Mathematical Inequalities with Applications to the Beta and Gamma Mappings - I

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Abstract

Some inequalities for beta and gamma functions, using fundamental inequalities, such as Čebyšev's integral inequality for synchronous (asynchronous) mappings, Hölder's integral inequality and Grüss's integral inequality, are presented. Consistent applications of these inequalities to the beta and gamma mappings are considered.

Key words and phrases: Čebyšev's integral inequality, Hölder's integral inequality, Grüss's integral inequality, Beta function, Gamma function.

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1. INTRODUCTION

We follow Weierstrass [8,p.9] in defining the function $\Gamma(z)$:

$$\frac{1}{\Gamma(z)} = \frac{1}{\Gamma(z)} ze^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(-\frac{z}{n}\right) \right], \qquad (1.1)$$

in which γ is Euler's constant, i.e., $\gamma = \lim_{n\to\infty} (H_n - \log n)$ and $Hn = \sum_{k=1}^{n} \frac{1}{k}$.

Also is known that the function $\Gamma(z)$ in (1.1) is identical with the Euler's integral, i.e.,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, Re(z) > 0. \qquad (1.2)$$

Further it is known that [8, p. 11]:

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right], \qquad (1.3)$$

which is the Euler's product for $\Gamma(z)$. Note that for real $x > 0, \Gamma(x) > 0$.

An important functional identity for Γ is [8, p. 12]:

$$\Gamma(z+1) = z\Gamma(z), \qquad (1.4)$$

which, in particular, gives

$$\Gamma(m+1) = m!, \qquad (1.5)$$

for any positive integer *m*.

Some other important properties of Γ mappings [8, p.21] are:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{2^{z-1}n(\pi 2)}, \qquad (1.6)$$

where z is non-integral.

We have Legendre's duplication formula [8, p. 244],

$$\Gamma(2z) = \frac{2}{\sqrt{\delta\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \qquad (1.7)$$

and

$$\Gamma(3z) = \frac{1}{2\pi}3^{3z-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right), \qquad (1.8)$$

also

$$\Gamma(z) = \lim_{k\to\infty}\frac{1.2.3...k}{z(z+1)...(z+k)}k^2, \qquad (1.9)$$

where $z \neq 0, -1, -2, ...$ [8, p. 244].

Finally, we have the Gauss multiplication theorem [8, p. 26]:

$$\prod_{s=1}^{k}\Gamma\left(z+\frac{\delta-1}{k}\right) = (2\pi)^{\frac{1}{2}(k-1)}k^{\frac{1}{2}-kz}\Gamma(kz). \qquad (1.10)$$

We define the beta function [8, p. 18]:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, Re(p) > 0, Re(q) > 0. \qquad (1.11)$$

Another useful form for this function obtained by putting $t = \min^2\theta$ is:

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} \overset{2p-1}{\min}\theta \cos^{2q-1} d\theta, Re(p) > 0, Re(q) > 0. \qquad (1.12)$$

The connection between Γ and B is given by:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad (1.13)$$

for $Re(p) > 0$ and $Re\,(q) > 0$.

Some other functional properties of Beta and Gamma mappings are [8, p. 31]:

$$pB(p,q+1) = qB(p+1,q), \qquad (1.14)$$

$$B(p,q) = B(p+1,q) + B(p,q+1), \qquad (1.15)$$

$$(p+q)B(p,q+1) = qB(p,q), \qquad (1.16)$$

$$B(p,q)B(p+q,r) = B(q,r)B(q+r,p), \qquad (1.17)$$

For other properties of the mappings of the Euler's beta, reference is made to the Chapter 2 of the classical book by E.D. Rainville [8].

The main aim of this survey paper is to point out some inequalities for beta and gamma mappings in the case when p, q, z are positive real numbers by using some fundamental inequalities such as: Čebyšev's integral inequality or synchronous (asynchronous) mappings, Hölder's integral inequality and Grüss integral inequality.

2. Inequalities via Čebyšev's Integral Inequality

We start with the following result [2]:

Theorem 2.1. *Let m, n, p, and q be positive real numbers, such that* $(p-m)(q-n) \le (\ge)\ 0$. *Then,*

$$B(p,q)B(m,n) \ge (\le) B(p,n)B(m,q), \qquad (2.1)$$

and

$$\Gamma(p+n)\Gamma(q+m) \le (\ge) \Gamma(p+q)\Gamma(m+n). \qquad (2.2)$$

Proof. Define the mappings : $f, g, h : [0,1] \to [0,\infty)$, given by

$$f(x) = x^{p-m},\ g(x) = (1-x)^{q-n} \text{ and } h(x) = x^{m-1}(1-x)^{n-1}. \qquad (2.3)$$

As $(p-m)(q-n) \leq (\geq) 0$, the mappings f and g are the same (opposite) monotonic on $[0,1]$ and h is non-negative on $[0,1]$.

Applying the well known Čebyšev's integral inequality for synchronous (asynchronous) mappings [5] , i.e.,

$$\int_a^b h(x)dx \int_a^b h(x)f(x)g(x)dx \geq (\leq) \int_a^b h(x)f(x)dx \int_a^b h(x)g(x)dx, \qquad (2.4)$$

according as $(p-m)(q-n) \leq (\geq)\ 0$, we can write the inequality

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}(1-x)^{q-n}dx \geq (\leq)$$
$$\int_0^1 x^{m-1}(1-x)^{n-1}x^{p-m}dx \int_0^1 x^{m-1}(1-x)^{n-1}(1-x)^{q-n}dx,$$

i.e.,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx \int_0^1 x^{p-1}(1-x)^{q-1}dx \geq (\leq)$$
$$\int_0^1 x^{p-1}(1-x)^{n-1}dx \int_0^1 x^{m-1}(1-x)^{q-1}dx. \qquad (2.5)$$

Now using (1.11), the inequality (2.1) is proved.

□

The inequality (2.2) is established from (2.1) by taking into account (1.13) that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

for all $p,q > 0$. We shall omit the details.

