MATHEMATICAL INEQUALITIES WITH APPLICATIONS TO THE BETA AND GAMMA MAPPINGS - II

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ABSTRACT

Fundamental inequalities, such as Čebyšev's integral inequality for synchronous (asynchronous) mappings, Hölder's integral inequality and Grüss's integral inequality, are applied to the estimation of moments and moment ratios of the beta and gamma random variables.

Key words and phrases: Čebyšev's integral inequality, Hölder's integral inequality, Grüss's integral inequality, Beta random variable, Gamma random variable, Moments, Moment ratios.

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1. INTRODUCTION

The beta and gamma distributions have shown useful representations of many physical situations. For example - these distributions are being applied in the statistical analysis of reliability and life testing models [1] and to make realistic adjustments to exponential distributions in representing life-testing situations; in the theory of random counters and other topics associated with random processes in time, in particular in meteorological precipitation processes [3,9]. The use of gamma distribution in approximating the distribution of quadratic forms (in particular positive definite quadratic forms), in multi-normally distributed random variables is well-established and widespread. One of the earliest examples refers to its use in approximating the distribution of the denominator in a test criterion for difference between values of two populations with unequal variances. Gamma distributions are also discussed to represent distributions of range and quasi-ranges in random samples drawn from a normal population [8, p. 59]. Gamma distribution may be used in place of normal distribution as *parent* distribution in expansions of Gram-Charlier type [8, p. 16].

A random variable (*r.v.*) X has a beta probability density function (*pdf*) with parameters (a,b), denoted as $X \sim B(a,b)$, and its *pdf* given by :

$$f(x : a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}; 0 < x < 1, \qquad (1.1)$$

where $\Omega = \{a,b : a > 0, b > 0\}$ and $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

A random variable (*r.v.*) X has a standard gamma probability density function:

$$f(x : \alpha) = \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}, \ \alpha > 0, \ \ x > 0. \qquad (1.2)$$

For $\alpha = 1$, this is an exponential distribution and if α is a positive integer, (1.2) represents an Erlang distribution [8, p. 222]. In most applications, two parameter form is used and is given by

$$f(x : \alpha,\theta) = \frac{\theta^{\alpha}x^{\alpha-1}e^{-x\theta}}{\Gamma(\alpha)}, \quad \alpha > 0, \ \theta > 0; \ x > 0. \qquad (1.3)$$

These beta and gamma distributions possess a number of statistical properties [8,12], a few being: (i) members of Pearson family, (ii) exponential, (iii) monotone likelihood ratio, (iv) unimodal.

We now consider the applications of the Čebyšev's integral inequality for synchronous (asynchronous) mappings, Hölder's integral inequality and Grüss's integral inequality to the estimation of moments and moment ratios of the beta and gamma random variables.

2. APPLICATIONS TO THE BETA PROBABILITY DISTRIBUTIONS

A distribution function determines a set of moments when they exist. The first moment about origin, recognized as the mean or center of gravity and the second moment about mean, a measure of the spread or dispersion of the population, are frequently studied parameters of a population . Other properties such as skewness and kurtosis are defined in terms of the higher moments. The r^{th} moment of the r.v. X about the origin is defined by $\mu_r^{'}(X) = E(X)^r$, where E denotes the mathematical expectation. The r^{th} central moment of the r.v. X , $\mu_r(X)$, can be derived from

$$\mu_r(X) = \sum_{i=0}^{r}(-1)^i \binom{r}{i} \mu_{r-1}^{'}(X)\mu_1^{'i}(X), r = 1,2,...$$

2.1. Inequalities for Moments

We first present the following estimation of the moments of the beta random variables [5]:

Theorem 2.1. *Let the r.v. X and Y be such that* $X \sim B(p,q)$ *and* $Y \sim B(m,n)$, $p,q,m,n > 0$. *Further, let the r.v. U and V be defined as* $U \sim B(p,n)$ *and* $V \sim B(m,q)$. *Then, for* $(p-m)(q-n) \leq (\geq)\ 0$,

$$\frac{E(X)^r E(Y)^r}{E(U)^r E(V)^r} \geq (\leq) \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)}, r = 1,2,... \qquad (2.2)$$

Proof. Choose the mappings

$$f(x) = x^{p-m}, g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{r+m-1}(1-x)^{n-1}. \qquad (2.3)$$

The inequalities (2.2) follows on substituting these mappings in the following Čebyšev's integral inequality for synchronous (asynchronous) mappings:

$$\int_a^b h(x)dx \int_a^b h(x)f(x)g(x)dx \geq (\leq) \int_a^b h(x)f(x)dx \int_a^b h(x)g(x)dx, \qquad (2.4)$$

according as $(p-m)(q-n) \leq (\geq)\ 0$.

