SOME INEQUALITIES INVOLVING BETA AND GAMMA FUNCTIONS

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ABSTRACT

An inequality for the Euler's Beta -function is established. Properties of the Beta probability distributions like mean , variance, moment ratios, are considered to prove some more inequalities.

1. INTRODUCTION

Beta probability distributions have established their usefulness in the statistical analysis of reliability, life testing models and in many other applications [Bain (1978)]. A beta random variable (r.v.) X with parameters (a,b) has the probability density function (*pdf*)

$$f(x\,:\,a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)};0 < x < 1 \quad (1.1)$$

where $\Omega = \{a,b\,:\,a > 0, b > 0\}$ and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

These distributions possess a number of statistical properties [Johnson and Kotz (1970), p.41; Ord (1972), p. 6], some of interesting being : these are (i) members of Pearson family, (ii) exponential (a) wrt a (b known) , (b) wrt b (a known), (c) wrt both a and b, (iii) monotone likelihood ratio (MLR) in (a)$T_1(x) = \log x$, (b known) , (b) $T_2(x) = \log(1-x)$, (a known), (iv) unimodal $(a > 1, b > 1)$; not unimodal $[(a < 1, b < 1), or, (a-1)(b-1) \leq 0]$. These have (i) increasing failure rate (IFR) $(a \geq 1, b = 1),$ not IFR$(0 < a < 1)$, not decreasing failure rate (DFR) $(0 < a < 1), r(x) = \frac{ax^{a-1}}{1-x^a}$, for $b = 1$, and (ii) IFR $(a = 1), r(x) = \frac{b}{1-x}$, where $r(x)$ is the failure rate.

The present paper aims at first establishing an inequality on the Beta - function and then, to use moments and moment ratios of the Beta random variables to derive some more inequalities.

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2. AN INEQUALITY FOR THE EULER'S BETA - FUNCTION

We start with the following lemma.

Lemma. *Let m, n, p, and q be positive real numbers, such that* (p-m) (q-n) $\leq$ ($\geq$) 0. *Then,*

$$B(p,q)B(m,n) \geq (\leq) B(p,n)B(m,q) \quad (2.1)$$

and

$$\Gamma(p+n)\Gamma(q+m) \leq (\geq) \Gamma(p+q)\Gamma(m+n) \quad (2.2)$$

Proof. Define the mappings : f , g , h : [0,1]→ $[0,\infty)$, given by

$$f(x) = x^{p-m}, g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{m-1}(1-x)^{n-1} \quad (2.3)$$

As$(p-m)(q-n) \leq (\geq) 0$, the mappings f and g are the same (opposite) monotonic on $[0,1]$ and h is non-monotonic on $[0,1]$.

Applying the well known Cebysev's integral inequality for synchronous (asynchronous) mappings [Lebedev (1957)] , i.e.,

$$\int_a^b h(x)dx \int_a^b h(x)f(x)g(x)dx \geq (\leq) \int_a^b h(x)f(x)dx \int_a^b h(x)g(x)dx \quad (2.4)$$

we can write the inequality

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}(1-x)^{q-n}dx \geq (\leq)$$

$$\int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}dx \int_0^1 x^{m-1}(1-x)^{n-1}(1-x)^{q-n}dx$$

i.e.,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \int_0^1 x^{p-1}(1-x)^{q-1}dx \geq (\leq)$$

$$\int_0^1 x^{p-1}(1-x)^{n-1}dx \int_0^1 x^{m-1}(1-x)^{q-1}dx$$

and, by virtue of (1.1), the inequality (2.1) is proved.

The inequality (2.2) follows from (2.1) by taking into account that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (2.5)$$

for all $p, q > 0$. We shall omit the details.

The following interesting corollaries may be noted as well:

Corollary 2.2. *Let p, m>0.Then,we have the inequalities*

$$B(p,p)B(m,m) \leq B^2(p,m) \quad (2.6)$$

and

$$\Gamma^2(p+m) \leq \Gamma(2p)\Gamma(2m) \quad (2.7)$$

Proof. In the above lemma, if we choose $p = q$, and $m = n$, we have $(p-m)(q-n) = (p-m)^2 \geq 0$, and thus,

$$B(p,p)B(m,m) \leq B(p,m)B(m,p)$$

which proves the inequality (2.6).

The inequality (2.7) follows from (2.6) through (2.5).

Corollary 2.3. *Given two positive real numbers u and v*>0*, the geometric mean of* $\Gamma(u)$ *and* $\Gamma(v)$ *is greater than the gamma of the arithmetic mean of u and v*.