The following interesting corollaries from Theorem 2.1 may be noted as well:

Corollary 2.2. *Let $p,m > 0$. Then, we have the inequalities*

$$B(p,p)B(m,m) \leq B^2(p,m), \qquad (2.6)$$

and

$$\Gamma^2(p+m) \leq \Gamma(2p)\Gamma(2m). \qquad (2.7)$$

Proof. In the above theorem 2.1, if we choose $q = p$, and $n = m$, we have

$(p-m)(q-n) = (p-m)^2 \geq 0$, and thus,

$$B(p,p)B(m,m) \leq B(p,m)B(m,p)$$

which proves the inequality (2.6).The inequality (2.7) follows from (2.6) through (1.13).

□

Corollary 2.3. *For two positive real numbers* $u, v > 0$, *the geometric mean of* $\Gamma(u)$ *and* $\Gamma(v)$ *is greater than or equal to* $\Gamma($*arithmetic mean of* u *and* $v)$.

By setting $2p = u$ and $2m = v$ in (2.7), we obtain

$$\Gamma\left(\frac{u+v}{2}\right) \leq \sqrt{\Gamma(u)\Gamma(v)},$$

and hence, the corollary.

We continue with the following theorem [6]:

Theorem 2.4. *Let* m, p, *and* k *be real numbers with* $m, p > 0$, *and* $p > k > -m$. *Then, if* $k(p-m-k) \geq (\leq)\ 0$, *we have*

$$\Gamma(p)\,\Gamma(m) \geq (\leq)\ \Gamma(p-k)\,\Gamma(m+k), \qquad (2.8)$$

and

$$B(p,m) \geq (\leq)\ B(p-k, m+k). \qquad (2.9)$$

Proof. Define the mappings

$$f(x) = x^{p-k-m},\ g(x) = x^k \text{ and } h(x) = x^{m-1}e^{-x}, \qquad (2.10)$$

for $x \in [0,\infty)$. As the mappings f and g are similarly (oppositely) ordered and h is non-negative, we apply the well known Čebyšev's integral inequality for synchronous (asynchronous) mappings , i.e., the inequality (2.4). Then, for $k(p-m-k) \geq (\leq)\ 0$, we can write the inequality

$$\int_0^\infty x^{m-1}e^{-x}dx \int_0^\infty x^{p-k-m}x^k x^{m-1}e^{-x}dx \geq (\leq)$$
$$\int_0^\infty x^{p-k-m}x^{m-1}e^{-x}dx \int_0^\infty x^k x^{m-1}e^{-x}dx, \qquad (2.11)$$

i.e.,

$$\int_0^\infty x^{m-1}e^{-x}dx \int_0^\infty x^{p-1}e^{-x}dx \geq (\leq) \int_0^\infty x^{p-k-1}e^{-x}dx \int_0^\infty x^{m+k-1}e^{-x}dx,$$

hence, the inequality (2.8).

On the other hand, since

$$B(p,m) = \frac{\Gamma(p)\Gamma(m)}{\Gamma(p+m)} \text{ and } B(p-k,m+k) = \frac{\Gamma(p-k)\Gamma(m+k)}{\Gamma(p+m)},$$

we deduce the inequality (2.9) from (2.8).

□

The following corollaries arise from Theorem 2.4 and may be noted as well:

Corollary 2.5. *Let* $p > 0$ *and* $q \in R$ *with* $|\ q\ | < p$. *Then , we have the inequalities*

$$\Gamma^2(p) \leq \Gamma(p-q)\ \Gamma(p+q), \qquad (2.12)$$

and

$$B(p,p) \leq B(p-q,p+q). \qquad (2.13)$$

Proof. In the above Theorem 2.4, if we choose $m = p$, and $q = k$, we have $k(p-m-k) \leq 0$, and thus, the inequality (2.12) is proved.

□

The inequality (2.13) follows from (2.9).

Corollary 2.6. *Let* p *and* q *be as above*. *Then, the geometric mean of* $\Gamma(p+q)$ *and* $\Gamma(p-q)$ *is greater than* $\Gamma[$*arithmetic mean of* $(p+q)$ *and* $(p-q)]$.

Proof. From (2.12), we have

$$\Gamma(p) = \Gamma(\frac{(p-q)+(p+q)}{2}) \leq \sqrt{\Gamma(p-q)\Gamma(p+q)}, \qquad (2.14)$$

and hence, the corollary.

□

Let us consider the following definition :

Definition 2.7. The positive real numbers a and b will be called *similarly* (*oppositely*) *unitary*, if $(a-1)(b-1) \geq (\leq)\ 0$.

We now prove the following theorem:

Theorem 2.8. *Let* $a, b > 0$ *be similarly* (*oppositely*) *unitary*. *Then, we have the inequalities*

$$\Gamma(a+b) \geq (\leq)\ ab\ \Gamma(a)\ \Gamma(b), \qquad (2.15)$$

and

$$B(a,b) \leq (\geq)\ \frac{1}{ab}. \qquad (2.16)$$

Proof. Define the mappings

$$f(t) = t^{a-1},\ g(t) = t^{b-1} \text{ and } h(t) = te^{-t}, \qquad (2.17)$$

for $t \in [0,\infty)$. As the mappings f and g are *similarly* (*oppositely*) ordered and h is non-negative, by applying the well known Čebyšev's integral inequality for synchronous (asynchronous) mappings , i.e.,

$$\int_l^u h(t)dt \int_l^u h(t)f(t)g(t)dt \geq (\leq) \int_l^u h(t)f(t)dt \int_l^u h(t)g(t)dt,$$

for $(a-1)(b-1) \geq (\leq)\ 0,$ we can write the inequality

$$\int_0^\infty te^{-t}dt \int_0^\infty t^{a+b-1}e^{-t}dt \geq (\leq) \int_0^\infty t^a e^{-t}dt \int_0^\infty t^b e^{-t}dt, \qquad (2.18)$$

i.e.,

$$\Gamma(2)\ \Gamma(a+b) \geq (\leq)\ \Gamma(a+1)\ \Gamma(b+1).$$

Now using the functional identities (1.4) and (1.5) of $\Gamma,$ we have

$$\frac{\Gamma(a+1)\ \Gamma(b+1)}{\Gamma(2)} = ab\ \Gamma(a)\ \Gamma(b),$$

and hence, the inequality (2.15).

On the other hand, since $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$ we obtain the inequality (2.16) from (2.15).