■

We further deduce some results on moments of the beta *r.v.'s* from (2.2):

Corollary 2.2. *For* $q = p$, $n = m > 0$, we have from (2.2)

$$\frac{E(X)^r E(Y)^r}{E^r(U)\, E^r(V)} \leq \frac{\Gamma(2p)\,\Gamma(2m)}{\Gamma^2(p+m)},\ r = 1,2,\ldots \qquad (2.5)$$

Corollary 2.3. *In* (2.2), *letting* $m = n = 1$, *the* r^{th} *moment about the origin of beta* $r.v.$ $X \sim B(p,q)$, *satisfies*

$$\mu_r'(X) = E(X)^r \geq (\leq) \frac{B(r+p,1)\, B(r+1,q)}{B(r+1,1)\, B(p,q)}, r = 1,2,\ldots \qquad (2.6)$$

according as $(p-1)(q-1) \leq (\geq)\ 0$.

Corollary 2.4. *Set* $q = p,\ n = m > 0$ in (2.5). *Then*

$$E(X)^r \leq \frac{\Gamma(r+p)\,\Gamma(2p)}{\Gamma(r+p+m)\,\Gamma(p)}, r = 1,2,\ldots \qquad (2.7)$$

Remark. Estimations for the absolute moments of a $r.v.\ X$, i.e, $\upsilon_r(X) = E\ |\ X\ |^r$ and the factorial moments, $\mu_{(r)}'(X) = E(X)^{(r)}$, about origin, may be obtained from Theorem 2.1, on replacing $\mu_r'(.)$ by $\upsilon_r(.)$ and $\mu_r'(.)$, respectively. Similarly, corresponding inequalities for the moment generating function, $M_x(t) = E(e^{tx})$, and characteristic function, $\phi_x(t) = E(e^{itx})$ may be easily obtained from (2.2).

The following inequality for moments of two beta $r.v.'s$ also holds [5]:

Theorem 2.5. *Let the* $r.v.$ *X and Y be such that* $X \sim B(p,q)$ *and* $Y \sim B(p,n)$. *Then for* $p,q,m,n > 0$,

$$\frac{E(X)^r}{E(Y)^r} \geq (\leq) \frac{\Gamma(p+q)\Gamma(m+n)}{\Gamma(p+n)\Gamma(m+q)}, \qquad r = 1,2,\ldots \qquad (2.8)$$

according as $(p-m)(q-n) \leq (\geq)\ 0$.

Proof. We choose in (2.3) the mappings,

$$f(x) = x^{r+p-m},\ g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{m-1}(1-x)^{n-1}.$$

Then, substituting these mappings in (2.4) results in the desired inequality (2.8). ■

Theorem 2.5 also leads to the following corollary:

Corollary 2.6. *For* $q = p,\ n = m \geq 0$ *in* (3.8),

$$E(X)^r\,\Gamma^2(p+m) \leq E(Y)^r\,\Gamma(2p)\,\Gamma(2m), \quad r = 1,2,\ldots \qquad (2.9)$$

2.2. Inequalities For Harmonic Means

We start with the following result on the harmonic means of two beta $r.v.'s$ [5]:

Theorem 2.7. *Let the r.v. X and Y be such that* $X \sim B(p,q)$ *and* $Y \sim B(p,n)$. *Denote the harmonic means of r.v. X and Y by* $HM(X) = E(\frac{1}{X})$ *and* $HM(Y) = E(\frac{1}{Y})$. *Then, for* $p,q,m,n > 0$,

$$\frac{HM(X)}{HM(Y)} \geq (\leq) \frac{B(p,n)B(m,q)}{B(p,q)B(m,n)}\ , \qquad (2.10)$$

according as $(p-m)(q-n) \leq (\geq)\ 0$.

Proof. We choose in (2.3),

$$f(x) = x^{-1+p-m},\ g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{m-1}(1-x)^{n-1}. \qquad (2.11)$$

Then , on substituting these mappings in Čebyšev’s integral inequality (2.4), we get the desired inequalities (2.10).

■

Theorem 2.7 further results in the following corollary:

Corollary 2.8. *For* $q = m, n = m > 0$, *inequality* (2.10) *yields*

$$\frac{HM(X)}{HM(Y)} \leq \frac{\Gamma(2p)\,\Gamma(2m)}{\Gamma^2(p+m)}. \qquad (2.12)$$

2.3. Inequalities For Variances

The following inequality holds for the variances of two beta $r.v.'s$ [5]:

Theorem 2.9. *Let the r.v. X and Y be such that* $X \sim B(p,q)$ *and* $Y \sim B(p,n)$. *Denote the variances of r.v. X and Y by* $V(X) = \mu_2'(X) - \mu_1'^2(X)$ *and* $V(Y) = \mu_2'(Y) - \mu_1'^2(Y)$. *Then, for* $p,q,m,n > 0$,

$$V(X)\,B(m,n)\,B(p,q) - V(Y)\,B(m,q)\,B(p,n) \geq (\leq)$$

$$\frac{B(m,q)\,B^2(p+1,n)}{B(p,n)} - \frac{B(m,n)\,B^2(p+1,q)}{B(p,q)}, \qquad (2.13)$$

according as $(p-m)(q-n) \le (\ge)\ 0$.

