Hermite-Hadamard Inequalities and Their Applications in Estimating Moments

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Summary . We consider the Hermite-Hadamard inequalities and related results to establish newinequalities involving moments of a random variable whose probability function is a convex function on the interval of real numbers. More results are derived using integral inequalities due to Grüss and Hölder. Some applications to special means are also considered.

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1. Introduction

Set X to denote a random variable whose probability function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval of real numbers I and let $a, b \in I$, $a < b$. Denote by $M_r(c)$ the r^{th} moment about any arbitrary point c of the random variable X, $r = 1, 2, 3, \ldots,$ defined as

$$M_r(c) = \int_a^b (x-c)^r f(x)\,dx. \qquad (1.1)$$

In what follows now when reference is made to the r^{th} moment of a particular distribution, we assume that the appropriate integral converges for that distribution.

2. Hermite-Hadamard Inequalities

The following inequalities and results are for the ready reference.

The Hermite-Hadamard $(H-H)$ inequalities [3,4] which provide a necessary and sufficient condition for a function f to be convex in $[a,b]$:

$$(b-a) f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,dx \le (b-a)\frac{f(a)+f(b)}{2}, \qquad (2.1)$$

according to as f is *convex* (*concave*).

Fejér’s inequalities which generalize $H-H$ inequalities [3]:

Consider the integral $\int_a^b f(x)g(x)\,dx$, where f is a *convex* function in the interval $[a,b]$ and g is a positive function in the same interval such that

$$g(a+t) = g(b-t), 0 \le t \le \frac{a+b}{2},$$

i.e., $y = g(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{a+b}{2}, 0)$ and is normal to the x-axis. Then,

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \qquad (2.2)$$

The following inequality is valid for a *convex* functions [6]:

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx, \qquad (2.3)$$

or

$$\frac{2}{b-a}\int_a^b f(x)\,dx \le \frac{1}{2}\left[f(a)+f(b)+2f\left(\frac{a+b}{2}\right)\right], \qquad (2.4)$$

which is equivalent to

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx + \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)\,dx$$
$$\le \frac{1}{2}\left[f(a)+f\left(\frac{a+b}{2}\right)\right] + \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+f(b)\right]. \qquad (2.5)$$

When $a = -1, b = 1$ in (2.5), we have the Bullen's inequality [6].

3. Estimation of Moments and Moment-Inequalities

We establish results on moment-estimation and moment-inequalities in the following theorems.

Theorem 1. *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I and let $a,b \in I$, $a < b$. If $f' \in L_1[a,b]$, then for $r = 0,1,2,\ldots,$*

$$(r+1)M_r(c) = (b-c)^{r+1}f(b) - (a-c)^{r+1}f(a) - \int_a^b (x-c)^{r+1}f'(x)\,dx. \qquad (3.1)$$

Proof Consider the integrable function

$$g(x) := (x-c)^{r+1}f(x), \quad x \in [a,b].$$

Integrating by parts

$$\int_a^b (x-c)^{r+1}f'(x)\,dx = (b-c)^{r+1}f(b) - (a-c)^{r+1}f(a) - (r+1)\int_a^b (x-c)^{r}f(x)\,dx, \qquad (3.2)$$

and using (1.1), we prove (3.1).

Corollary 1. *For $c = \frac{a+b}{2}, r = 0$ in* (3.2), *we have*

$$\int_a^b xf'(x)\,dx = (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(x)\,dx, \qquad (3.3)$$

an identity by Dragomir and Pearce [3,*p*.30].

Corollary 2. *For $c = 0, r = 1,2,3,\ldots$ in* (3.1), *the identity for the central moments*:

$$(r+1)M_r(0) = b^{r+1}f(b) - a^{r+1}f(a) - \int_a^b x^{r+1}f'(x)\,dx. \qquad (3.4)$$

The following theorem provides the estimation of $M_r(c)$:

Theorem 2. *Let the mapping $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I and let $a,b \in I$, $a < b$. Further let the new mapping*

$$x \;:\; x \quad c^{\,r\;1} f \; x\,, \qquad 3.5$$

be convex on the interval a,b *. Then for* $r \quad 0,1,2,\ldots$

$$b \quad c^{\,r\;1} f\, b \qquad a \quad c^{\,r\;1} f\, a \qquad \frac{b \quad a}{4} \; b \quad c^{\,r\;1} f\; b \qquad a \quad c^{\,r\;1} f\; a$$

$$r \quad 1\, M^{r}\, c \qquad b \quad c^{\,r\;1} f\, b \qquad a \quad c^{\,r\;1} f\, a \;. \qquad 3.6$$

Proof Applying the H-H and Bullen's inequalities to the mapping , i.e.,

$$\frac{1}{2} \quad c \qquad \frac{a \qquad b}{2} \qquad \frac{1}{b \quad a} \; {}_{a}^{b} \; x\, dx \qquad c\,, \qquad 3.7$$

we have

$$\frac{1}{2}\, 0 \quad \frac{a \quad c^{\,r\;1} f\; a \qquad b \quad c^{\,r\;1} f\; b}{2}$$

$$\frac{1}{b \quad a} \; {}_{a}^{b}\, x \quad c^{\,r\;1} f\; x\, dx \qquad 0. \qquad 3.8$$

From (3.1) and (3.8),

$$\frac{1}{4} \quad a \quad c^{\,r\;1} f\; a \qquad b \quad c^{\,r\;1} f\; b$$

$$\frac{1}{b \quad a} \qquad b \quad c^{\,r\;1} f\, b \qquad a \quad c^{\,r\;1} f\, a \qquad r \quad 1\, M_{r}\, c \qquad 0, \qquad 3.9$$

hence the theorem.