□

The following interesting corollaries may also be seen from Theorem 2.8:

Corollary 2.9. *The mapping ln* Γ *is superadditive on the interval* $[1,\infty)$.

Proof. For every $a,b \in [1,\infty)$ in Theorem 2.8, we have from (2.15)

$$\ln\Gamma(a+b) \geq \ln a + \ln b + \ln\Gamma(a) + \ln\Gamma(b) \geq \ln\Gamma(a) + \ln\Gamma(b), \qquad (2.19)$$

and hence, the corollary 2.9.

□

Corollary 2.10. *Let a and b be as above. Then, it follows from* (2.19) *that the arithmetic mean of ln* $\Gamma(a)$ *and ln* $\Gamma(b)$ *has the upper bound ln* $[\Gamma(a+b)]^{\frac{1}{2}}$.

Corollary 2.11. *For every* $n \in N, n \geq 1$ *and* $a > 0,$ *we have*

$$\Gamma(na) \geq (n-1)!\ a^{2(n-1)}\ [\Gamma(a)]^n. \qquad (2.20)$$

Proof. We can write from (2.15)

$$\Gamma(2a) \geq a^2\ \Gamma(a)\ \Gamma(a)$$

$$\Gamma(3a) \geq 2\ a^2\ \Gamma(2a)\ \Gamma(a)$$

....

$$\Gamma(na) \geq (n-1)!\, a^2\, \Gamma[(n-1)a]\, \Gamma(a).$$

By multiplying these inequalities, we reach at (2.20).

□

Corollary 2.12. *For all* $a > 0$, *we obtain*

$$\Gamma(a) \leq \frac{2^{2a-1}}{\sqrt{\pi}\, a^2}\, \Gamma(a + \frac{1}{2}). \qquad (2.21)$$

Proof. We refer to the identity [4, p. 45]:

$$2^{2a-1}\, \Gamma(a)\, \Gamma(a + \frac{1}{2}) = \sqrt{\pi}\, \Gamma(2a), \quad a > 0.$$

Since $\Gamma(2a) \geq a^2\, \Gamma^2(a)$, we reach at

$$2^{2a-1}\, \Gamma(a)\, \Gamma(a + \frac{1}{2}) \geq \sqrt{\pi}\, \Gamma(2a), \quad a > 0,$$

and hence, the desired inequality (2.21).

□

3. Inequalities via Hölder's Inequality

We now prove the following result for gamma functions [5]:

Theorem 3.1. *Let* $a, b \geq 0$ *with* $a + b = 1$ *and* $x, y > 0$. *Then,*

$$\Gamma(ax + by) \leq [\Gamma(x)]^a\, [\Gamma(y)]^b, \qquad (3.1)$$

implying, that the mapping Γ *is logarithmically convex on* $[0,\infty)$.

Proof. Define the non-negative mappings f, g and h given by

$$f(t) = t^{a(x-1)},\ g(t) = t^{b(y-1)} \text{ and } h(t) = e^{-t}, \text{ for } t \in [0,\infty). \qquad (3.2)$$

Denote by $p = \frac{1}{a}$ and $q = \frac{1}{b}$. Then $p \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Now applying the Hölder's integral inequality for p and q as above, we can write

$$\int_l^u f(t)g(t)h(t)dt \leq [\int_l^u \{f(t)\}^{\frac{1}{a}} h(t)dt]^a\, [\int_l^u \{g(t)\}^{\frac{1}{b}} h(t)dt]^b, \qquad (3.3)$$

i.e.,

$$\int_0^\infty t^{a(x-1)+b(y-1)} e^{-t} dt \leq [\int_0^\infty t^{x-1} e^{-t} dt]^a [\int_0^\infty t^{y-1} e^{-t} dt]^b,$$

and hence, the inequality (3.1), which implies that the mapping Γ is logarithmically convex on $[0,\infty)$.

□

Remark 3.2. Γ being logarithmically convex on $[0,\infty)$ is obviously convex on $[0,\infty)$.

We now prove the following theorems for beta functions:

Theorem 3.3. *The mapping B is logarithmically convex on* $[0,\infty)^2$ *as a function of two variables.*

Proof. Let $(p,q),\ (m,n) \in [0,\infty)^2$ and $a,b \geq 0,$ with $a+b=1$. Then

$$B[a(p,q)+b(m,n)] \ = B(ap+bm, aq+bn)$$

$$= \int_0^1 t^{ap+bm-1}(1-t)^{aq+bn-1}dt = \int_0^1 t^{a(p-1)+b(m-1)}(1-t)^{a(q-1)+b(n-1)}dt$$

$$= \int_0^1 [t^{p-1}(1-t)^{q-1}]^a[t^{p-1}(1-t)^{q-1}]^b dt. \qquad (3.4)$$

Define the non-negative mappings

$$f(t) = [t^{p-1}(1-t)^{q-1}]^a,\ g(t) = [t^{m-1}(1-t)^{n-1}]^b \text{ and } h(t) = 1,$$

for $p = \frac{1}{a},\ q = \frac{1}{b},\ (\frac{1}{p} + \frac{1}{q} = 1$ and $p \geq 1)$.

Now applying the Hölder's integral inequality (3.3), we have

$$\int_0^1 [t^{p-1}(1-t)^{q-1}]^a[t^{p-1}(1-t)^{q-1}]^b dt$$

$$\leq [\int_0^1 t^{p-1}(1-t)^{q-1}dt]^a[\int_0^1 t^{m-1}(1-t)^{n-1}dt]^b, \qquad (3.5)$$

and, thus, from (1.11)

$$B[a(p,q)+b(m,n)] \ \leq [B(p,q)]^a\ [B(m,n)]^b, \qquad (3.6)$$

which shows the logarithmic convexity of B on $[0,\infty)^2$.

□

Now the following result on the logarithmic derivative of the Γ function [3]:

Theorem 3.4. *Define the mapping* $\Psi : [0,\infty) \to [0,\infty)$ *, given by* $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ *and called it the logarithmic derivative of the* Γ *function*. *Then,* Ψ *is monotonic non-decreasing and concave on* $[0,\infty)$.

