The Ostrowski Type Moment Integral Inequalities and Moment-Bounds for Continuous Random Variables

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Abstract—We establish Ostrowski type integral inequalities involving moments of a continuous random variable defined on a finite interval. We also derive bounds for moments from these inequalities. Further, we discuss applications of these bounds to the Euler’s beta mappings and illustrate their behaviour. © 2005 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $X$ be a random variable whose probability density function is $f : [a,b] \rightarrow \mathbb{R}$ and $M_r(c)$ represents the $r$th moment about $c \in \mathbb{R}$ of $X$ defined as $M_r(c) = \int_a^b (x - c)^r f(x) \, dx$, for any positive integer $r$. It may be noted that for $c = 0, M_r(0)$ produces moments about origin and for $c = M_1(0) = \mu$, $M_r(\mu)$ generates the central moments of $X$.

Ostrowski [1] proved the following integral inequality which is well known in the literature as the Ostrowski’s inequality.

**Theorem 1.1.** Let mapping $f : [a,b] \rightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ whose derivative $f' : (a,b) \rightarrow \mathbb{R}$ be bounded on $(a,b)$, i.e., $|f'(x)|_\infty := \sup_{t \in (a,b)} |f'(t)\, dt| \leq M (< \infty)$. Then, for all $x \in [a,b]$

$$\left| f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{M}{b - a} \left( \left( \frac{b - a}{2} \right)^2 + \left( x - \frac{a + b}{2} \right)^2 \right),$$

(1.1)

Dragomir et al. [2] proved the following version of the Ostrowski’s inequality using the Grüss inequality.

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Theorem 1.2. Let mapping \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping in the interior of \( I \) and let \( a, b \in \text{int}(I) \) with \( a < b \). If \( f' \in L_1[a, b] \) and \( \gamma \leq f'(x) \leq \Gamma \) for all \( x \in [(a, b)] \), then for all \( x \in [a, b] \)

\[
|f(x) - \frac{1}{b - a} \int_a^b f(t) \, dt - \frac{f(b) - f(a)}{b - a} (x - a + \frac{b}{2})| \leq \frac{1}{4} (b - a)(\Gamma - \gamma), \tag{1.2}
\]

Now, consider the following function \( p(x, t) \) of a variable \( t \) for constants \( A \) and \( B \) and any real numbers \( a < b \),

\[
p(x, t) = \begin{cases} t - a + A, & \text{if } a \leq t < x, \\ t + b - B, & \text{if } x < t \leq b, \end{cases}
\]

such that

1. \( p(x, t) \) has the jump \( [p]_x = (B - A) - (b - a) \) at the point \( t = x \) and \( (d/dt)p(x, t) = 1 + [p]_x \delta(t - x) \);
2. let \( M_x := \sup_{t \in (a, b)} p(x, t) \) and \( m_x := \inf_{t \in (a, b)} p(x, t) \), then
   (a) For \( B - A \leq 0 \), we have \( M_x - m_x = -[p]_x \);
   (b) For \( B - A > 0 \), \( M_x - m_x \) can be evaluated as follows:
      (i) if \( 0 \leq B - A \leq (b - a)/2 \),
      \[
      M_x - m_x = \begin{cases} -x + b, & \text{for } a \leq x \leq a + (B - A), \\ -[p]_x, & \text{for } a + (B - A) \leq b - (B - A), \\ x - a, & \text{for } (B - A) \leq x \leq b; \end{cases}
      \]
      (ii) if \( (b - a)/2 < B - A \leq b - a \),
      \[
      M_x - m_x = \begin{cases} -x + b, & \text{for } a \leq x < b - (B - A), \\ B - A, & \text{for } (b - (B - A)) \leq x < a + (B - A), \\ x - a, & \text{for } a + (B - A) \leq x \leq b; \end{cases}
      \]
      (iii) if \( B - A > b - a \), then \( M_x - m_x = B - A \).

Fedotov et al. [3] proved the following generalization of the Ostrowski type inequality.