This result follows by re-writing (2.7) as

$$\Gamma\left(\frac{u+v}{2}\right) \leq \sqrt{\Gamma(u)\Gamma(v)}$$

where $u = 2p$ and $v = 2m$.

3. INEQUALITIES FOR MOMENTS OF BETA RANDOM VARIABLES

A distribution function determines a set of moments when they exist. The first moment about origin, recognized as the mean or center of gravity and the second moment about mean, a measure of the spread or dispersion of the population, are frequently studied parameters of a population . Other properties such as skewness and kurtosis are defined in terms of the higher moments. The $r-th$ moment of the r.v. X about the origin is defined by $\mu_r^{'}(X) = E(X)^r$, where E denotes the mathematical expectation. The $r-th$ central moment of the r.v. X , $\mu_r(X)$, can be derived from

$$\mu_r(X) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\mu_{r-1}^{'}(X)\mu_1^{'i}(X), r = 1,2,...$$

Now, we present a theorem on the moments of the random variables which follow Beta pdfs.

Theorem 3.1. *Let the r.v. X and Y be such that X~* $B(p,q)$ *and Y~* $B(m,n)$, $p,q,m,n > 0$. *Further, let the r.v. U and V be defined as U~* $B(p,n)$ *and V~* $B(m,q)$. *Then, for* $(p\text{-}m)(q\text{-}n) \leq (\geq)\ 0$,

$$\frac{E(X)^r E(Y)^r}{E(U)^r E(V)^r} \geq (\leq) \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)}, r = 1,2,... \quad (3.1)$$

Proof. In (2.3) of lemma 2.1, we choose

$$f(x) = x^{p-m}, g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{r+m-1}(1-x)^{n-1} \quad (3.2)$$

Then , on substituting these mappings in (2.4), we reach at the inequality in Theorem 3.1.

Remark 3.2. The inequalities for the absolute moments of r.v. X , $\upsilon_r(X) = E\ |\ X\ |^r$ and the factorial moments, $\mu_{(r)}^{'}(X) = E(X)^{(r)}$, about origin, may be obtained from Theorem 3.1, on replacing $\mu_r^{'}(.)$ by $\upsilon_r(.)$ and $\mu_r^{'}(.)$, respectively. Similarly, corresponding inequalities for the moment generation functions , $M_x(t) = E(e^{tx})$, and characteristic function, $\phi_x(t) = E(e^{itx})$ may be easily obtained from lemma 2.1.

An interesting result from this theorem follows as :

Corollary 3.3. *For* $p(=q), m(=n) > 0$,

$$\frac{E(X)^r E(Y)^r}{E^r(U)E^r(V)} \leq \frac{\Gamma(2p)\Gamma(2m)}{\Gamma^2(p+m)}, r = 1,2,... \quad (3.3)$$

4. INEQUALITIES FOR MOMENTS OF TWO BETA RANDOM VARIABLES

Theorem 4.1. *Let the r.v. X and Y be such that X~* $B(p,q)$ *and Y~* $B(p,n)$. *Then, for* $p,q,m,n > 0$

$$\frac{E(X)^r}{E(Y)^r} \geq (\leq) \frac{\Gamma(p+q)\Gamma(m+n)}{\Gamma(p+n)\Gamma(m+q)}, r = 1,2,... \quad (4.1)$$

according as $(p\text{-}m)(q\text{-}n) \leq (\geq) 0$.

Proof. We choose in (2.3) of lemma 2.1,

$$f(x) = x^{r+p-m}, g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{m-1}(1-x)^{n-1} \quad (4.2)$$

Then , substituting these mappings in (2.4) results in the desired expression in Theorem 4.1.

Corollary 4.2. *For* $q(=p)$, $m(=n) > 0$,

$$E(X)^r\Gamma^2(p+n) \leq E(Y)^r\Gamma(2p)\Gamma(2n), r = 1,2,... \quad (4.3)$$

5. INEQUALITIES FOR HARMONIC MEANS OF TWO BETA RANDOM VARIABLES

Theorem 5.1. *Let the r.v. X and Y be such that X~* $B(p,q)$ *and Y~* $B(p,n)$. *Denote the harmonic means of r.v. X and r.v. Y by HM(X)=* $E(\frac{1}{X})$ *and HM(Y) =* $E(\frac{1}{Y})$. *Then, for* $p,q,m,n > 0$,

$$\frac{HM(X)}{HM(Y)} \geq (\leq) \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)} \quad (5.1)$$

according as $(p\text{-}m)(q\text{-}n) \leq (\geq) 0$.