Proof. We choose $r = 2$ in the inequality (2.8) and rewrite $\mu_2'(.)$ in terms of $V(.)$. Then, we get

$$[V(X) + \mu_1'^2(X)]B(m,n)B(p,q) \ge (\le)[V(Y) + \mu_1'^2(Y)]B(m,q)B(p,n).$$

Since for beta $r.v.'s$ X and Y, $\mu_1'(X) = \frac{p}{p+q}$ and $\mu_1'(Y) = \frac{p}{p+n}$, we prove the desired inequality (2.13).

■

The inequality involving coefficients of variation of two beta $r.v.'s$ also follows from (2.13):

Corollary 2.10. *Denoting coefficients of variation of the r.v.X and Y by* $CV(X)$ *and* $CV(Y)$ *where* $CV(.) = \frac{\sqrt{V(.)}}{\mu_1'(.)}$, *ineqalities hold*

$$\frac{CV^2(X)+1}{CV^2(Y)+1} \ge (\le) \frac{(p+q)\Gamma(m+n)\Gamma(p+q+1)}{(p+n)\Gamma(m+q)\Gamma(p+n+1)}. \qquad (2.14)$$

according as $(p-m)(q-n) \le\ (\ge)\ 0$.

Proof. From (2.13), we have for $(p-m)(q-n) \le (\ge)\ 0$,

$$\frac{\frac{V(X)}{\mu_1'^2(X)}+1}{\frac{V(Y)}{\mu_1'^2(Y)}+1} \ge (\le) \frac{\Gamma(m+n)\Gamma(p+q)}{\Gamma(m+q)\Gamma(p+n)}\,\frac{\mu_1'^2(Y)}{\mu_1'^2(X)}.$$

Now substituting $\mu_1'(X) = \frac{p}{p+q}$ and $\mu_1'(Y) = \frac{p}{p+n}$ in the above expression, we prove the corollary.

■

Another interesting case follows from Corollary 2.10 as :

Corollary 2.11. *For* $q = p$ *and* $n = m$ *in* (2.14),

$$\frac{CV^2(X)+1}{CV^2(Y)+1} \le 4p^2 \bullet \frac{\Gamma(2p)\Gamma(2m)}{\Gamma^2(p+m+1)}. \qquad (2.15)$$

Now from [5], using the logarithmic convexity of beta function on $[0,\infty)^2$,i.e.,

$$B[a(p,q)+b(m,n)] \ \leq [B(p,q)]^{a}[B(m,n)]^{b},$$

we prove the following theorem:

Theorem 2.12. *Let* $(p,q),\ (m,n) \in [0,\infty)^{2}$ *and* $a,b \geq 0,$ *with* $a+b=1$. *Define the beta r.v.*$'$*s* $U \sim B(ap+bm, aq+bn)$ *and* $V \sim B(p,q)$. *Then*

$$\frac{E(U)^{ar}}{[E(V)^{r}]^{a}} \leq \frac{[B(p,q)]^{a}[B(m,n)]^{b}}{B(ap+bm,aq+bn)},\ r=1,2,... \qquad (2.16)$$

Proof. We choose the mappings

$$f(t)=[t^{r+p-1}(1-t)^{q-1}]^{a},\ g(t)=[t^{m-1}(1-t)^{n-1}]^{b},\ h(t)=1,$$

for $p=\frac{1}{a},\ q=\frac{1}{b},\ (\frac{1}{p}+\frac{1}{q}=1$ and $p \geq 1)$.

Substituting these mappings in the Hölder’s integral inequality,

$$\int_{l}^{u} f(t)g(t)h(t)dt \leq [\int_{l}^{u}\{f(t)\}^{\frac{1}{a}}h(t)dt]^{a}[\int_{l}^{u}\{g(t)\}^{\frac{1}{b}}h(t)dt]^{b},$$

we obtain the desired inequality (2.16).

■

2.4. Inequalities For Mean Deviation

Grüss (1935) established an integral inequality which provides an estimation for integral of a product in terms of the product of integrals [11, p. 293] as :

Theorem 2.13. *Let f and g be two functions defined and integrable on* $[a,b]$. *If*

$$\Psi \leq f(x) \leq \Phi,\ \gamma \leq g(x) \leq \Gamma, \qquad (2.17)$$

for each $x \in [a,b]$ *where* Ψ, Φ, γ *and* Γ *are given real constants, then*

$$\left| \frac{1}{b-a}\int_{a}^{b} f(x)\, g(x)\, dx - \frac{1}{b-a}\int_{a}^{b} f(x)\, dx \cdot \frac{1}{b-a}\int_{a}^{b} g(x)\, dx \right|$$
$$\leq \frac{1}{4}(\Phi-\Psi)\,(\Gamma-\gamma), \quad (2.18)$$

and the constant $\frac{1}{4}$ *is the best possible.*

Now, an application of the Grüss’s integral inequality (2.18) results in the following estimation of the mean deviation of a beta random variable [16]:

Theorem 2.14. *Let* $p,q > 1$ *and* $x \in [0,1]$. *Then, for the mean deviation of a r.v.* $X \sim B(p,q),$ *holds*

$$\frac{2p^p q^q}{(p+q)^{p+q}} \cdot \frac{1}{[\frac{1}{pq}+\frac{1}{4}]} \leq MD(X) \leq \frac{2p^p q^q}{(p+q)^{p+q}} \cdot \frac{1}{[\frac{1}{pq}-\frac{1}{4}]}, \text{ for } pq < 4. \qquad (2.19)$$

Proof. Consider in (2.18) the mappings

$$f(x) = x^{p-1} \text{and } g(x) = (1-x)^{q-1},\ p,q,> 1. \qquad (2.20)$$

Since

$$\Psi = \inf_{x \in\in [0,1]} f(x) = 0,\ \gamma = \inf_{x \in [0,1]} g(x) = 0,$$

and

$$\Phi = \sup_{x \in [0,1]} f(x) = 1, \Gamma = \sup_{x \in [0,1]} g(x),$$

we deduce from (2.18)

$$\left| \int_0^1 x^{p-1}(1-x)^{q-1}dx - \int_0^1 x^{p-1}dx \int_0^1 (1-x)^{q-1}dx \right| \leq \frac{1}{4}, \qquad (2.21)$$

or

$$\left| B(p,q) - \frac{1}{pq} \right| \leq \frac{1}{4}. \qquad (2.22)$$

Since, for a *r.v.* $X \sim B(p,q),$ mean deviation is

$$MD(X) = \frac{2p^p q^q}{B(p,q)(p+q)^{p+q}}, \ (2.23)$$

the inequality (2.22) through (2.23) leads to (2.19) and hence the theorem.

■

Another estimation for the mean deviation of a beta *r.v.* is established in the following theorem [16]:

Theorem 2.15. *Let p and q be positive real numbers. Then, for the mean deviation of a r.v.* $X \sim B(p,q),$

$$MD(X) \geq (\leq) \frac{2p^{p+1}q^{q+1}}{(p+q)^{p+q}}, \qquad (2.24)$$

according as $(p-1)(q-1) \geq (\leq) 0$.

Proof. Define the mappings : $f, g, h : [0,1] \to [0,\infty)$, given by

$$f(x) = x^{p-1}, \; g(x) = (1-x)^{q-1} \text{ and } h(x) = 1. \quad (2.25)$$

As $(p-1)(q-1) \geq (\leq) 0$, the mappings f and g are the same (opposite) monotonic on $[0,1]$ and h is non-negative on $[0,1]$.

Applying the well known Čebyšev's integral inequality, we write

$$\int_0^1 1dx \int_0^1 x^{p-1}(1-x)^{q-1}dx \leq (\geq) \int_0^1 x^{p-1}dx \int_0^1 (1-x)^{q-1}dx, \qquad (2.26)$$

or,

$$B(p,q) \leq (\geq) \frac{1}{pq}. \qquad (2.27)$$

Now, by virtue of (2.23), the inequality (2.24) follows from (2.27). ■

In what follows now, another estimation for the mean deviation of a beta *r.v.* is obtained [16]:

Theorem 2.16. *Let* p *and* q *be real numbers with* $p, q > 0$. *Then, if* $p-q-1 \geq (\leq) 0$, *the mean deviation of a* $r.v. X \sim B(p,q)$ *satisfies*

$$MD(X) \leq (\geq) \frac{2p^p q^q}{(p+q)^{p+q}} \cdot \frac{\Gamma(p+q)}{\Gamma(p-1)\Gamma(q+1)}. \qquad (2.28)$$

Proof. Define the mappings

$$f(x) = x^{p-q-1}, \; g(x) = x \text{ and } h(x) = x^{q-1}e^{-x}, \qquad (2.29)$$

for $x \in [0,\infty)$.

As the mappings f and g are similarly (oppositely) ordered and h is non-negative, we can apply the well known Čebyšev's integral inequality for synchronous (asynchronous) mappings. For $p-q-1 \geq (\leq) 0$, we can write the inequality

$$\int_0^\infty x^{q-1}e^{-x}dx \int_0^\infty x^{p-1}e^{-x}dx \geq (\leq) \int_0^\infty x^{p-2}e^{-x}dx \int_0^\infty x^q e^{-x}dx,$$

or

$$\Gamma(q)\Gamma(p) \geq (\leq) \Gamma(p-1)\Gamma(q+1). \qquad (2.30)$$

Noting that $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, we rewrite (2.30) as

$$B(p,q) \geq (\leq) \frac{\Gamma(p-1)\Gamma(q+1)}{\Gamma(p+q)}, \qquad (2.31)$$

according as $p - q - 1 \geq (\leq)\ 0$.