Theorem 1.3. Let mapping \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\) with \( a < b \), such that \( \gamma \leq f'(t) \leq \Gamma \) for all \( t \in ((a, b)) \), where \( \gamma \) and \( \Gamma \) are real numbers. Then, for \( A, B, M_x \) and \( m_x \) as above and for all \( x \in [a, b] \),

\[
\left| (C(x) - A)f(a) + (B - C(x))f(b) - (b - a - B + A)f(x) - \int_a^b f(t) \, dt \right| \leq \frac{1}{4} (b - a)(\Gamma - \gamma)(M_x - m_x), \tag{1.3}
\]

where

\[
C(x) = \frac{1}{2(b - a)} [(x - a)(x - a + 2A) - (x - b)(x - b + 2B)].
\]

Dragomir et al. [4] established some results on the weighted version of the Ostrowski’s inequality for the Hölder type mappings and proved.

Theorem 1.4. Let mappings \( f, w : (a, b) \subseteq \mathbb{R} \to \mathbb{R} \) be such that \( w(s) \geq 0, w \) is integrable on \((a, b), \int_a^b w(s) \, ds > 0, f \) is of \( R - H \) Hölder type, i.e., \( |f(x) - f(y)| \leq H|x - y|^R \) for all \( x \in (a, b) \) and \( H > 0 \) and \( R \in (0, 1) \). If \( w \in L_1[a, b] \), then for all \( x \in [a, b] \)

\[
\left| f(x) - \frac{1}{\int_a^b w(s) \, ds} \int_a^b w(s) f(s) \, ds \right| \leq H \frac{1}{\int_a^b w(s) \, ds} \int_a^b |x - s|^R w(s) \, ds. \tag{1.4}
\]
The constant factor $C = 1$ in the right-hand side is sharp in the sense that this cannot be replaced by a smaller one.

If $R = 1$, i.e., the mapping $f$ is Lipschitzian with constant $L > 0$, then from (1.4)

$$
|f(x) - \frac{1}{f_a} \int_a^b w(s) ds| \leq L \frac{1}{f_a} \int_a^b |s| w(s) ds.
$$

(1.5)

Kumar [5–7] applied integral inequalities of Grüss, Hölder and Hermite-Hadamard and Korkine to establish inequalities involving moments and to evaluate bounds for moments of continuous random variables defined over a finite interval. In what follows now, we prove some results for the Ostrowski type integral inequalities involving moments.

2. OSTROWSKI TYPE INEQUALITIES INVOLVING MOMENTS $M_r(c)$

An inequality which provides estimation of $M_r(c)$ follows from (1.1).

**Theorem 2.1.** Let $X$ be a random variable whose probability density function $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous mapping with $c \in \mathbb{R}$ and $|f'(x)| \leq M$ for all $x \in [a, b], a < b$. Then, for any positive integer $r$,

$$
M_r(c) \leq M \left( \frac{b-a}{2} + \frac{1}{M(b-a)} \right) \left( \frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1} \right)
- M \left( \frac{(b-c)^{r+2} + (a-c)^{r+2}}{(r+1)(r+2)} \right) + 2M \left( \frac{(b-c)^{r+3} - (a-c)^{r+3}}{(r+1)(r+2)(r+3)} \right). 
$$

(2.1)

**Proof.** The reverse inequality from (1.1) provides for all $x \in [a, b]$,

$$
M_r(c) \leq M \left( \frac{b-a}{2} + \left( x - \frac{a + b}{2} \right)^2 \right).
$$

(2.2)

Multiplying both sides of (2.2) by $(x-c)^r$ and integrating and since $\int_a^b f(t) dt = 1, f$ being the probability density function, we get

$$
\int_a^b (x-c)^r f(x) dx - \frac{1}{b-a} \int_a^b (x-c)^r dx \leq \frac{M(b-a)}{4} \int_a^b (x-c)^r dx
+ \frac{M}{b-a} \int_a^b (x-c)^r \left( x - \frac{a + b}{2} \right)^2 dx.
$$

(2.3)