Corollary 3. *Choosing* $r \quad 0$ *and* $c \quad \frac{a\;b}{2}$ *in* (3.8), *we get*

$$\frac{b \quad a}{2} \; \frac{f \;\, b \qquad f \;\, a}{2} \qquad \frac{f\, a \qquad f\, b}{2} \qquad {}_{a}^{b} f\, x\; dx \qquad 0, \qquad 3.10$$

a result established by Dragomir and Pearce [3,*p*.30].

Corollary 4. *The estimation of central moments of random variable X follows from* (3.6) *by taking* $c \quad 0$. *For* $r \quad 1,2,\ldots$

$$b^{r\;1} f\, b \qquad a^{r\;1} f\, a \qquad \frac{b \quad a}{4}\; b^{r\;1} f \;\, b \qquad a^{r\;1} f \;\, a$$

$$r \quad 1\, M_{r}\; 0 \qquad b^{r\;1} f\, b \qquad a^{r\;1} f\, a \;. \qquad 3.11$$

Now we state a lemma without proof [3,p.30] that provides a refinement of the Chebychev's integral inequality and that we will use to establish new moment inequalities.

Lemma 1. *Let* $f,g \;:\; a,b \qquad \mathbb{R}$ *be two integrable mappings which are synchronous, i.e.,* $f\,x \quad f\,y \quad g\,x \quad g\,y \qquad 0$ *for all* $x,y \qquad a,b$ *. Then*

$$C\; f,g \qquad max \;\; |C\; |f|, |g|\;|, |C\; |f|, g\;|, |C\; f, |g|\;| \qquad 0, \qquad 3.12$$

where

$$C\; f,g \;:\; \quad b \quad a \quad {}_{a}^{b} f\, x\; g\; x\; dx \qquad {}_{a}^{b} f\; x\; dx \quad {}_{a}^{b} g\; x\; dx. \qquad 3.13$$

We have the following theorem using the H-H inequality and the above lemma 1:

Theorem 3. *Let* $f : I \quad \mathbb{R} \quad \mathbb{R}$ *be a differentiable convex mapping on I and* $a,b \quad I$. *Further let new mapping*

$$x \;:\; x \quad c^{\,r\;1} f \; x\,, \qquad 3.14$$

be convex in the interval $[a,b]$. *Then for* $r=0,1,2,\ldots$

$$\left|\frac{1}{b-a}\left[(b-c)^{r+1}f(b)-(a-c)^{r+1}f(a)\right]-(r+1)M_r(c)\right|$$
$$\geq \max\{|A|,|B|,|C|\}\geq 0, \qquad (3.15)$$

where

$$A:=\int_a^b |x-c|^{r+1}|f'(x)|dx-\frac{(c-a)^{r+2}+(b-c)^{r+2}}{(r+2)(b-a)}\int_a^b |f'(x)|dx,$$

$$B:=\Big|(b-c)^{r+2}f'(b)-(a-c)^{r+2}f'(a)$$
$$-(r+1)\left[\int_a^c (c-x)^r f'(x)\,dx+\int_c^b (x-c)^r f'(x)\,dx\right]$$
$$-\frac{(c-a)^{r+2}+(b-c)^{r+2}}{(r+2)(b-a)}\left[f(b)-f(a)\right]\Big|,$$

$$C:=\int_a^b (x-c)^{r+1}|f'(x)|dx-\frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+2)(b-a)}\int_a^b |f'(x)|dx.$$

$$A:=(b-a)\int_a^b |x-c|^{r+1}|f'(x)|dx-\int_a^b |x-c|^{r+1}dx\int_a^b |f'(x)|dx,$$

$$B:=(b-a)\int_a^b |x-c|^{r+1}f'(x)\,dx-\int_a^b |x-c|^{r+1}dx\int_a^b f'(x)\,dx,$$

$$C:=(b-a)\int_a^b (x-c)^{r+1}|f'(x)|dx-\int_a^b (x-c)^{r+1}dx\int_a^b |f'(x)|dx$$

Proof As f is convex on I, the mappings f' and $(x-c)^{r+1}, r=0,1,2,\ldots,$ are synchronous on $[a,b]$. Applying the lemma 1, we have:

$$(b-a)\int_a^b (x-c)^{r+1}f'(x)\,dx-\int_a^b (x-c)^{r+1}dx\int_a^b f'(x)\,dx$$
$$\geq \max\{|A_1|,|B_1|,|C_1|\}\geq 0, \qquad (3.16)$$

where

$$A_1:=(b-a)\int_a^b |x-c|^{r+1}|f'(x)|dx-\int_a^b |x-c|^{r+1}dx\int_a^b |f'(x)|dx,$$

$$B_1:=(b-a)\int_a^b |x-c|^{r+1}f'(x)\,dx-\int_a^b |x-c|^{r+1}dx\int_a^b f'(x)\,dx,$$

$$C_1:=(b-a)\int_a^b (x-c)^{r+1}|f'(x)|dx-\int_a^b (x-c)^{r+1}dx\int_a^b |f'(x)|dx.$$