Now, from (2.23) and (2.31), we prove the inequality (2.28).

■

3. APPLICATIONS TO GAMMA PROBABILITY DISTRIBUTIONS

3.1. Inequalities for Moments

We start with a theorem on the moments of the gamma random variables [6]:

Theorem 3.1. *Let the r.v. X be such that* $X \sim G(a+b)$, *where* $a,b > 0$ *and are similarly (oppositely) unitary. Further, define the r.v.'s U and V as* $U \sim G(a+1)$ *and* $V \sim G(b+1)$. *Then, if* $(a-1)(b-1) \geq (\leq)\ 0$,

$$\frac{E(X)^r}{E(U)^r E(V)^r} \geq (\leq) \ \frac{\Gamma(a+1)\ \Gamma(b+1)}{\Gamma(r+2)\ \Gamma(a+b)},\ r = 1,2,\ldots \qquad (3.3)$$

Proof. We choose mappings

$$f(t) = t^{a-1},\ g(t) = t^{b-1} \text{ and } h(t) = t^{r+1}e^{-t}. \qquad (3.4)$$

On substituting these mappings in Čebyšev's integral inequality, we obtain

$$\int_0^\infty t^{r+1}e^{-t}dt \int_0^\infty t^{r+a+b-1}e^{-t}dt \geq (\leq) \int_0^\infty t^{r+a}e^{-t}dt \int_0^\infty t^{r+b}e^{-t}dt,$$

i.e.,

$$\Gamma(r+2)\ \Gamma(a+b)\ E(X)^r \geq (\leq)\ \Gamma(a+1)\ E(U)^r\ \Gamma(b+1)\ E(V)^r],\ r = 1,2,\ldots$$

according as $(a-1)(b-1) \geq (\leq)\ 0$, and hence, the theorem.

■

Another result which establishes an inequality for moments of two gamma *r. v.'s* follows from this theorem as:

Corollary 3.2. *For $b = a > 0$, inequality* (3.3) *becomes*

$$\Gamma(2a)\Gamma(r+2)E(X)^r \geq \Gamma(a+1)E(U)^r,\ r = 1,2,... \qquad (3.5)$$

We obtain another estimation for moment ratios of two gamma *r. v.* [14] as:

Theorem 3.3. *Let the r.v. $X \sim G(a+b)$ and $Z \sim G(a-1)$. Then, if $(a-1)(b-1) \geq (\leq)\ 0$,*

$$\frac{E(X)^r}{E(Z)^r} \geq (\leq) \frac{\Gamma(a-1)\Gamma(b-1)}{\Gamma(a+b)},\ r = 1,2,... \qquad (3.6)$$

Proof. We choose

$$f(t) = t^{r+a-1},\ g(t) = t^{b-1} \text{ and } h(t) = te^{-t}. \qquad (3.7)$$

Then , substituting these mappings in the Čebyšev's integral inequality gives (3.6) and hence, completes the proof.

■

Another estimation of moment ratios of Gamma *r. v.'s* is [14]:

Theorem 3.4. *Let the r.v. X and Y be such that $X \sim G(p-k)$ and $Y \sim G(m+k)$, for the real numbers p, m, k such that $p, m > 0$ and $p > k > -m$. Further, let the r.v. U and V be defined as $U \sim G(p)$ and $V \sim G(m)$. Then, if $k(p-m-k) \geq (\leq) 0$,*

$$\frac{E(U)^r E(V)^r}{E(X)^r E(Y)^r} \geq (\leq) \frac{\Gamma(p-k)\Gamma(m+k)}{\Gamma(p)\Gamma(m)},\ r = 1,2,... \qquad (3.8)$$

Proof. Choose the mappings as

$$f(x) = x^{p-k-m},\ g(x) = x^k \text{ and } h(x) = x^{r+m-1}e^{-x}. \qquad (3.9)$$

Then, on substituting these mappings in inequality due to Čebyšev, we obtain the desired expression (3.8).

■

An interesting corollary arises from this theorem as :

Corollary 3.5. *For $m = p > 0$ in* (3.8),

$$\Gamma^2(p)E^2(U)^r \leq \Gamma(p-k)\Gamma(m+k)E(X)^r E(Y)^r,\ r = 1,2,... \qquad (3.10)$$

Another estimation of moment ratios of two Gamma *r.v.'s* follows as [14]:

Theorem 3.6. *Let the r.v. U and Y be as above, i.e.,* $U \sim G(p)$ *and* $Y \sim G(m+k)$. *Then, if* $k(p-m-k) \geq (\leq)\ 0$,

$$\Gamma(p)\Gamma(m)E(U)^r \geq (\leq)\Gamma(p-k)\Gamma(m+k)E(Y)^r,\ r = 1,2,... \qquad (3.11)$$

Proof. Consider the mappings

$$f(x) = x^{p-k-m},\ g(x) = x^{r+k} \text{ and } h(x) = x^{m-1}e^{-x}. \qquad (3.12)$$

Then , substitution of these mappings in the Čebyšev's integral inequality completes the proof.