Setting

$$
I := \int_a^b (x-c)^r \left( x - \frac{a + b}{2} \right)^2 dx,
$$

and integrating by parts, we get

$$
I = \left( \frac{b-a}{2} \right)^2 \left( \frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1} \right) - (b-a) \left( \frac{(b-c)^{r+2} + (a-c)^{r+2}}{(r+1)(r+2)} \right)
+ \frac{(b-c)^{r+3} - (a-c)^{r+3}}{(r+1)(r+2)(r+3)}.
$$

Thus, (2.3) simplifies to

$$
M_r(c) \leq \frac{1}{b-a} \left( \frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1} \right) \leq \frac{M(b-a)}{4} \left( \frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1} \right)
+ \frac{M}{b-a} \left( \frac{b-a}{2} \right)^2 \left( \frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1} \right)
- (b-a) \left( \frac{(b-c)^{r+2} + (a-c)^{r+2}}{(r+1)(r+2)} \right)
+ \frac{(b-c)^{r+3} - (a-c)^{r+3}}{(r+1)(r+2)(r+3)};
$$

which results in (2.1) and proves the theorem.
The $r$th moment about origin, $M_r(0)$, is obtained by taking $c = 0$ in (2.1) and is given by.

**Corollary 2.1.** Let $X$ be a random variable whose probability function $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous mapping with $|f'(x)| \leq M$ for all $x \in [a, b]$, $a < b$. Then, for any positive integer $r$,

$$M_r(0) \leq M \left(\frac{b-a}{2} + \frac{1}{M(b-a)}\right) \left(\frac{r^{r+1} - a^{r+1}}{r+1}\right) - M \left(\frac{b^{r+2} + a^{r+2}}{(r+1)(r+2)}\right) + 2M \left(\frac{b^{r+3} - a^{r+3}}{(r+1)(r+2)(r+3)}\right).$$

Taking $r = 1$ in (2.4), the mean $\mu$ on the random variable $X$ has an upper bound

$$M_1(0) = \mu \leq \left(\frac{a + b}{2}\right) \left(1 + \frac{M(b-a)^2}{3}\right),$$

and $r = 2$, $c = \mu$ in (2.1), the variance $\sigma^2$ has the upper bound

$$M_2(\mu) = \sigma^2 \leq M \left(\frac{b-a}{2} + \frac{1}{M(b-a)}\right) \left(\frac{(b-\mu)^3 - (a-\mu)^3}{3}\right) - M \left(\frac{(b-\mu)^4 + (a-\mu)^4}{4}\right) + M \left(\frac{(b-\mu)^5 - (a-\mu)^5}{30(b-a)}\right).$$

Note that, the following inequality which provides the lower bound for $M_r(c)$ follows immediately from inequality (1.1) and (2.1).

**Theorem 2.2.** Let $X$ be a random variable whose probability density function $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous mapping with $c \in \mathbb{R}$ and $|f'(x)| \leq M$ for all $x \in [a, b]$, $a < b$. Then, for any positive integer $r$,

$$M_r(c) \geq M \left(\frac{(b-c)^r + (a-c)^r}{(r+1)(r+2)}\right) - M \left(\frac{b-a}{2} + \frac{1}{M(b-a)}\right) \left(\frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1}\right) - 2M \left(\frac{(b-c)^{r+3} - (a-c)^{r+3}}{(r+1)(r+2)(r+3)}\right).$$

Now, we present an inequality for moments $M_r(c)$ by using the Ostrowski and Grüss inequalities (1.2).