Since

$$\int_a^b (x-c)^{r+1}dx = \frac{(b-c)^{r+2}-(a-c)^{r+2}}{r+2},$$

$$\int_a^b |x-c|^{r+1}dx = \int_a^c (c-x)^{r+1}dx + \int_c^b (x-c)^{r+1}dx = \frac{(c-a)^{r+2}+(b-c)^{r+2}}{r+2},$$

$$\int_a^b f'(x)\,dx = f(b)-f(a),$$

$$\int_a^b |x-c|^{r+1}f'(x)\,dx = -\int_a^c (c-x)^{r+1}f'(x)\,dx + \int_c^b (x-c)^{r+1}f'(x)\,dx$$
$$= (b-c)^{r+2}f(b) - (a-c)^{r+2}f(a)$$
$$-(r+1)\left[\int_a^c (c-x)^r f(x)\,dx + \int_c^b (x-c)^r f(x)\,dx\right]$$

we have

$$A_1 := (b-a)\int_a^b |x-c|^{r+1}|f'(x)|dx - \frac{(c-a)^{r+2}+(b-c)^{r+2}}{r+2}\int_a^b |f'(x)|dx,$$
$$B_1 := (b-a)\Big[(b-c)^{r+2}f(b)-(a-c)^{r+2}f(a)$$
$$-(r+1)\left(\int_a^c (c-x)^r f(x)\,dx + \int_c^b (x-c)^r f(x)\,dx\right)\Big]$$
$$-\frac{(c-a)^{r+2}+(b-c)^{r+2}}{r+2}\,[f(b)-f(a)],$$
$$C_1 := (b-a)\int_a^b (x-c)^{r+1}|f'(x)|dx - \frac{(b-c)^{r+2}-(a-c)^{r+2}}{r+2}\int_a^b |f'(x)|dx.$$

Using inequality (3.16), we have

$$\left|\frac{1}{b-a}\int_a^b (x-c)^{r+1}f'(x)\,dx\right| \le \max\{|A|,|B|,|C|\} \ge 0,$$

where A, B and C are as given in theorem 3. From the identity (3.1), we get (3.15) and hence the theorem.

Corollary 5. *Choosing* $r = 0$ *and* $c = \frac{a+b}{2}$ *in* (3.15), *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le max\{|A_2|,|B_2|,|C_2|\} \ge 0, \qquad (3.17)$$

where

$$A_2 := \frac{1}{b-a}\int_a^b \left|x-\frac{a+b}{2}\right||f'(x)|dx - \frac{1}{4}\int_a^b |f'(x)|dx,$$
$$B_2 := \frac{f(b)-f(a)}{4} - \frac{1}{b-a}\left[-\int_a^{\frac{a+b}{2}} f(x)\,dx + \int_{\frac{a+b}{2}}^b f(x)\,dx\right],$$
$$C_2 := \frac{1}{b-a}\int_a^b \left(x-\frac{a+b}{2}\right)|f'(x)|dx,$$

the inequality established in [3,*p*.31].

In what follows we apply the Hölder's integral inequality to derive another estimation for $M_r(c)$.

Theorem 4. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable convex mapping on I and* $a, b \in I$ *such that* $a < b$ *and*

$p > 1$. *If* $|f'|$ *is q-integrable where* $q = \frac{p}{p-1}$, *then for* $r = 0,1,2\ldots$

$$|(b-c)^{r+1}f(b) - (a-c)^{r+1}f(a) - (r+1)M_r(c)|$$
$$\le \left[\frac{(b-c)^{p(r+1)+1} + (c-a)^{p(r+1)+1}}{p(r+1)+1}\right]^{\frac{1}{p}}\left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.18)$$

Proof For $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, using the Hölder's integral inequality we have

$$\left|\frac{1}{b-a}\int_a^b (x-c)^{r+1}f'(x)dx\right| \le \left(\frac{1}{b-a}\int_a^b |(x-c)^{r+1}|^p dx\right)^{\frac{1}{p}}\left(\frac{1}{b-a}\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.19)$$

Since

$$\int_a^b |(x-c)^{r+1}|^p dx = \int_a^c (c-x)^{p(r+1)}dx + \int_c^b (x-c)^{p(r+1)}dx$$
$$= \frac{(b-c)^{p(r+1)+1} + (c-a)^{p(r+1)+1}}{p(r+1)+1}, \qquad (3.20)$$

we get (3.18) using the identity (3.1). This completes the proof.

Corollary 6. *For* $r = 0$ *and* $c = \frac{a+b}{2}$ *in* (3.19), *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^{\frac{1}{p}}}{2(p+1)^{\frac{1}{p}}}\left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}, \qquad (3.21)$$

the inequality established by Dragomir and Pearce [3,*p*.33].