■

Functional properties of gamma mappings [6],i.e.,

$$\Gamma(ax+by) \leq [\Gamma(x)]^a\ [\Gamma(y)]^b,$$

$a,b \geq 0$ *with* $a+b = 1$ *and* $x,y > 0$, *implying, that the mapping* Γ *is logarithmically convex on* $[0,\infty)$, result in the following estimation for moments of two gamma *r.v.'s*:

Theorem 3.7. *Let the r.v. U and V be such that* $U \sim G(ax+by)$ *and* $V \sim G(x)$, *where* $a,b \geq 0$ *with* $a+b = 1$ *and* $x,y > 0$. *Then*

$$\frac{E(U)^{ar}}{E^a(V)^r} \leq \frac{\Gamma^a(x)\Gamma^b(y)}{\Gamma(ax+by)},\ r = 1,2,... \qquad (3.13)$$

Proof. We choose

$$f(t) = t^{a(x-1)+ar},\ g(t) = t^{b(y-1)} \text{ and } h(t) = e^{-t}, \qquad (3.14)$$

for $t \in [0,\infty]$. On substituting these mappings in Hölder's integral inequality, we obtain

$$\int_0^\infty t^{ar+a(x-1)+b(y-1)}e^{-t}dt \leq [\int_0^\infty t^{r+(x-1)}e^{-t}dt]^a[\int_0^\infty t^{y-1}e^{-t}dt]^b,$$

i.e.,

$$\Gamma(ax+by)[E(U)^{ar}] \leq [\Gamma(x)E(V)^r]^a[\Gamma(y)]^b,\ r = 1,2,...,$$

and hence, the theorem.

■

Another result may be noted from this theorem as well :

Corollary 3.8. *For* $a,b(=a) > 0$,

$$[\Gamma\{a(x+y)\}E(U)^{ar}]^{\frac{1}{a}} \leq \Gamma(x)\Gamma(y)E(V)^r,\ r = 1,2,... \qquad (3.15)$$

The following theorem applying properties of Gamma functions also holds:

Theorem 3.9. *For $x, y \geq 0$ and $m > 0$, let there be Gamma r.v.'s $U \sim G(m)$, $V \sim G(x+y+m)$, $W \sim G(x+m)$ and $Z \sim G(y+m)$. Then,*

$$\frac{E(U)^r E(V)^r}{E(W)^r E(Z)^r} \geq \frac{\Gamma_m(x)\Gamma_m(y)}{\Gamma_m(x+y)},\ r = 1,2,... \qquad (3.16)$$

Proof. We choose the mappings

$$f(t) = t^y,\ g(t) = t^x \text{ and } h(t) = t^{r+m-1}e^{-t}, \qquad (3.17)$$

for $t \in [0,\infty)$.

On substituting these mappings in Čebyšev's integral inequality, we obtain

$$[\Gamma(m)E(U)^r][\Gamma(x+y+m)E(V)^r] \geq [\Gamma(x+m)E(W)^r][\Gamma(y+m)E(Z)^r],\ r = 1,2,...$$

and hence the theorem.

■

Following theorem provides estimation of moment ratios of two Gamma *r.v.'s*:

Theorem 3.10. *For $x, y \geq 0$ and $m > 0$, let there be gamma r.v.'s $V \sim G(x+y+m)$ and $Z \sim G(y+m)$. Then,*

$$\Gamma(m)\Gamma(x+y+m)E(V)^r \geq \Gamma(x+m)\Gamma(y+m)E(Z)^r,\ r = 1,2,... \qquad (3.18)$$

Proof. The mappings

$$f(t) = t^y,\ g(t) = t^{r+x} \text{ and } h(t) = t^{m-1}e^{-t},\ t \in [0,\infty),$$

in the Čebyšev's integral inequality results in the deisred inequalities.

■

3.2. Inequalities For Harmonic Means

We now discuss some estimations for the harmonic means of gamma *r.v.* which are based on the applications of Čebyšev's and Hölder's integral inequalities. The following theorem is on the moment ratios:

Theorem 3.11. *Let the r.v. $X \sim G(a+b)$ and $Z \sim G(a-1)$. Then, if $(a-1)(b-1) \geq (\leq) 0$,*

$$\frac{E(1/X)^r}{E(1/Z)^r} \geq (\leq) \frac{\Gamma(a-1)\Gamma(b-1)}{\Gamma(a+b)},\ r = 1,2,\ldots \qquad (3.19)$$

Proof. On substituting the mappings

$$f(t) = t^{-r+a-1},\ g(t) = t^{b-1} \text{ and } h(t) = te^{-t},\ t \in [0,\infty), \qquad (3.20)$$

in the Čebyšev's integral inequality, we have the desired estimation (3.19).