**Theorem 2.3.** Let $X$ be a random variable whose probability density function $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous mapping with $c \in \mathbb{R}$ and $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$, $a < b$. Then, for any positive integer $r$,

$$M_r(c) \leq \left[\frac{1}{b-a} + \frac{(b-a)(\Gamma - \gamma)}{4}\right] - \left(\frac{a+b}{2}\right) \left(f(b) - f(a)\right) \left(\frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1}\right)$$

$$\left(\frac{f(b) - f(a)}{b-a}\right) \left(\frac{(b-c)^{r+2} - (a-c)^{r+2}}{(r+1)(r+2)}\right).$$

**Proof.** From (1.2), we have for all $x \in [a, b]$,

$$f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(b) - f(a)}{b-a} \left(x - \frac{a+b}{2}\right) \leq \frac{1}{4}(b-a)(\Gamma - \gamma),$$

or

$$f(x) \leq \left[\frac{1}{b-a} + \frac{1}{4}(b-a)(\Gamma - \gamma) - \frac{f(b) - f(a)}{b-a} \left(\frac{a+b}{2}\right)\right] + \frac{f(b) - f(a)}{b-a} x.$$  

(2.8)
Multiplying both sides of (2.8) by \((x - c)^r\) and integrating and since \(\int_a^b f(t) \, dt = 1\), we get

\[
\int_a^b (x - c)^r f(x) \, dx \leq \left[ \frac{1}{b-a} + \frac{1}{4} \frac{(b-a)(\Gamma - \gamma)}{4} - \frac{f(b) - f(a)}{b-a} \right] \int_a^b (x-c)^r \, dx
\]

or,

\[
M_r(c) \leq \left[ \frac{1}{b-a} + \frac{1}{4} \frac{(b-a)(\Gamma - \gamma)}{4} - \frac{f(b) - f(a)}{b-a} \right] \frac{\left( b(c) + (c-a)^r \right) }{r+1} \leq \frac{1}{b-a} \int_a^b x(x-c)^r \, dx.
\]

which proves the theorem.

The \(r^{th}\) moment about origin, \(M_r(0)\), is obtained in the following corollary by taking \(c = 0\) in (2.7).

**Corollary 2.2.** Let \(X\) be a random variable whose probability function \(f : [a, b] \to \mathbb{R}\) is an absolutely continuous mapping with \(\gamma \leq f'(x) \leq \Gamma\) for all \(x \in [a, b], a < b\). Then, for any positive integer \(r\),

\[
M_r(0) \leq \left[ \frac{1}{b-a} + \frac{1}{4} \frac{(b-a)(\Gamma - \gamma)}{4} - \frac{f(b) - f(a)}{b-a} \right] \left( \frac{b^{r+1} - a^{r+1}}{r+1} \right) - \frac{f(b) - f(a)}{b-a} \left( \frac{b^{r+2} - a^{r+2}}{r+2} \right).
\]

Taking \(r = 1\) in (2.9), the mean \(\mu\) of \(X\) follows

\[
\mu \leq \frac{a+b}{2} + \frac{(b-a)^2(a+b)(\Gamma - \gamma)}{8} + \left( \frac{f(b) - f(a)}{b-a} \right) \left( \frac{b^3 - a^3}{3} - \frac{(b-a)(a+b)^2}{4} \right),
\]

and \(r = 2, c = \mu\) in (2.7), the variance \(\sigma^2\)

\[
\sigma^2 \leq \left[ \frac{1}{b-a} + \frac{(b-a)(\Gamma - \gamma)}{4} - \frac{a+b}{2} \right] \left( \frac{f(b) - f(a)}{b-a} \right) \left( \frac{(b-\mu)^3 - (a-\mu)^3}{3} \right)
\]

\[
+ \left( \frac{f(b) - f(a)}{b-a} \right) \left( \frac{b(b-\mu)^3 - a(a-\mu)^3}{3} \right) - \frac{b-\mu}{12} - \frac{(b-\mu)^4 - (a-\mu)^4}{12}.
\]

The following inequality which provides the lower bound for \(M_r(c)\) follows immediately from inequality (1.2) and (2.7).