Proof. In Theorem 3.1, we have proved that $\Gamma(ax+by) \leq [\Gamma(x)]^a\ [\Gamma(y)]^b$, where $x,y > 0$ and $a,b \geq 0$ with $a+b=1$, which shows that the mapping $\ln \Gamma$ is convex. From this result follows that its derivative is monotonic non-decreasing. Further, since $\frac{d}{dt}[\ln\Gamma(t)] \ = \frac{\Gamma'(t)}{\Gamma(t)} = \Psi(t),\ t > 0$, the monotonicity of Ψ is proved.

To prove the concavity of Ψ, we use the following known representation of Ψ [4, p. 21]:

$$\Psi(x) + \gamma = \int_0^1 \frac{1 - t^{x-1}}{1 - t} dt$$

for $x > 0$, where γ is the Euler's constant.

Now, let $x, y > 0$ and $a, b \geq 0$ with $a + b = 1$. Then

$$\Psi(ax + by) + \gamma = \int_0^1 \frac{1 - t^{ax+by-1}}{1 - t} dt = \int_0^1 \frac{1 - t^{a(x-1)+b(y-1)}}{1 - t} dt.$$

As the mapping $R \ni x \longmapsto a^x \in [0, \infty)$ is convex for $a \in (0, 1)$, we have

$$t^{a(x-1)+b(y-1)} \leq at^{x-1} + bt^{y-1}, \ for\ t \in [0, 1],\ x, y > 0.$$

Thus,

$$\int_0^1 \frac{1 - t^{ax+by-1}}{1 - t} dt \geq \int_0^1 \frac{1 - (at^{x-1} + bt^{y-1})}{1 - t} dt$$

$$= \int_0^1 \frac{a(1 - t^{x-1}) + b(1 - t^{y-1})}{1 - t} dt = a \int_0^1 \frac{1 - t^{x-1}}{1 - t} dt + b \int_0^1 \frac{1 - t^{y-1}}{1 - t} dt$$

$$= a[\Psi(ax + by) + \gamma] + b[\Psi(ax + by) + \gamma] = a\Psi(x) + b\Psi(y) + \gamma,$$

from where follows the concavity of the mapping Ψ.

□

We now present the following for gamma functions [7]:

Theorem 3.5. *For $m > 0$ and $x, y \geq 0$,*

$$\Gamma(m)\ \Gamma(x + y + m) \geq \Gamma(x + m)\ \Gamma(y + m), \qquad (3.7)$$

implying, that the mapping Γ_m is supremultiplicative on $[0, \infty)$, where

$$\Gamma_m(z) = \frac{\Gamma(z + m)}{\Gamma(m)}. \qquad (3.8)$$

Proof. Define the mappings $f(t) = t^x$ and $g(t) = t^y$, which are monotonic non-decreasing on $[0, \infty)$ and $h(t) = t^{m-1}e^{-t}$, non-negative on $[0, \infty]$, for $t \in [0, \infty)$.

Now applying the Čebyšev's integral inequality for *f* and *g* with the weight *h*, we can write

$$\int_0^\infty t^{m-1}e^{-t}dt \int_0^\infty t^{x+y+m-1}e^{-t}dt \geq \int_0^\infty t^{x+m-1}e^{-t}dt \int_0^\infty t^{y+m-1}e^{-t}dt, \qquad (3.9)$$

and hence, the inequality (3.7) and the desired result that the mapping Γ_m is supremultiplicative on $[0, \infty)$.

Using (3.8), the inequality (3.7) can be expressed in terms of the mapping Γ_m as

$$\Gamma_m(x + y) \geq \Gamma_m(x)\Gamma_m(y). \qquad (3.10)$$

□

4. Some Properties of the Mapping $l_{a,b}$

Let $a,b > 0$ and $l_{a,b} : [0,1] \to R,\ l_{a,b}(x) = x^a(1-x)^b$. Then

$$l'_{a,b}(x) = x^{a-1}(1-x)^{b-1}[a-(a+b)x], \qquad (4.1)$$

$$m_{a,b} := \inf_{x\in[a,b]} l_{a,b}(x) = 0, \qquad (4.2)$$

and

$$M_{a,b} := \sup_{x\in[a,b]} l_{a,b}(x) = l_{a,b}\left(\frac{a}{a+b}\right) = \frac{a^a b^b}{(a+b)^{a+b}}. \qquad (4.3)$$

Also, we have

$$\|l_{a,b}\|_\infty = \frac{a^a b^b}{(a+b)^{a+b}}, a,b > 0, \qquad (4.4)$$

$$\|l_{a,b}\|_\infty = B(a+1,b+1), a,b > 0, \qquad (4.5)$$

and

$$\|l_{a,b}\|_p = [B(pa+1,\ pb+1)]^{\frac{1}{p}}, p > 1, a,b > 0. \qquad (4.6)$$

Now observe that

$$\begin{aligned}\|l'_{a,b}(x)\| &\le x^{a-1}(1-x)^{b-1}\ \mid\ a-(a+b)x\ \mid \\ &\le \max\{a,b\} l_{a-1,b-1}(x), a,b > 0, x \in (0.1).\end{aligned}$$

Then we have the estimations

$$\|l'_{a,b}\|_\infty = \max\{a,b\}\frac{(a-1)^{a-1}(b-1)^{b-1}}{(a+b-2)^{a+b-2}},\ if\, a,b > 1, \qquad (4.7)$$

$$\|l'_{a,b}\|_1 = \max\{a,b\}B(a,b),\ if\, a,b > 0, \qquad (4.8)$$

$$\|l'_{a,b}\|_p = \max\{a,b\}[B(p(a-1)+1,\ p(b-1)+1)]^{\frac{1}{p}},\ if\, p > 1 \text{ and } a,b > 1 \qquad (4.9).$$

Now let observe that

$$\begin{aligned}l''_{a,b}(x) &= [l_{a-1,b-1}(x)]'[a-(a+b)x] - l_{a-1,b-1}(x)(a+b) \\ &= l_{a-2,b-2}(x)\ [a-1-(a-1+b-1)x] - l_{a-1,b-1}(x)(a+b) \\ &= l_{a-2,b-2}(x)\ [(a+b)x^2 - 2(a+b-1)x + a - 1].\end{aligned}$$