Proof. We choose in (2.3) of lemma 2.1,

$$f(x) = x^{-1+p-m}, g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{m-1}(1-x)^{n-1} \quad (5.2)$$

Then , substituting these mappings in (2.4) proves Theorem 5.1.

Corollary 5.2. *For* $q(=p)$, $m(=n) > 0$,

$$\frac{HM(X)}{HM(Y)} \leq \frac{\Gamma(2p)\Gamma(2n)}{\Gamma^2(p+n)} \quad (5.3)$$

6. INEQUALITIES FOR VARIANCES OF TWO BETA RANDOM VARIABLES

Theorem 6.1. *Let the r.v. X and Y be such that* $X\sim B(p,q)$ *and* $Y\sim B(p,n)$. *Denote the variances of r.v. X and r.v. Y by* $V(X)=\mu_2'(X) - \mu_1'^2(X)$ *and* $V(Y) = \mu_2'(Y) - \mu_1'^2(Y)$. *Then, for* $p,q,m,n > 0$

$$V(X)B(m,n)B(p,q) - V(Y)B(m,q)B(p,n) \geq (\leq)$$

$$\frac{B(m,q)B^2(p+1,n)}{B(p,n)} - \frac{B(m,n)B^2(p+1,q)}{B(p,q)} \quad (6.1)$$

according as $(p\text{-}m)(q\text{-}n)\leq (\geq)\ 0$.

Proof. We consider the inequality in theorem 4.1 by choosing $r = 2$ and rewrite $\mu_2'(.)$ in terms of V(.).Then, we get

$$[V(X) + \mu_1'^2(X)]B(m,n)B(p,q) \geq (\leq)[V(Y) + \mu_1'^2(Y)]B(m,q)B(p,n) \quad (6.2)$$

Now substituting $\mu_1'(X) = \frac{p}{p+q}$ and $\mu_1'(Y) = \frac{p}{p+n}$ in the above expression, we reach at the desired inequality in Theorem 6.1.

Corollary 6.2. *Denoting coefficients of variation of the r.v. X and Y by* $CV(X)$*and* $CV(Y)$ *where* $CV(.) = \frac{\sqrt{V(.)}}{\mu_1'(.)}$, *the inequality for* $CV\ (X)$ *and* $CV(Y)$ *follows*:

$$\frac{CV^2(X)+1}{CV^2(Y)+1} \geq (\leq) \frac{(p+q)\Gamma(m+n)\Gamma(p+q+1)}{(p+n)\Gamma(m+q)\Gamma(p+n+1)} \quad (6.3)$$

according as $(p\text{-}m)(q\text{-}n)\leq (\geq)\ 0$.

Proof. From (6.1), we have

$$\frac{\frac{V(X)}{\mu_1'^2(X)}+1}{\frac{V(Y)}{\mu_1'^2(Y)}+1} \geq (\leq) \frac{\Gamma(m+n)\Gamma(p+q)}{\Gamma(m+q)\Gamma(p+n)} \frac{\mu_1'^2(Y)}{\mu_1'^2(X)}$$

Now, substituting $\mu_1'(X) = \frac{p}{p+q}$ and $\mu_1'(Y) = \frac{p}{p+n}$, in the above expression, we prove the corollary.

7. SOME MORE INEQUALITIES FOR GAMMA FUNCTIONS

We note that the mean and variance of a Beta $r.\ v.\ Z$ with parameters u and v are $\frac{u}{u+v}$ and $\frac{uv}{(u+v+1)(u+v)^2}$, respectively. Then, we have for Beta $r.v's$. X and Y, defined as above,

$$E(X) = \frac{p}{p+q},\ E(Y) = \frac{p}{p+n}$$

$$V(X) = \frac{pq}{(p+q+1)(p+q)^2},\ V(Y) = \frac{pn}{(p+n+1)(p+n)^2}$$

Using these values, the inequality (4.1) and (6.3) yield

$$\Gamma(p+n+1)\Gamma(m+q) \geq (\leq)\ \Gamma(p+q+1)\Gamma(m+n) \quad (7.1)$$

and

$$\frac{(p+n)[q+p(p+q+1)]}{(p+q)[n+p(p+n+1)]} \geq (\leq) \frac{\Gamma(p+q+2)\Gamma(m+n)}{\Gamma(p+n+2)\Gamma(m+q)} \quad (7.2)$$

according as $(p-m)(q-n) \leq (\geq)\ 0$, where $p, q, m, n > 0$.

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