**Theorem 2.4.** Let \(X\) be a random variable whose probability density function \(f : [a, b] \to \mathbb{R}\) is an absolutely continuous mapping with \(\gamma \leq f'(x) \leq \Gamma\) for all \(x \in [a, b], a < b\). Then, for any positive integer \(r\),

\[
M_r(c) \geq \left( \frac{a+b}{2} \right) \left( \frac{f(b) - f(a)}{b-a} \right) - \frac{1}{b-a} - \frac{(b-a)(\Gamma - \gamma)}{4} \left( \frac{b(c)^{r+1} - (a-c)^{r+1}}{r+1} \right)
\]

\[
+ \left( \frac{f(b) - f(a)}{b-a} \right) \left( \frac{(b-c)^{r+2} - (a-c)^{r+2}}{r+1}(r+2) \right) - \frac{b(b-c)^{r+1} - a(a-c)^{r+1}}{r+1}.
\]

Now, using the generalized Ostrowski type inequality (1.3), we have the following results.
Theorem 2.5. Let \( X \) be a random variable whose probability density function \( f : [a, b] \to \mathbb{R} \) is an absolutely continuous mapping with \( c \in \mathbb{R} \) and \( \gamma \leq f'(x) \leq \Gamma \) for all \( x \in [a, b] \), \( a < b \). Then, for any positive integer \( r \),

\[
(b - a - B + A)M_r(c) \leq R_1 \left( \frac{(b - c)^{r+1} + (a - c)^{r+1}}{r + 1} \right) + R_2 \left( (b - a)^2 \left( \frac{(b - c)^{r+1} - (a - c)^{r+1}}{r + 1} \right) - 2(b - a) \left( \frac{(b - c)^{r+2} - (a - c)^{r+2}}{(r + 1)(r + 2)} \right) \right) + 2(B - A) \left( \frac{(b - c)^{r+1} - (a - c)^{r+1}}{r + 1} \right) - \frac{(b - c)^{r+2} - (a - c)^{r+2}}{(r + 1)(r + 2)} \right) \right],
\]

where

\[
R_1 = 1 + \frac{(M_x - m_x)(b - a)(\Gamma - \gamma)}{4} - Bf(b) + Af(a), \quad R_2 = \frac{f(b) - f(a)}{2(b - a)},
\]

and \( A, B, M_x \) and \( m_x \) are as above.

Proof. From (1.3) and since \( \int_a^b f(t) \, dt = 1 \), we have

\[
(b - a - B + A)f(x) \leq \left[ 1 + Af(a) - Bf(b) + \frac{1}{4} (b - a)(\Gamma - \gamma)(M_x - m_x) \right] + [f(b) - f(a)]C(x).
\]

Multiplying (2.13) by \((x - c)^r\) and integrating, we get

\[
(b - a - B + A) \int_a^b (x - c)^r f(x) \, dx \leq \left[ 1 + Af(a) - Bf(b) + \frac{1}{4} (b - a)(\Gamma - \gamma)(M_x - m_x) \right] \times \int_a^b (x - c)^r \, dx + [f(b) - f(a)] \int_a^b C(x)(x - c)^r \, dx,
\]

or

\[
(b - a - B + A)M_r(c) \leq \left[ 1 + Af(a) - Bf(b) + \frac{1}{4} (b - a)(\Gamma - \gamma)(M_x - m_x) \right] \times \left( \frac{(b - c)^{r+1} + (a - c)^{r+1}}{r + 1} \right) + [f(b) - f(a)]I,
\]

where

\[
I := \int_a^b C(x)(x - c)^r \, dx
= \int_a^b \left[ \frac{1}{2(b - a)}(x - a)(x - a + 2A) - (x - b)(x - b + 2B) \right] (x - c)^r \, dx
= \frac{1}{2(b - a)} \left[ \int_a^b (x - a)^2(x - c)^r \, dx - \int_a^b (x - b)^2(x - c)^r \, dx \right] + 2A \int_a^b (x - a)(x - c)^r \, dx + 2B \int_a^b (x - b)(x - c)^r \, dx.
\]
Integrating by parts, we evaluate integrals in (2.14) as

\[ I_1 := \int_a^b (x-a)^2(x-c)^r \, dx = \frac{(b-a)^2(b-c)^{r+1}}{r+1} \]

\[ - \frac{2}{r+1} \left[ \frac{(b-a)(b-c)^{r+2}}{r+2} - \frac{(b-c)^{r+3} - (a-c)^{r+3}}{(r+2)(r+3)} \right] \]