Consider the mapping $g_{a,b} : [0,1) \to R$ given by

$$g_{a,b}(x) := (a+b)x^2 - 2(a+b-1)x + a - 1.$$

We have

$$g_{a,b}(0) = a - 1 \text{ and } g_{a,b}(1) = 1 - b.$$

If $a > 1, b > 1$, then $g_{a,b}$ has a solution on the interval $(0,1)$ and other one in $(1,\infty)$. Also the coordinates of the vertices are

$$x_v = \frac{2(a+b-1)}{2(a+b)} = \frac{a+b-1}{a+b} < 1,$$

$$y_v = -\frac{b^2+ab-a-b+1}{a+b} = -(b-1+\frac{1}{a+b}).$$

Consequently

$$|\ g_{a,b}(x)\ | \le \max\{g_{a,b}(a), |\ y_v\ |\} = \max\{a-1, b-1+\frac{1}{a+b}\} = \max\{a, b+\frac{1}{a+b}\} - 1,$$

and then we get

$$\|l''_{a,b}(x)\| \le [\max\{a, b+\frac{1}{a+b}\} - 1]\ l_{a-2,b-2}(x),\ a,b > 1,\ x \in (0,1). \qquad (4.10)$$

If $a, b > 2$, we have

$$\|l''_{a,b}\|_\infty \le [\max\{a, b+\frac{1}{a+b}\} - 1]\ \frac{(a-2)^{b-2}(b-2)^{b-2}}{(a+b-4)^{a+b-4}}. \qquad (4.11)$$

From (4.10), if $a, b > 1$, we get

$$\|l''_{a,b}\|_1 \le [\max\{a, b+\frac{1}{a+b}\} - 1]\ B(a-1, b-1), \qquad (4.12)$$

and, if $a, b > 2$,

$$\|l''_{a,b}\|_p \le [\max\{a, b+\frac{1}{a+b}\} - 1]\ [B(p(a-2)+1, p(b-2)+1)]^{\frac{1}{p}}. \qquad (4.13)$$

5. Grüss's Integral Inequality and Beta and Gamma Mappings

Grüss (1935) established an integral inequality which provides an estimation for the integral of a product in terms of the product of integrals [5, p. 296]. We provide the inequality with its proof in the following lemma:

Lemma 5.1. *Let f and g be two functions defined and integrable on $[a,b]$. If*

$$\Psi \le f(x) \le \Phi, \tau \le g(x) \le \Gamma, \text{ for each } x \in [a,b], \qquad (5.1)$$

where Ψ, Φ, τ and Γ are given real constants, then

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx \right|$$
$$\leq \frac{1}{4}(\Phi - \Psi)(\Gamma - \tau), \qquad (5.2)$$

and the constant $\frac{1}{4}$ *is the best possible*.

Proof. Let us note that the following equality is valid :

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b [f(x)-f(y)][g(x)-g(y)]dxdy.$$

Applying Cauchy-Buniakowski-Schwartz's integral inequality for double integrals, we have

$$\left[\frac{1}{2(b-a)^2}\int_a^b\int_a^b \{f(x)-f(y)\}\{g(x)-g(y)\}dxdy\right]^2$$
$$\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b \{f(x)-f(y)\}^2 dxdy \times$$
$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b \{g(x)-g(y)\}^2 dxdy$$
$$= \left[\frac{1}{(b-a)}\int_a^b f^2(x)dx - \left\{\frac{1}{(b-a)}\int_a^b f(x)dx\right\}^2\right] \times$$
$$\left[\frac{1}{(b-a)}\int_a^b g^2(x)dx - \left\{\frac{1}{(b-a)}\int_a^b g(x)dx\right\}^2\right] \qquad (5.3)$$

The following equality also holds :

$$\frac{1}{(b-a)}\int_a^b f^2(x)dx - \left\{\frac{1}{(b-a)}\int_a^b f(x)dx\right\}^2$$
$$= \left[\Phi - \frac{1}{(b-a)}\int_a^b f(x)dx\right]\left[\frac{1}{(b-a)}\int_a^b f(x)dx - \Psi\right] -$$
$$\frac{1}{(b-a)}\int_a^b [\Phi - f(x)][f(x)-\Psi]dx.$$

Since by (5.1), $[\Phi - f(x)][f(x)-\Psi] \geq 0$, for each $x \in [a,b]$, we have

$$\frac{1}{(b-a)}\int_a^b f^2(x)dx - \left\{\frac{1}{(b-a)}\int_a^b f(x)dx\right\}^2 \leq$$
$$\left[\Phi - \frac{1}{(b-a)}\int_a^b f(x)dx\right]\left[\frac{1}{(b-a)}\int_a^b f(x)dx - \Psi\right], \qquad (5.4)$$

and

$$\frac{1}{(b-a)}\int_a^b g^2(x)dx - \{\frac{1}{(b-a)}\int_a^b g(x)dx\}^2 \leq$$
$$[\Gamma - \frac{1}{(b-a)}\int_a^b g(x)dx][\frac{1}{(b-a)}\int_a^b g(x)dx - \gamma]. \qquad (5.5)$$

Now from (5.3) through (5.5),

$$\left|\ \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\ \right|$$
$$\leq [\Phi - \frac{1}{(b-a)}\int_a^b f(x)dx][\frac{1}{(b-a)}\int_a^b f(x)dx - \Psi]$$
$$[\Gamma - \frac{1}{(b-a)}\int_a^b g(x)dx][\frac{1}{(b-a)}\int_a^b g(x)dx - \gamma]. \qquad (5.6)$$

Using the elementary inequality for real numbers $4pq \leq (p+q)^2,\ p,q \in R,$ we can state that

$$4[\Phi - \frac{1}{(b-a)}\int_a^b f(x)dx][\frac{1}{(b-a)}\int_a^b f(x)dx - \Psi] \leq (\Phi - \Psi)^2, \qquad (5.7)$$

and

$$4[\Gamma - \frac{1}{(b-a)}\int_a^b g(x)dx][\frac{1}{(b-a)}\int_a^b g(x)dx - \gamma] \leq (\Gamma - \gamma)^2. \qquad (5.8)$$

Combining (5.6) with (5.7) and (5.8), we prove the lemma.

