{(2r+1)(r+1)^2}}\|f'\|_2, \tag{7.6}$$

where

$$\|f'\|_2 = (\alpha-1)\left(\frac{2\Gamma(2\alpha-2)\Gamma(2\alpha-3)}{\Gamma(4\alpha-4)}\right)^{1/2},$$

and $\Gamma(n) = (n-1)!$.

When $r = 1, 2$ and $\alpha, \beta > 0$, the upper bounds for μ and σ^2 from (7.3),

$$\begin{aligned} \mu &\leq \frac{1}{2}\left(1 + \frac{\alpha-1}{\alpha+\beta-2}\sqrt{\frac{1}{12}}\right), \\ \sigma^2 + \mu^2 &\leq \frac{1}{3}\left(1 + \frac{\alpha-1}{\alpha+\beta-2}\sqrt{\frac{1}{5}}\right), \end{aligned} \tag{7.7}$$

and from (7.4),

$$\begin{aligned} \mu &\leq \frac{1}{2}\left(1 + \frac{1}{\pi\sqrt{3}}\|f'\|_2\right), \\ \sigma^2 + \mu^2 &\leq \frac{1}{3}\left(1 + \frac{2}{\pi\sqrt{5}}\|f'\|_2\right), \end{aligned} \tag{7.8}$$

where

$$\|f'\|_2 = \big((\alpha-1)^2 B(2\alpha-3, 2\beta-1) + (\beta-1)^2 B(2\alpha-1, 2\beta-3) \\ -2(\alpha-1)(\beta-1B(2\alpha-2, 2\beta-2)\big)^{1/2}.$$

For the beta random variables with $\alpha = \beta$, we have from (7.7),

$$\begin{aligned} \mu &\leq \frac{1}{2}\left(1 + \frac{1}{4\sqrt{3}}\right), \\ \sigma^2 + \mu^2 &\leq \frac{1}{3}\left(1 + \frac{1}{2\sqrt{5}}\right), \end{aligned} \tag{7.9}$$

and from (7.8),

$$\begin{aligned} \mu &\leq \frac{1}{2}\left[1 + \frac{1}{\pi}\left(\frac{2}{3}\frac{(\alpha-1)\Gamma(2\alpha-2)\Gamma(2\alpha-3)}{\Gamma(4\alpha-4)}\right)^{1/2}\right], \\ \sigma^2 + \mu^2 &\leq \frac{1}{3}\left[1 + \frac{2}{\pi}\left(\frac{2}{5}\frac{(\alpha-1)\Gamma(2\alpha-2)\Gamma(2\alpha-3)}{\Gamma(4\alpha-4)}\right)^{1/2}\right](\|f'\|_2), \end{aligned} \tag{7.10}$$

Table 1. Exact values of μ and σ^2 and upper bounds for $\alpha, \beta = 2, 3, 4$.

			U. Bound	U. Bound		U. Bound	U. Bound
α	β	μ	(7.7)	(7.8)	σ^2	(7.7)	(7.8)
2	2	0.50	0.57	0.55	0.05	0.0805	0.0823
3	3	0.50	0.57	0.51	0.04	0.0805	0.0836
4	4	0.50	0.57	0.50	0.03	0.0805	0.0834
2	3	0.40	0.55	0.53	0.04	0.0826	0.0833
2	4	0.33	0.54	0.53	0.03	0.0832	0.0835
3	4	0.43	0.56	0.66	0.03	0.0819	0.0617
4	3	0.57	0.59	0.66	0.03	0.0787	0.0617

where

$$\|f'\|_2 = (\alpha - 1) \left(\frac{2\Gamma(2\alpha - 2)\Gamma(2\alpha - 3)}{\Gamma(4\alpha - 4)} \right)^{1/2}.$$

The exact values of μ and σ^2 and their upper bounds from (7.7) and (7.8) for some choices of α and β are evaluated in Table 1.

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