\[ I_2 := \int_a^b (x-b)^2(x-c)^r \, dx = -\frac{(b-a)^2(a-c)^{r+1}}{r+1} \]

\[ - \frac{2}{r+1} \left[ \frac{(b-a)(a-c)^{r+2}}{r+2} - \frac{(b-c)^{r+3} - (a-c)^{r+3}}{(r+2)(r+3)} \right] \]

\[ I_3 := 2A \int_a^b (x-a)(x-c)^r \, dx = 2A \left[ \frac{(b-a)(b-c)^{r+1}}{r+1} - \frac{(b-c)^{r+2} - (a-c)^{r+2}}{(r+1)(r+2)} \right] \]

\[ I_4 := 2B \int_a^b (x-b)(x-c)^r \, dx = 2B \left[ \frac{(b-a)(a-c)^{r+1}}{r+1} - \frac{(b-c)^{r+2} - (a-c)^{r+2}}{(r+1)(r+2)} \right]. \]

Substituting the values of the above integrals in (2.14), we arrive at (2.12), and hence, the theorem.

The inequality involving the \( r \)th moment about origin, \( M_r(0) \), follows from Theorem 2.3 by setting \( c = 0 \).

**Corollary 2.3.** Let \( X \) be a random variable whose probability density function \( f : [a, b] \to \mathbb{R} \) is an absolutely continuous mapping \( \gamma \leq f'(x) \leq \Gamma \) for all \( x \in [a, b] \), \( a < b \). Then, for any positive integer \( r \),

\[
(b - a - B + A)M_r(0) \leq R_1 \left( \frac{b^{r+1} + a^{r+1}}{r+1} \right) + R_2 \left[ \left( \frac{b^{r+1} - a^{r+1}}{r+1} \right) \cdot (b-a)^2 \right] - 2(b-A) \left[ \left( \frac{b^{r+1} - a^{r+1}}{r+1} \right) \cdot (b-a) \right]
\]

\[
= 2 \left[ \left( b-a \right)^{r+1} \right] \left( \frac{\left( b-a \right)^2}{r+1} \right) - 2(b-A) \left( \frac{\left( b-a \right)^2}{r+1} \right)
\]

(2.15)

where \( R_1, R_2, A, B, M_r \) and \( m_x \) are defined above.

Setting \( r = 1 \) in (2.15), the mean \( \mu \) of \( X \) has the upper bound

\[
\mu \leq \left( \frac{a^2 + b^2}{2} \right) R_1 + \left[ \frac{(b-a)^2(a+b)}{2} - \frac{(b-a)(b^3 - a^3)}{3} \right] R_2,
\]

(2.16a)

and \( r = 2, c = \mu \) in (2.12), the upper bound for the variance \( \sigma^2 \) is

\[
(b - a - B + A)\sigma^2 \leq \left( \frac{(b - \mu)^3 + (a - \mu)^3}{3} \right) R_1 + \left[ \frac{(b - \mu)^2(a-b)}{2} - \frac{(b-\mu)^4 - (a-\mu)^4}{12} \right] R_2
\]

\[
+ 2(b-A) \left[ \left( \frac{(b - \mu)^3 - (a - \mu)^3}{3} \right) - \frac{(b-\mu)^4 - (a-\mu)^4}{12} \right]
\]

(2.16b)

Note that, the following inequality which provides the lower bound for \( M_r(c) \) follows immediately from inequality (1.3) and (2.12).
Theorem 2.6. Let $X$ be a random variable whose probability density function $f : [a, b] \to \mathbb{R}$ is an absolutely continuous mapping with $c \in \mathbb{R}$ and $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$, $a < b$. Then, for any positive integer $r$,

$$
(b - a - B + A)M_r(c)
\geq R_2 \left[ 2(b - a) \left( \frac{(b - c)^{r+2} - (a - c)^{r+2}}{(r+1)(r+2)} \right) - (b - a)^2 \left( \frac{(b - c)^{r+1} - (a - c)^{r+1}}{r+1} \right) \right]
+ 2(B - A) \left\{ (b - a) \left( \frac{(b - c)^{r+1} - (a - c)^{r+1}}{r+1} \right) - \frac{(b - c)^{r+2} - (a - c)^{r+2}}{(r+1)(r+2)} \right\} \tag{2.17}
$$

where $R_1, R_2, A, B, M_x$ and $m_x$ are defined above.