Corollary 7. *With the above assumptions and provided that f is convex on I, from* (3.18) *the following reverse inequality holds*:

$$0 \le (b-c)^{r+1}f(b) - (a-c)^{r+1}f(a) - (r+1)M_r(c)$$
$$\le \left[\frac{(b-c)^{p(r+1)+1} + (c-a)^{p(r+1)+1}}{p(r+1)+1}\right]^{\frac{1}{p}}\left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.22)$$

Corollary 8. *Let* $c = 0$ *in* (3.18). *Then inequality involving the central moments of the random variable X for* $r = 0,1,2\ldots$

$$|b^{r+1}f(b) - a^{r+1}f(a) - (r+1)M_r(0)|$$
$$\le \left[\frac{b^{p(r+1)+1} - a^{p(r+1)+1}}{p(r+1)+1}\right]^{\frac{1}{p}}\left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.23)$$

Corollary 9. *From* (3.18) *with* $c = 0$, *the reverse inequality involving the central moments of the random variable X for* $r = 0,1,2\ldots$

$$0 \le b^{r+1}f(b) - a^{r+1}f(a) - (r+1)M_r(0)$$
$$\le \left[\frac{b^{p(r+1)+1} - a^{p(r+1)+1}}{p(r+1)+1}\right]^{\frac{1}{p}}\left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.24)$$

Now we state the well known Grüss integral inequality as a lemma:

Lemma 2. *Let* $f,g : [a,b] \to R$ *be two integrable functions such that* $\phi \le f(x) \le \Phi$ *and* $\gamma \le g(x) \le \Gamma$ *for all* $x \in [a,b]$. *Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{(\Phi-\phi)(\Gamma-\gamma)}{4}. \qquad (3.25)$$

The following theorem involving moments and based on the above lemma 2 holds:

Theorem 5. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a,b \in I$ *with* $a < b$ *and* $m \le f'(x) \le M$ *for* *all* $x \in [a,b]$. *If* $f' \in L_1[a,b]$, *then for* $r = 0,1,2,\ldots$

$$\left|(b-c)^{r+1}f(b) - (a-c)^{r+1}f(a) - (r+1)M_r(c) - \frac{[f(b)-f(a)][(b-c)^{r+2}-(a-c)^{r+2}]}{(b-a)(r+2)}\right|$$
$$\le \frac{(M-m)(b-a)^2}{4}. \qquad (3.26)$$

Proof Set the mapping

$$g(x) = (x-c)^{r+1}, x \in [a,b].$$

Then

$$(a-c)^{r+1} \le g(x) \le (b-c)^{r+1}, \text{for all } x \in [a,b].$$

Applying the Grüss integral inequality (3.25), we get

$$\left|\frac{1}{b-a}\int_a^b (x-c)^{r+1}f'(x)\,dx - \frac{1}{b-a}\int_a^b (x-c)^{r+1}\,dx \cdot \frac{1}{b-a}\int_a^b f'(x)\,dx\right| \le \frac{(M-m)(b-a)}{4}.$$

Since

$$\int_a^b (x-c)^{r+1}\,dx = \frac{(b-c)^{r+2}-(a-c)^{r+2}}{r+2}, \int_a^b f'(x)\,dx = f(b)-f(a),$$

and using the identity (3.1), we deduce (3.26) that proves the theorem.

Corollary 10. *For* $r = 0$ *and* $c = \frac{a+b}{2}$ *in* (3.26), *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(M-m)(b-a)}{4}, \qquad (3.27)$$

the inequality established in [3,*p*.34].

Corollary 11. *With the above assumptions and provided that f is convex on I, from* (3.26) *the following reverse inequality holds*:

$$0 \le \frac{[(b-c)^{r+2}-(a-c)^{r+2}][f(b)-f(a)]}{(b-a)(r+2)} - [(b-c)^{r+1}f(b) - (a-c)^{r+1}f(a) - (r+1)M_r(c)]$$
$$\le \frac{(M-m)(b-a)^2}{4}. \qquad (3.28)$$

Corollary 12. *Let* $c = 0$ *in* (3.26). *Then inequality involving the central moments of the random variable X for* $r = 0,1,2\ldots$

$$\left|b^{r+1}f(b) - a^{r+1}f(a) - (r+1)M_r(0) - \frac{[f(b)-f(a)](b^{r+2}-a^{r+2})}{(b-a)(r+2)}\right|$$
$$\le \frac{(M-m)(b-a)^2}{4}. \qquad (3.29)$$

Corollary 13. *From* (3.26) *with* $c = 0$, *the reverse inequality involving the central moments of the random variable X for* $r = 0,1,2\ldots$

$$b^{r+1}f(b) - a^{r+1}f(a) - (r+1)M_r(0) \ge \frac{(b^{r+2}-a^{r+2})[f(b)-f(a)]}{(b-a)(r+2)}$$
$$- \frac{(M-m)(b-a)^2}{4}. \qquad (3.30)$$

Now follows an useful identity in terms of moments:

Lemma 3. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable convex mapping on I and* $a,b \in I$ *such that* $a < b$ *and* $f' \in L_1[a,b]$. *Then for* $r = 0,1,2,\ldots$

$$r \quad 1 M_r\, c \qquad c \quad a^{\,r\ 1} \quad c \quad b^{\,r\ 1} f\, c \qquad {}_a^{b} p\; x\, f\;\; x\; dx, \qquad 3.31$$

where

$$p\; x \quad \begin{cases} x \quad a^{\,r\ 1}, x \quad a, c\;, \\ x \quad b^{\,r\ 1}, x \quad c, b\;. \end{cases}$$

Proof Integrating by parts,

$${}_a^{c} x \quad a^{\,r\ 1} f\;\; x\; dx \qquad c \quad a^{\,r\ 1} f\, c \qquad r \quad 1 \;\; {}_a^{c} x \quad a^{\,r} f\, x\;\; dx,$$

$${}_c^{b} x \quad b^{\,r\ 1} f\;\; x\; dx \qquad c \quad b^{\,r\ 1} f\, c \qquad r \quad 1 \;\; {}_c^{b} x \quad b^{\,r} f\, x\;\; dx,$$

and adding

$${}_a^{c} x \quad a^{\,r\ 1} f\;\; x\; dx \quad {}_a^{c} x \quad a^{\,r\ 1} f\;\; x\; dx \qquad c \quad a^{\,r\ 1} \quad c \quad b^{\,r\ 1} f\, c \qquad r \quad 1 \;\; {}_a^{b} x \quad c^{\,r} f\, x\;\; dx.$$

Using (1.1) and since

$${}_a^{c} x \quad a^{\,r\ 1} f\;\; x\; dx \quad {}_a^{c} x \quad a^{\,r\ 1} f\;\; x\; dx \qquad {}_a^{b} p\; x\;\; f\;\; x\; dx,$$

(3.31) is proved.

Corollary 14. *For* $r \quad 0$ *and* $c \quad \frac{a\ b}{2}$ *in* (3.31), *we have*

$$f\, \frac{a \quad b}{2} \qquad \frac{1}{b \quad a}\; {}_a^{b} f\, x\; dx \quad \frac{1}{b \quad a}\; {}_a^{b} p\; x\, f\;\; x\; dx, \qquad 3.32$$

where

$$p\; x \quad \begin{cases} x \quad a\,, x \quad a, \frac{a\ b}{2}\;, \\ x \quad b, x \quad \frac{a\ b}{2}, b\;, \end{cases}$$

the identity established by Dragomir and Pearce [3,*p*.35].

The following theorem also holds good:

Theorem 6. *Let* $f : I \quad \mathbb{R} \quad \mathbb{R}$ *be a differentiable mapping on* I, $a, b \quad I$ *with* $a \quad b$ *and* $p \quad 1$. *If* $|f|$ *is* q*-integrable on* a, b *where* $q \quad \frac{p}{p\ 1}$, *then*

$$| \; c \quad a^{\,r\ 1} \quad c \quad b^{\,r\ 1} f\, c \qquad r \quad 1\, M_r\, c \;|$$
$$\frac{c \quad a^{\,p\ 1} \quad b \quad c^{\,p\ 1}}{p \quad 1}^{\frac{1}{p}} \qquad {}_a^{b} |f \;\; x\, |^q dx^{\,\frac{1}{q}}. \qquad 3.33$$

Proof By applying the Holder's integral inequality

$$|\frac{1}{b \quad a}\; {}_a^{b} p\; x\, f\;\; x\; dx| \qquad \frac{1}{b \quad a}\; {}_a^{b} |p\; x\, |^p dx^{\,\frac{1}{p}} \qquad \frac{1}{b \quad a}\; {}_a^{b} |f \;\; x\, |^q dx^{\,\frac{1}{q}}.$$

Since

$${}_a^{b} |p\; x\, |^p dx \qquad {}_a^{c} |x \quad a|^p dx \qquad {}_c^{b} |x \quad b|^p dx$$
$$\frac{c \quad a^{\,p\ 1} \quad b \quad c^{\,p\ 1}}{p \quad 1},$$

and using identity (3.1), we establish (3.33) and hence the theorem.

Corollary 15. *For* $r = 0$ *and* $c = \frac{a+b}{2}$ *in* (3.33), *we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^{\frac{1}{p}}}{2(p+1)^{\frac{1}{p}}}\left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}, \qquad (3.34)$$

the inequality established in [3,*p*.36].

Corollary 16. *With the above assumptions and provided that f is convex on I, from* (3.33) *the following reverse inequality holds*:

$$0 \le \left[(c-a)^{r+1} - (c-b)^{r+1}\right] f(c) - (r+1) M_r(c)$$
$$\le \left(\frac{(c-a)^{p+1} + (b-c)^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.35)$$

Corollary 17. *Let* $c = 0$ *in* (3.33). *Then inequality involving the central moments of the random variable X for* $r = 0,1,2\ldots$

$$\left|\left[(-a)^{r+1} - (-b)^{r+1}\right] f(0) - (r+1) M_r(0)\right|$$
$$\le \left(\frac{(-a)^{p+1} + b^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.36)$$