■

The following theorem also holds for the harmonic means of two gamma *r.v.'s* [14]:

Theorem 3.12. *Let the r.v. U and Y be such that* $U \sim G(p)$ *and* $Y \sim G(m+k)$. *Then, if* $k(p-m-k) \geq (\leq)\ 0$,

$$\frac{E(1/U)^r}{E(1/Y)^r} \geq (\leq) \frac{\Gamma(p-k)\Gamma(m+k)}{\Gamma(p)\Gamma(m)},\ r = 1,2,\ldots \qquad (3.21)$$

Proof. Choose the mappings,

$$f(x) = x^{p-k-m}, g(x) = x^{-r+k} \text{ and } h(x) = x^{m-1}e^{-x},$$

which applied to the Čebyšev's integral inequality results in (3.21).

■

An application of Hölder's integral inequality provides the following estimation for the harmonic means of two gamma *r.v.'s*:

Theorem 3.13. *Let the r.v.* $U \sim G(ax+by)$ *and* $V \sim G(x)$, *where* $a,\ b \geq 0$ *with* $a+b = 1$ *and* $x,y > 0$. *Then,*

$$\Gamma(ax+by)[E(\frac{1}{U})^{ra}] \leq [\Gamma(x)E(\frac{1}{V})^r]^a[\Gamma(y)]^b,\ r = 1,2,\ldots \qquad (3.22)$$

Proof. We consider mappings

$$f(t) = t^{a(x-1)+ar},\ g(t) = t^{b(y-1)} \text{ and } h(t) = e^{-t},\ t \in [0,\infty), \qquad (3.23)$$

and apply the Holder's integral inequality to prove the estimation (3.22).

■

Finally, we also have the following estimation from the application of the Čebyšev's integral inequality to Gamma functions:

Theorem 3.14. *Let the r.v.'s V and Z be such that* $V \sim G(x+y+m)$ *and* $Z \sim G(y+m)$.

Then,

$$\frac{E(\frac{1}{V})^r}{E(\frac{1}{Z})^r} \geq \frac{\Gamma_m(x)\Gamma_m(y)}{\Gamma_m(x+y)}, \; r = 1, 2, \ldots \qquad (3.24)$$

The proof follows from (3.18).

3.3. Inequalities For Variances

The Čebyšev's integral inequality for synchronous (asynchronous) mappings, Hölder's integral inequality and Grüss's integral inequality have been applied to obtain estimation of variances for gamma random variables. We start with the following result:

Theorem 3.15. *Let the r.v.'s* $X \sim G(a+b)$ *and* $Z \sim G(a-1)$. *Denote the variances of r.v.'s X and Z by* $V(X) = E(X)^2 - [E(X)]^2$ *and* $V(Z) = E(Z)^2 - [E(Z)]^2$, *respectively. Then, if* $(a-1)(b-1) \geq (\leq)\; 0$,

$$\Gamma(a+b)V(X) - \Gamma(a-1)\Gamma(b-1)V(Z) \geq (\leq)(a-1)\Gamma(a)\Gamma(b-1) - (a+b)\Gamma(a+b+1) \quad (3.25)$$

Proof. We consider (3.6)

$$\frac{E(X)^r}{E(Z)^r} \geq (\leq) \frac{\Gamma(a-1)\Gamma(b-1)}{\Gamma(a+b)}, \; r = 1, 2, \ldots$$

and choosing $r = 2$. On rewriting $E(.)$ in terms of $V(.)$, we get

$$[V(X) + E^2(X)]\Gamma(a+b) \geq (\leq)[V(Z) + E^2(Z)]\Gamma(a-1)\Gamma(b-1). \qquad (3.26)$$

Now, since X and Z are gamma r.v.'s with parameters $(a+b)$ and $(a-1)$, respectively, $E(X) = a+b$, and $E(Z) = a-1$. On substituting these values in (3.26), we reach at the desired inequality (3.25).