We apply the weighted Ostrowski inequality for the Lipschitzian mappings of H"{o}lder type (1.5) to prove the following theorem.

Theorem 2.7. Let mapping $f$ be Lipschitzian with constant $L > 0$ and $f, w : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be such that $w(s) \geq 0$, $w$ is integrable on $(a, b)$, $\int_a^b w(s) \, ds > 0$. If $wf \in L_1[a, b]$, then for all $x \in [a, b]$ and for any positive integer $r$

$$
M_r(c) \leq \left( \frac{(b - c)^{r+1} - (a - c)^{r+1}}{(r+1)} \right) \frac{\int_a^b w(s)f(s) \, ds}{\int_a^b w(s) \, ds} + \frac{L \int_a^b |x - s|(c - c)^r w(s) \, ds}{\int_a^b w(s) \, ds} \tag{2.18}
$$

Proof. From (1.5), we can write the inequality

$$
f(x) \leq \frac{\int_a^b w(s)f(s) \, ds}{\int_a^b w(s) \, ds} + \frac{L \int_a^b |x - s|w(s) \, ds}{\int_a^b w(s) \, ds}.
$$

Multiplying it by $(x - c)^r$ and integrating we obtain (2.18).

Corollary 2.4. If $f$ is differentiable on $(a, b)$ and its derivative $f'$ is bounded on $(a, b)$, i.e.,

$$
\|f'\|_{\infty} := \sup_{t \in (a, b)} |f'(t)| < \infty,
$$

then $L = \|f'\|_{\infty}$ and for any positive integer $r$

$$
M_r(c) \leq \left( \frac{(b - c)^{r+1} - (a - c)^{r+1}}{(r+1)} \right) \frac{\int_a^b w(s)f(s) \, ds}{\int_a^b w(s) \, ds} + \|f'\|_{\infty} \frac{\int_a^b |x - s|(c - c)^r w(s) \, ds}{\int_a^b w(s) \, ds} \tag{2.19}
$$

The inequality for the $r^{th}$ moment about origin, $M_r(0)$, follows by setting $c = 0$ in (2.19).

Corollary 2.5. If $f$ is differentiable on $(a, b)$ and its derivative $f'$ is bounded on $(a, b)$, i.e.,

$$
\|f'\|_{\infty} := \sup_{t \in (a, b)} |f'(t)| < \infty,
$$

then $L = \|f'\|_{\infty}$ and for any positive integer $r$

$$
M_r(0) \leq \left( \frac{b^{r+1} - a^{r+1}}{(r+1)} \right) \frac{\int_a^b w(s)f(s) \, ds}{\int_a^b w(s) \, ds} + \|f'\|_{\infty} \frac{\int_a^b \left( \frac{f_b}{f_a} |x - s|w(s) \, ds \right) x^r \, dx}{\int_a^b w(s) \, ds} \tag{2.20}
$$

The following inequality which provides the lower bound for $M_r(c)$ follows immediately from inequality (1.5) and (2.18).

Theorem 2.8. Let mapping $f$ be Lipschitzian with constant $L > 0$ and $f, w : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$ be such that $w(s) \geq 0$, $w$ is integrable on $(a, b)$, $\int_a^b w(s) \, ds > 0$. If $wf \in L_1[a, b]$, then for all $x \in [a, b]$ and for any positive integer $r$

$$
M_r(c) \geq \left( \frac{(a - c)^{r+1} - (b - c)^{r+1}}{(r+1)} \right) \frac{\int_a^b w(s)f(s) \, ds}{\int_a^b w(s) \, ds} - \frac{L \int_a^b |x - s|(c - c)^r w(s) \, ds}{\int_a^b w(s) \, ds} \tag{2.21}
$$