Corollary 18. *From* (3.18) *with* $c = 0$, *the reverse inequality involving the central moments of the random variable X for* $r = 0,1,2\ldots$

$$\left[(-a)^{r+1} - (-b)^{r+1}\right] f(0) - (r+1) M_r(0)$$
$$\le \left(\frac{(-a)^{p+1} + b^{p+1}}{p+1}\right)^{\frac{1}{p}} \left(\int_a^b |f'(x)|^q dx\right)^{\frac{1}{q}}. \qquad (3.37)$$

We present another theorem involving moments by applying the Grüss integral inequality:

Theorem 7. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable convex mapping on I and* $a, b \in I$ *such that* $a < b$ *and* $m \le f'(x) \le M$ *for all* $x \in [a,b]$. *If* $f' \in L_1[a,b]$, *then for* $r = 0,1,2,\ldots$

$$\left|\left[(c-a)^{r+1} - (c-b)^{r+1}\right] f(c) - (r+1) M_r(c) - \frac{\left[(c-a)^{r+2} - (c-b)^{r+2}\right]\left[f(b) - f(a)\right]}{(b-a)(r+2)}\right|$$
$$\le \frac{(M-m)(b-a)^2}{4}. \qquad (3.38)$$

Proof Applying the Grüss integral inequality (3.25), we get

$$\left|\frac{1}{b-a}\int_a^b p(x) f'(x)\,dx - \frac{1}{b-a}\int_a^b p(x)\,dx \cdot \frac{1}{b-a}\int_a^b f'(x)\,dx\right| \le \frac{(M-m)(b-a)}{4}.$$

Since

$$\int_a^b p(x)\,dx = \int_a^c (x-a)^{r+1}dx + \int_c^b (x-b)^{r+1}dx$$
$$= \frac{(c-a)^{r+2} - (c-b)^{r+2}}{r+2},$$
$$\int_a^b f'(x)\,dx = f(b) - f(a),$$

and identity (3.231), we establish the required inequality (3.38)

Corollary 19. *For* $r = 0$ *and* $c = \frac{a+b}{2}$ *in* (3.38), *we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(M-m)(b-a)}{4}, \qquad (3.39)$$

the inequality established in [3,*p*.36].

Corollary 20. *With the above assumptions and provided that f is convex on I, from* (3.38) *the following reverse inequality holds*:

$$0 \le \left[(c-a)^{r+1}-(c-b)^{r+1}\right]f'(c)-\frac{\left[(c-a)^{r+2}-(c-b)^{r+2}\right]\left[f'(b)-f'(a)\right]}{(b-a)(r+2)}-(r+1)M_r(c)$$
$$\le \frac{(M-m)(b-a)^2}{4}. \qquad (3.40)$$

Corollary 21. *Let* $c=0$ *in* (3.38). *Then inequality involving the central moments of the random variable X for* $r=0,1,2...$

$$\left|\left[(-a)^{r+1}-(-b)^{r+1}\right]f'(0)-(r+1)M_r(0)-\frac{\left[(-a)^{r+2}-(-b)^{r+2}\right]\left[f'(b)-f'(a)\right]}{(b-a)(r+2)}\right|$$
$$\le \frac{(M-m)(b-a)^2}{4}. \qquad (3.41)$$

Corollary 22. *From* (3.38) *with* $c=0$, *the reverse inequality involving the central moments of the random variable X for* $r=0,1,2...$

$$\left[(-a)^{r+1}-(-b)^{r+1}\right]f'(0)-(r+1)M_r(0)-\frac{\left[(-a)^{r+2}-(-b)^{r+2}\right]\left[f'(b)-f'(a)\right]}{(b-a)(r+2)}$$
$$\le \frac{(M-m)(b-a)^2}{4}. \qquad (3.42)$$

4. Applications to Special Means

We now consider the following convex mappings that result in special means:

f(x)	**Mean**		
x^p	Arithmetic:	$A(a^p,b^p)$	$=\frac{a^p+b^p}{2}, a,b>0$
$\frac{1}{x}$	Logarithmic:	$L(a,b)$	$=\begin{cases}\frac{b-a}{\ln b-\ln a}, \text{if } a\ne b, a,b>0\\ a, \text{ if } a=b, a,b>0\end{cases}$
$\frac{1}{x}$	Harmonic:	$H(a,b)$	$=\frac{2}{\frac{1}{a}+\frac{1}{b}}, a,b>0$
$\ln x$	Identric:	$I(a,b)$	$=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, \text{if } a\ne b, a,b>0\\ a, \text{ if } a=b, a,b>0\end{cases}$
$\ln x$	Geometric:	$G(a,b)$	$=\sqrt{ab}, a,b>0$

4.1. Mapping $f(x)=x^p, p>1, x>0, a,b\in R$ with $0<a<b$

We have

$$pa^{p-1}\le f'(x)=px^{p-1}\le pb^{p-1},\ x\in[a,b],$$

$$f\left(\frac{a+b}{2}\right)=A^p(a,b),\ \frac{f(a)+f(b)}{2}=A(a^p,b^p),\ \frac{1}{b-a}\int_a^b f(x)\,dx=L_p^p(a,b). \qquad (4.1)$$