□

Now the following application of the Grüss integral inequality for the beta mappings holds [1]:

Theorem 5.2. *Let m,n,p and $q > 0$. Then holds the inequality for the beta mappings*

$$|\ B(m+p+1,n+q+1) - B(m+1,n+1)B(p+1,q+1)\ |$$
$$\leq \frac{1}{4}\cdot\frac{p^pq^q}{(p+q)^{p+q}}\cdot\frac{m^mn^n}{(m+n)^{m+n}}. \qquad (5.9)$$

Proof. Consider the mappings

$$l_{m,n}(x) = x^m(1-x)^n,\ l_{p,q}(x) = x^p(1-x)^q, x \in [0,1]. \qquad (5.10)$$

Then from (4.3),we have

$$0 \leq l_{m,n}(x) \leq \frac{m^mn^n}{(m+n)^{m+n}}, x \in [0,1],$$

and

$$0 \leq l_{p,q}(x) \leq \frac{p^pq^q}{(p+q)^{p+q}}, x \in [0,1].$$

Applying the Gruss's integral inequality for the mappings $l_{m,n}(x)$ and $l_{p,q}(x)$, we get

$$| \int_0^1 l_{m+p,n+q}(x)dx - \int_0^1 l_{m,n}(x)dx \cdot \int_0^1 l_{p,q}(x)dx \ |$$
$$\leq \frac{1}{4} \cdot \frac{p^p q^q}{(p+q)^{p+q}} \cdot \frac{m^m n^n}{(m+n)^{m+n}},$$

which is exactly the desired inequality (5.9).

□

Another estimation for Euler's beta function is obtained as [1]:

Theorem 5.3. *Let $p,q > 0$. Then, we have the inequality*

$$| \ B(p+1,q+1) - \frac{1}{(p-1)(q-1)} \ | \leq \frac{1}{4}, \qquad (5.11)$$

or, equivalently

$$\max\{0, \frac{3-pq-p-q}{(p-1)(q-1)}\} \ \leq B(p-1,q-1) \leq \frac{pq+p+q+5}{(p-1)(q-1)}. \qquad (5.12)$$

Proof. Consider the mappings

$$f(x) = x^p, \ g(x) = (1-x)^q, p,q > 0. \qquad (5.13)$$

Then

$$0 \leq f(x) \leq 1, \ 0 \leq g(x) \leq 1, \int_0^1 f(x)dx = \frac{1}{p+1}, \int_0^1 g(x)dx = \frac{1}{q+1}.$$

Using the Grüss's integral inequality for $f(x)$ and $g(x)$, we reach at

$$| \ \int_0^1 f(x)g(x)dx - \int_0^1 f(x)dx \cdot \int_0^1 g(x)dx \ | \leq \frac{1}{4},$$

which is equal to (5.11). The second inequality (5.12) follows from (5.11) by simple computation.

□

Remark 5.4. Taking into account that $B(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}$, the inequality (5.11) can be written as

$$| \ \frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)} - \frac{1}{(p+1)(q+1)} \ | \leq \frac{1}{4},$$

and as $\Gamma(p+1) = (p+1)\Gamma(p), \ \Gamma(q+1) = (q+1)\Gamma(q),$ we deduce

$$| \ \Gamma(p+q+2) - \Gamma(p)\Gamma(q) \ | \leq \frac{1}{4}\Gamma(p+1)\Gamma(q+1). \qquad (5.14)$$

From (5.12), we obtain

$$\max\{0, 3-pq-p-q\}\Gamma(p)\Gamma(q) \leq \Gamma(p+q+2) \leq (pq+p+q+5)\Gamma(p)\Gamma(q). \ (5.15)$$

The weighted version of Grüss integral inequality states that if

$$m_1 \le f_1(x) \le M_1, m_2 \le f_2(x) \le M_2 \text{ for all } x \in [a,b],$$

and $h : [a,b] \to (0,\infty)$ is integrable, then

$$| \int_a^b h(x)dx \int_a^b h(x)f_1(x)f_2(x)dx - \int_a^b h(x)f_1(x)dx \int_a^b h(x)f_2(x)dx \ |$$
$$\le \frac{1}{4}(M_1 - m_1)(M_2 - m_2)[\int_a^b h(x)dx]^2, \qquad (5.16)$$

and using (5.16) we can state the following:

Theorem 5.5. *Let* $m,n,p,q > 0$ *and* $r,s > -1$. *Then we have*

$$| \ B(r+1,s+1)B(m+p+r+1,n+q+s+1) - B(m+r+1,n+s+1)$$
$$B(p+r+1,q+s+1) \ \ | \le \frac{1}{4} \cdot \frac{p^p q^q}{(p+q)^{p+q}} \cdot \frac{m^m n^n}{(m+n)^{m+n}} \cdot B^2(r+1,s+1). \qquad (5.17)$$

The proof follows from the inequality (5.16) by letting

$$h(x) = l_{r,s}(x),\ f_1(x) = l_{m,n}(x),\ f_2(x) = l_{p,q}(x).$$

Now applying the same inequality (5.16) but to the mappings

$$h(x) = l_{r,s}(x),\ f_1(x) = x^p,\ f_2(x) = (1-x)^q,$$

we deduce the following generalization:

Theorem 5.6. *Let* $p,q > 0$ *and* $r,s > -1$. *Then holds the inequality*

$$| \ B(r+1,s+1)B(p+r+1,q+s+1) - B(r+1,q+s+1)B(p+r+1,s+1) \ |$$
$$\le \frac{1}{4} \cdot \ B^2(r+1,s+1). \qquad (5.18)$$

6. Čebyšev’s Type Inequalities and Beta and Gamma Mappings

The following results are interesting to note from the Čebyšev’s type inequalities:

Theorem 6.1. *Let* $m,n,p,q > 1$ *and* $r,s > -1$. *Then we have the inequality*

$$| \ B(r+1,s+1)B(m+p+r+1,n+q+s+1) - B(m+r+1,n+s+1)B(p+r+1,q+s+1) \ |$$
$$\le M'_\infty(p,q)M'_\infty(m,n)[B(r+3,s+1)B(r+1,s+1) - B^2(r+2,s+1)], \qquad (6.1)$$

where

$$M'_\infty(p,q) = \max\{p,q\} \cdot \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}},\ p,q > 1.$$

Proof. Recall the following inequality generalizing Čebyšev's inequality:

$$\begin{aligned} &\left| \int_a^b h(x)dx \int_a^b h(x)f_1(x)f_2(x)dx - \int_a^b h(x)f_1(x)dx \int_a^b h(x)f_2(x)dx \right| \\ &\leq \| f' \|_\infty \ \| g' \|_\infty \left[\int_a^b x^2 h(x)dx \int_a^b h(x)dx - \left(\int_a^b xh(x)dx\right)^2\right], \qquad (6.2) \end{aligned}$$

provided that $h(x) > 0$ and f', g' are differentiable and noticing that the first derivatives are bounded on (a,b).