■

Another inequality for the coefficients of variation of *r.v.'s X* and *Z* follows immediately from this theorem as:

Corollary 3.16. *Denoting coefficients of variation of the r.v. X and Z by* $CV(X)$ *and* $CV(Z)$ *where* $CV(.) = \frac{\sqrt{V(.)}}{E(.)}$, *the inequalities for* $CV(X)$ *and* $CV(Z)$ *is*

$$\frac{1+CV^2(X)}{1+CV^2(Z)} \geq (\leq) \frac{(a-1)\Gamma(a)\Gamma(b-1)}{(a+b)\Gamma(a+b+1)}. \qquad (3.27)$$

according as $(a-1)(b-1) \geq (\leq)\ 0.$

Proof. From (3.26), we have

$$\frac{\frac{V(X)}{E^2(X)}+1}{\frac{V(Z)}{E^2(Z)}+1} \geq (\leq) \frac{\Gamma(a-1)\Gamma(b-1)}{\Gamma(a+b)} \cdot \frac{E^2(Z)}{E^2(X)}.$$

Since $E(X) = a+b,$ and $E(Z) = a-1,$ we reach at (3.27) and hence the corollary. ■

The following result also holds for variances of two gamma *r.v.'s* [14]:

Theorem 3.17. *Let the r.v.'s* $U \sim G(p)$ *and* $Y \sim G(m+k)$. *Denote the variances of r.v.'s* U *and* Y *by* $V(U) = E(U)^2 - [E(U)]^2$ *and* $V(Y) = E(Y)^2 - [E(Y)]^2,$ *respectively. Then, if* k $(p-m-k) \geq (\leq)\ 0,$

$$\Gamma(p)\Gamma(m)V(U) - \Gamma(p-k)\Gamma(m+k)V(Y) \geq (\leq)\ (m+k)\Gamma(p-k)\Gamma(m+k+1) - p\Gamma(m)\Gamma(p+1). \qquad (3.28)$$

Proof. We consider inequality (3.11)

$$\Gamma(p)\Gamma(m)E(U)^r \geq (\leq)\Gamma(p-k)\Gamma(m+k)E(Y)^r,\ r = 1,2,...$$

and choose $r = 2$. Rewriting $E(.)$ in terms of $V(.)$, we obtain

$$[V(U)+E^2(U)]\Gamma(p)\Gamma(m) \geq (\leq)[V(Y)+E^2(Y)]\Gamma(p-k)\Gamma(m+k). \qquad (3.29)$$

Now substituting $E(U) = p$ and $E(Y) = m+k$ in the above expression (3.29), we reach at the desired inequality (3.28).

Another estimation for the coefficients of variation of *r.v.'s* U and Y follows from this theorem as :

Corollary 3.18. *Denoting coefficients of variation of the r.v.* U *and* Y *by* $CV(U)$ *and* $CV(Y)$ *where* $CV(.) = \frac{\sqrt{V(.)}}{E(.)}$ *, the inequality for* $CV(U)$ *and* $CV(Y)$, *for* $k(p-m-k) \geq (\leq)\ 0,$ *is*

$$\frac{1+CV^2(U)}{1+CV^2(Y)} \geq (\leq) \frac{(m+k)\Gamma(m+k+1)\Gamma(p-k)}{p\Gamma(m)\Gamma(p+1)}. \qquad (3.30)$$

Proof. Rewriting (3.29) as

$$\frac{\frac{V(U)}{E^2(U)}+1}{\frac{V(Y)}{E^2(Y)}+1} \geq (\leq) \frac{\Gamma(m+k)\Gamma(p-k)}{\Gamma(m+q)\Gamma(p+n)} \cdot \frac{E^2(Y)}{E^2(U)},$$

and since $E(U) = p, E(Y)(m+k),$ we prove the corollary.

■

The following inequality using properties of gamma functions also holds for variances of gamma *r.v.'s* [6]:

Theorem 3.19. *Let the r.v.'s* $U \sim G(ax+by)$ *and* $V \sim G(x)$ *be as above. Denote the variances of r.v.'s U and V by* $Var(U) = E(U)^2 - [E(U)]^2$ *and* $Var(V) = E(V)^2 - [E(V)]^2$, *respectively. Then,*

$$\Gamma(ax+by)[Var(U^a)+E^2(U^a)] \leq \Gamma^a(x)\Gamma^b(y)[Var(V)+x^2]^a. \qquad (3.31)$$

Proof. We consider the inequality (3.13). Choose $r = 2$ and rewrite $E(.)$ in terms of $Var(.)$. On substituting $E(V) = x$ in (3.13), we reach at the desired inequality (3.31).

■

Finally, we can also state another inequality for the variances of two gamma $r.v's$:

Theorem 3.20. *Let the r.v.'s* $V \sim G(x+y+m)$ *and* $Z \sim G(y+m)$ *be as above. Denote the variances of r.v.'s V and Z by* $Var(V) = E(V)^2 - [E(V)]^2$ *and* $Var(Z) = E(Z)^2 - [E(Z)]^2$, *respectively. Then,*

$$\frac{Var(V)+(x+y+m)^2}{Var(Z)+(y+m)^2} \geq \frac{\Gamma(x+m)\Gamma(y+m)}{\Gamma(x+y+m)}. \qquad (3.32)$$

Proof. We consider the inequality (3.18), choose $r = 2$ and rewrite $E(.)$ in terms of $Var(.)$. On substituting $E(V) = x+y+m,$ and $E(Z) = y+m,$ in (3.18), we prove the theorem.

■

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