Proposition 1. *Let* $p \quad 1, q \quad \frac{p}{p\ 1}$ *and* $0 \quad a \quad b$. *Then for* $r \quad 0, 1, 2, \ldots$

$$0 \quad r \quad 1\, M_r\ c \qquad b \quad c^{\ r\ 1} b^p \quad a \quad c^{\ r\ 1} a^p$$
$$\frac{p\ b \quad a^{\frac{1}{q}}\ \ b \quad c^{\ p\ r\ 1\ \ 1} \quad c \quad a^{\ p\ r\ 1\ \ 1\ \frac{1}{p}}}{p\ r \quad 1 \quad 1^{\frac{1}{p}}}\ L_p\ a, b^{\ \frac{p}{q}}. \qquad 4.2$$

Proof For the convex mapping $f\ x \quad x^p$, we apply (3.22). Then

$$r \quad 1\, M_r\ c \qquad b \quad c^{\ r\ 1} b^p \quad a \quad c^{\ r\ 1} a^p$$
$$\frac{b \quad c^{\ p\ r\ 1\ \ 1} \quad c \quad a^{\ p\ r\ 1\ \ 1\ \frac{1}{p}}}{p\ r \quad 1 \quad 1^{\frac{1}{p}}} \quad \int_a^b |px^{p\ 1}|^q dx^{\ \frac{1}{q}}$$
$$\frac{p\ \ b \quad c^{\ p\ r\ 1\ \ 1} \quad c \quad a^{\ p\ r\ 1\ \ 1\ \frac{1}{p}}}{p\ r \quad 1 \quad 1^{\frac{1}{p}}} \quad \int_a^b x^{\ p\ 1\ q} dx^{\ \frac{1}{q}}.$$

Since

$$\int_a^b x^{\ p\ 1\ q} dx \quad \frac{b^{p\ 1} \quad a^{p\ 1}}{p \quad 1} \qquad b \quad a\ L_p^p\ a, b\ ,$$

we prove the required inequality (4.2).

Corollary 23. *For* $r \quad 0$ *and* $c \quad \frac{a\ b}{2}$ *in* (4.2), *we have*

$$0 \quad A\ a^p, b^p \quad L_p^p\ a, b \quad \frac{p\ b \quad a}{2\ p \quad 1^{\frac{1}{p}}}\ L_p\ a, b^{\ \frac{p}{q}}, \qquad 4.3$$

the inequality established by Dragomir and Pearce [3,*p*.37].

4.2. Mapping $f\ x \quad \frac{1}{x}, x \quad 0, a, b \quad R$ with $0 \quad a \quad b$

We have

$$\frac{1}{a^2} \quad f\ x \qquad \frac{1}{x^2} \qquad \frac{1}{b^2} \text{ for all } x \quad a, b\ ,$$
$$f\ \frac{a \quad b}{2} \qquad A^{\ 1}\ a, b\ , \ \frac{f\ a \quad f\ b}{2} \qquad H^{\ 1}\ a, b\ , \ \frac{1}{b \quad a} \int_a^b f\ x\ dx \quad L^{\ 1}\ a, b\ . \qquad 4.4$$

Proposition 2. *Let* $p \quad 1, q \quad \frac{p}{p\ 1}$ *and* $0 \quad a \quad b$. *Then for* $r \quad 0, 1, 2, \ldots$

$$0 \quad r \quad 1\, M_r\ c \qquad b \quad c^{\ r\ 1} \frac{1}{b} \quad a \quad c^{\ r\ 1} \frac{1}{a}$$
$$\frac{b \quad a^{\frac{1}{q}}\ \ b \quad c^{\ p\ r\ 1\ \ 1} \quad c \quad a^{\ p\ r\ 1\ \ 1\ \frac{1}{p}}}{p\ r \quad 1 \quad 1^{\frac{1}{p}}}\ L_{\frac{p\ 1}{1\ p}}\ a, b^{\ \frac{p\ 1}{p}}. \qquad 4.5$$

Proof For the convex mapping $f\ x \quad \frac{1}{x}$, we apply (3.22). Then

$$r \quad 1\, M_r\ c \qquad b \quad c^{\ r\ 1} \frac{1}{b} \quad a \quad c^{\ r\ 1} \frac{1}{a}$$
$$\frac{b \quad c^{\ p\ r\ 1\ \ 1} \quad c \quad a^{\ p\ r\ 1\ \ 1\ \frac{1}{p}}}{p\ r \quad 1 \quad 1^{\frac{1}{p}}} \quad \int_a^b \frac{1}{x^{2q}} dx^{\ \frac{1}{q}}.$$

Since

$$\int_a^b \frac{1}{x^{2q}}dx = (b-a)L_{1-2q}^{1-2q}(a,b),$$

and $1-2q = \frac{p+1}{1-p}$,

we prove (4.4).