Now applying the above inequality to the following mappings:

$$h(x) = l_{r,s}(x),\ f_1(x) = l_{m,n}(x),\ f_2(x) = l_{p,q}(x)$$

and taking into account

$$\| l'_{m,n} \|_\infty \ \leq M'_\infty(m,n), \quad \| l'_{p,q} \|_\infty \ \leq M'_\infty(p,q),$$

for all $m,n,p,q > 1,$ we deduce the desired inequality (6.1).

□

The following particular case also holds good:

Corollary 6.2. *Let* $m,n,p,q > 1$. *Then holds the inequality*

$$| B(m+p+1, n+q+1) - B(m+1,n+1)B(p+1,q+1) | \leq M'_\infty(p,q)M'_\infty(m,n). \qquad (6.3)$$

The proof follows from (6.1) on choosing $r = s = 0$.

Another result on beta mapping follows as:

Theorem 6.3. *Let* $p,q > 1$ *and* $r,s > -1$. *Then*

$$\begin{aligned} &| B(r+1,s+1)B(p+r+1,q+s+1) - B(p+r+1,s+1)B(r+1,q+s+1) | \\ &\leq pq[B(r+3,s+1)B(r+1,s+1) - B^2(r+2,s+1)]. \qquad (6.4) \end{aligned}$$

Proof. Consider the mappings

$$h(x) = l_{r,s}(x),\ f(x) = x^p,\ g(x) = (1-x)^q.$$

Then

$$f'(x) = px^{p-1},\ g'(x) = -q(1-x)^q, \quad \| f' \|_\infty \ = p, \quad \| g' \|_\infty \ = q.$$

Applying the inequality (6.2) to the mappings h,f,g as above, we can state that

$$\left| \int_0^1 l_{r,s}(x)dx \int_0^1 l_{r+p,s+q}(x)dx - \int_0^1 l_{r+p,s}(x)dx \int_0^1 l_{r,s+q}(x)dx \right|$$
$$\leq pq[\int_0^1 l_{r,s}(x)dx \int_0^1 l_{r+2,s}(x)dx - \{\int_0^1 l_{r+1,s}(x)dx\}^2],$$

which is equivalent to (6.4).

□

The following corollary for the beta mapping is interesting:

Corollary 6.4. *Let* $p,q > 1$. *Then from* (6.4) *by letting* $r = s = 0$, *we have the inequality*

$$\left| B(p+1,q+1) - \frac{1}{(p+1)(q+1)} \right| \leq \frac{pq}{12}. \qquad (6.5)$$

Note that the above inequality is equivalent to

$$\max\{0, \frac{11 - p^2q^2 - p^2q - pq^2 - pq}{12(p+1)(q+1)}\}$$
$$\leq B(p+1,q+1)$$
$$\leq \frac{1 + p^2q^2 + p^2q + pq^2 + pq}{12(p+1)(q+1)}, \; p,q > 1. \qquad (6.6)$$

Further, the inequality (6.5) is equivalent to the following one for the gamma mapping:

$$\left| \Gamma(p+q+2) - \Gamma(p)\Gamma(q) \right| \leq \frac{pq}{12}\Gamma(p+1)\Gamma(q+1). \; (6.7)$$

7. Some Other Inequalities for Beta Mappings

The following inequality for beta mappings is based on the Grüss integral inequality:

Theorem 7.1. *Let* $m,n > 1$ *and* $p,q > 0$. *Then*

$$\left| B(m+p+1,n+q+1) - B(m+1,n+1)B(p+1,q+1) \right|$$
$$\leq \begin{cases} M'_\infty(m,n)B(p+1,q+1) \\ [\frac{2}{(\alpha+1)(\alpha+2)}]^{\frac{1}{\alpha}} M'_\infty(m,n)[B(\beta p+1,\beta q+1)]^{\frac{1}{\beta}}, if \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \frac{1}{3}M'_\infty(m,n)\frac{p^pq^q}{(p+q)^{p+q}} \end{cases} \qquad (7.1)$$

where

$$M'_\infty(m,n) = \max\{m,n\} \cdot \frac{(m-1)^{m-1}(n-1)^{n-1}}{(m+n-2)^{m+n-2}}, \; m,n > 1.$$

Proof. We use the following inequality [3] which says: if f is differentiable on (a,b), then

$$\left|\ \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\ \right|$$

$$\leq \begin{cases} \| f' \|_\infty \ \| g \|_1 \text{ , provided } g \in L_1[a,b],\ f' \in L_\infty(a,b) \\ [\frac{2}{(\alpha+1)(\alpha+2)}]^{\frac{1}{\alpha}}(b-a)^{\frac{1}{\alpha}}\ \| f' \|_\infty \ \| g \|_\beta \text{ , provided } g \in L_\beta[a,b],\ f' \in L_\infty(a,b), \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1 \\ \frac{b-a}{3}\ \| f' \|_\infty \ \| g \|_\infty \text{ , provided } f', g \in L_\infty(a,b) \end{cases}$$