Corollary 24. *For* $r=0$ *and* $c=\frac{a+b}{2}$ *in* (4.5), *we have*

$$0 \le \frac{\frac{1}{a}+\frac{1}{b}}{2} - \frac{\ln b-\ln a}{b-a} \le \frac{b-a}{2(p+1)^{\frac{1}{p}}} L_{\frac{p+1}{1-p}}(a,b)^{\frac{p+1}{p}},$$

or

$$0 \le H^{-1}(a,b) - L^{-1}(a,b) \le \frac{b-a}{2(p+1)^{\frac{1}{p}}} L_{\frac{p+1}{1-p}}(a,b)^{\frac{p+1}{p}}, \qquad (4.6)$$

the inequality in [3,*p*.37].

4.3. For mapping $f(x) = -\ln x,\ x \in (0,\infty),\ 0<a<b$

We have

$$\frac{1}{b} \le |f'(x)| = \frac{1}{x} \le \frac{1}{a} \text{ for all } x \in (a,b),$$

$$f\left(\frac{a+b}{2}\right) = -\ln A(a,b),\quad \frac{f(a)+f(b)}{2} = -\ln G(a,b),\quad \frac{1}{b-a}\int_a^b f(x)\,dx = -\ln I(a,b). \qquad (4.7)$$

Proposition 3. *Let* $p>1, q=\frac{p}{p-1}$ *and* $0<a<b$. *Then for* $r=0,1,2,\ldots$

$$0 \le (r+1)M_r(c) + (a-c)^{r+1}\ln a - (b-c)^{r+1}\ln b$$
$$\le \frac{(b-a)^{\frac{1}{q}}\left[(b-c)^{p(r+1)+1} - (c-a)^{p(r+1)+1}\right]^{\frac{1}{p}}}{[p(r+1)+1]^{\frac{1}{p}}} L_{1-q}(a,b)^{\frac{1-q}{q}}. \qquad (4.8)$$

Proof For the convex mapping $f(x) = -\frac{1}{x}$, applying (3.22), we have

$$(r+1)M_r(c) - (b-c)^{r+1}\ln b + (a-c)^{r+1}\ln a$$
$$\le \frac{\left[(b-c)^{p(r+1)+1} - (c-a)^{p(r+1)+1}\right]^{\frac{1}{p}}}{[p(r+1)+1]^{\frac{1}{p}}}\left(\int_a^b \frac{1}{x^q}dx\right)^{\frac{1}{q}}.$$

Since

$$\int_a^b \frac{1}{x^q}dx = (b-a)L_{1-q}^{1-q}(a,b),$$

$$\left(\int_a^b \frac{1}{x^q}dx\right)^{\frac{1}{q}} = (b-a)^{\frac{1}{q}} L_{1-q}(a,b)^{\frac{1-q}{q}},$$

and thus we prove (4.8).

Corollary 25. *For* $r=0$ *and* $c=\frac{a+b}{2}$ *in* (4.8), *we have*

$$0 \le \frac{I(a,b)}{G(a,b)} \le \exp\left[\frac{b-a}{2(p+1)^{\frac{1}{p}}} L_{1-q}(a,b)^{\frac{1-q}{q}}\right], \qquad (4.9)$$

the inequality by Dragomir and Pearce [3,*p*.38].

We apply (3.26) that was established using the Grüss integral inequality to (4.1,4.4,4.7) and obtain the following results (proofs are straightforward, hence omitted):

Proposition 4. *Let* $p \quad 1, q \quad \frac{p}{p \ 1}$ *and* $0 \quad a \quad b$. *Then for* $r \quad 0,1,2,...$

$$0 \quad r \quad 1 \; M_r \; c \qquad b \quad c^{\; r \ 1} b^p \qquad a \quad c^{\; r \ 1} a^p \qquad \frac{b \quad c^{\; r \ 2} \qquad a \quad c^{\; r \ 2} \; b^p \quad a^p}{b \quad a \quad r \quad 2}$$

$$\frac{p \, p \quad 1 \quad b \quad a^{\; 3}}{4} L_{p \ 2} \; a, b^{\quad p \ 2}. \qquad 4.10$$

Proposition 5. *Let* $p \quad 1, q \quad \frac{p}{p \ 1}$ *and* $0 \quad a \quad b$. *Then for* $r \quad 0,1,2,...$

$$0 \quad r \quad 1 \; M_r \; c \qquad b \quad c^{\; r \ 1} \frac{1}{b} \qquad a \quad c^{\; r \ 1} \frac{1}{a} \qquad \frac{b \quad c^{\; r \ 2} \qquad a \quad c^{\; r \ 2} \; \frac{1}{b} \quad \frac{1}{a}}{b \quad a \quad r \quad 2}$$

$$\frac{b \quad a^{\; 2} \; b^2 \quad a^2}{4a^2 b^2}. \qquad 4.11$$

Proposition 6. *Let* $p \quad 1, q \quad \frac{p}{p \ 1}$ *and* $0 \quad a \quad b$. *Then for* $r \quad 0,1,2,...$

$$0 \quad r \quad 1 \; M_r \; c \qquad a \quad c^{\; r \ 1} \ln a \qquad b \quad c^{\; r \ 1} \ln b \qquad \frac{b \quad c^{\; r \ 2} \qquad a \quad c^{\; r \ 2} \; \ln a \quad \ln b}{b \quad a \quad r \quad 2}$$

$$\exp \; \frac{b \quad a^{\; 3}}{4ab} \quad . \qquad 4.12$$

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