Now we choose the mappings

$$f(x) = l_{m,n}(x),\ g(x) = l_{p,q}(x),\ x \in [0,1],$$

and then

$$\left|\ \int_0^1 l_{m+p,n+q}(x)dx - \int_0^1 l_{m,n}(x)dx \cdot \int_0^1 l_{p,q}(x)dx\ \right|$$

$$\leq \begin{cases} \| l'_{m,n} \|_\infty \ \| l_{p,q} \|_1 \text{ , } m,n > 1,\ p,q > 0 \\ [\frac{2}{(\alpha+1)(\alpha+2)}]^{\frac{1}{\alpha}}\ \| l'_{m,n} \|_\infty \ \| l_{p,q} \|_\beta \text{ , } m,n > 1,\ p,q > 0, \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1 \\ \frac{1}{3} M'_\infty(m,n)\ \| l_{p,q} \|_\infty \text{ , } m,n > 1,\ p,q > 0 \end{cases}$$

$$\leq \begin{cases} M'_\infty(m,n)B(p+1,q+1) \text{ , } m,n > 1,\ p,q > 0 \\ [\frac{2}{(\alpha+1)(\alpha+2)}]^{\frac{1}{\alpha}}\ M'_\infty(m,n)[B(\beta p+1,\beta q+1)]^{\frac{1}{q}},\ m,n > 1,\ p,q > 0, \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1 \\ \frac{1}{3} M'_\infty(m,n)\ \frac{p^p q^q}{(p+q)^{p+q}} \text{ , } m,n > 1,\ p,q > 0 \end{cases}$$

We also prove the following inequality for beta mappings:

Theorem 7.2. *Let $m,n > 1$ and $p,q > 0$. Then*

$$\left|\ B(p+1,q+1) - \frac{1}{(p+1)(q+1)}\ \right|$$

$$\leq \begin{cases} \frac{p}{q-1} \text{ , } m,n > 1, \text{ if } p > 1, q > -1 \\ [\frac{2}{(\alpha+1)(\alpha+2)}]^{\frac{1}{\alpha}}\ \frac{p}{(2\beta+1)^{\frac{1}{\beta}}},\ p > 1,\ q > 0, \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1 \\ \frac{1}{3}\ p,\ p > 1,\ q > 0 \end{cases} \qquad (7.3)$$

Proof. Consider the mappings

$$f(x) = x^p, g(x) = (1-x)^q.$$

We note that

$$f'(x) = px^{p-1}, \| f'' \| = p (p > 1), \| g \|_1 = \int_0^1 (1-x)^q dx = \frac{1}{q-1} (q > -1),$$
$$\| g \|_\beta = \left(\int_0^1 (1-x)^{q\beta} dx\right)^{\frac{1}{\beta}} = \left(\frac{1}{q\beta+1}\right)^{\frac{1}{\beta}}, \| g \|_\infty = 1 (q > 0).$$

Now applying the inequality (7.2) for the above mappings, we deduce the desired inequality (7.3).

The following inequality of Grüss type has been established [1]:

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right|$$
$$\leq \frac{1}{6} \| f'' \|_\alpha \| g'' \|_\beta (b-a), \quad (7.4)$$

provided that $f' \in L_\alpha(a,b)$ and $g' \in L_\beta(a,b)$ where $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

□

Using inequality (7.4), we can state the following result for the beta mappings:

Theorem 7.3. *Let* $m,n,p,q > 0$. *Then*

$$| B(m+p+1, n+q+1) - B(m+1,n+1)B(p+1,q+1) |$$
$$\leq \frac{1}{6} \max\{m,n\} \max\{p,q\} [B((m-1)\alpha+1,(n-1)\alpha+1)]^{\frac{1}{\alpha}} [B((p-1)\beta+1,(q-1)\beta+1)]^{\frac{1}{\beta}}, \quad (7.5)$$

where $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. The proof follows from (7.4) by choosing

$$f(x) = x^m(1-x)^n, \; g(x) = x^p(1-x)^q, \; m,n,p,q > 0,$$

and using the fact that

$$\| f'' \|_\alpha \leq \max\{m,n\} [B((m-1)\alpha+1,(n-1)\alpha+1)]^{\frac{1}{\alpha}},$$

and

$$\| g'' \|_\beta \leq \max\{p,q\} \; [B((p-1)\beta+1,(q-1)\beta+1)]^{\frac{1}{\beta}},$$

proved in [1].

□

Further, we have from [1]

$$\left| \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx \right|$$
$$\leq \frac{1}{6} \| f'' \|_\infty \ \| g'' \|_1 \ (b-a), \qquad (7.6)$$

Using this inequality, we can state another result on the beta mappings:

Theorem 7.4. *Let* $m,n > 1$ *and* $p,q > 0$. *Then*

$$| B(m+p+1,n+q+1) - B(m+1,n+1)B(p+1,q+1) |$$
$$\leq \frac{1}{6}\max\{m,n\}\max\{p,q\}\frac{(m-1)^{m-1}(n-1)^{n-1}}{(m+n-2)^{m+n-2}}B(p,q). \qquad (7.7)$$

Now using (7.4) and (7.6), we can point out the following estimation for the beta mappings:

Theorem 7.5. *The inequalities for beta mappings that hold good*:

$$\left| B(p+1,q+1) - \frac{1}{(p+1)(q+1)} \right|$$
$$\leq \frac{1}{6}pq\left[\frac{1}{\alpha(p-1)+1}\right]^{\frac{1}{\alpha}}\left[\frac{1}{\beta(q-1)+1}\right]^{\frac{1}{\beta}}, \alpha > 1, \frac{1}{\alpha}+\frac{1}{\beta} = 1, p,q > 1, \qquad (7.8)$$

and

$$\left| B(p+1,q+1) - \frac{1}{(p+1)(q+1)} \right| \leq \frac{1}{6}p,\ p > 1. \qquad (7.9)$$

Proof. Choose $f(x) = x^p,\ g(x) = (1-x)^q$. Then for $p,q > 1,$ we note that

$$\| f' \|_\alpha = p\left[\frac{1}{\alpha(p-1)+1}\right]^{\frac{1}{\alpha}}, \| g' \|_\beta = q\left[\frac{1}{\beta(q-1)+1}\right]^{\frac{1}{\beta}}.$$

Now using (7.4), we reach at (7.8).

$$\| f' \|_\alpha = p\left[\frac{1}{\alpha(p-1)+1}\right]^{\frac{1}{\alpha}}, \| g' \|_\beta = q\left[\frac{1}{\beta(q-1)+1}\right]^{\frac{1}{\beta}}.$$

Further, as $\| f' \|_\infty = p$ and $\| g' \|_1 = 1,$ (7.6) gives the inequality (7.9).

□

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