In what follows now, we provide results for some commonly employed weight functions.
2.1.1. Mapping $w(s) = 1$

**Corollary 2.1.1.** We obtain inequality in Theorem 2.1.

2.1.2. Logarithmic mapping $w(s) = \ln(1/s)$

**Corollary 2.1.2.** Let $f : (0, 1) \rightarrow \mathbb{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_0^1 \ln(1/s)f(s)\, ds$ is finite. Then, from all $x \in (0, 1)$ and for any positive integer $r$

$$M_r(0) \leq \frac{\int_0^1 \ln(1/s)f(s)\, ds}{(r+1)} + \|f'\|\infty \left( \frac{1}{4(r+1)} - \frac{1}{r+2} + \frac{3}{2r+5} \right). \quad (2.22)$$

**Proof.** We have $w(s) = \ln(1/s)$, $a = 0, b = 1$. Thus, $\int_0^1 \ln(1/s)\, ds = 1$, and for all $x \in (0, 1)$

$$\int_0^1 |x-s|\ln \left( \frac{1}{s} \right)\, ds = \int_0^x (s-x)\ln s\, ds + \int_x^1 (x-s)\ln s\, ds = x^2 \left( \frac{3}{2} - \ln x \right) - x + \frac{1}{4}.$$ 

Substituting these values in (2.19), we get (2.22). \hfill \blacksquare

2.1.3. Jacobi mapping $w(s) = 1/\sqrt{s}$

**Corollary 2.1.3.** Let $f : (0, 1) \rightarrow \mathbb{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_0^1 (f(s)/\sqrt{s})\, ds$ is finite. Then, for all $x \in (0, 1)$ and for any positive integer $r$

$$M_r(0) \leq \frac{\int_0^1 (f(s)/\sqrt{s})\, ds}{(r+1)} + 2\|f'\|\infty \left( \frac{1}{r+1} - \frac{3}{r+2} + \frac{8}{2r+5} \right). \quad (2.23)$$

**Proof.** We are given $w(s) = 1/\sqrt{s}$, $a = 0, b = 1$. Thus, $\int_0^1 (1/\sqrt{s})\, ds = 1$, and for all $x \in (0, 1)$

$$\int_0^1 \frac{|x-s|}{\sqrt{s}}\, ds = \frac{8x^{3/2} - 6x + 2}{3}.$$ 

Substituting these values in (2.19) provides (2.23) and hence the corollary. \hfill \blacksquare

2.1.4. Chebyshev mapping $w(s) = 1/\sqrt{1 - s^2}$

**Corollary 2.1.4.** Let $f : (-1, 1) \rightarrow \mathbb{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_{-1}^1 (f(s)/\sqrt{1 - s^2})\, ds$ is finite. Then, for all $x \in (-1, 1)$ and for any positive integer $r$

$$M_r(0) \leq \frac{(2r+1) - 1}{(r+1)\pi} \int_{-1}^1 \frac{f(s)}{\sqrt{1 - s^2}}\, ds + 2\|f'\|\infty \int_{-1}^1 \left( x^{r+1} \arcsin x + x^r \sqrt{1 - x^2} \right)\, dx. \quad (2.24)$$

**Proof.** We have $w(s) = 1/\sqrt{1 - s^2}$, $a = 0, b = 1$. Thus, $\int_{-1}^1 (1/\sqrt{1 - s^2})\, ds = \pi$, and for all $x \in (-1, 1)$

$$\int_{-1}^1 \frac{|x-s|}{\sqrt{1 - s^2}}\, ds = 2 \left( \arcsin x + \sqrt{1 - x^2} \right).$$ 

Substituting these values in (2.19) proves the corollary. \hfill \blacksquare

**Remark 1.** Let $f : (-1, 1) \rightarrow \mathbb{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_{-1}^1 ((f(s)/\sqrt{1 - s^2})\, ds$ is finite. Then, for all $x \in (-1, 1)$ and any odd positive integer $r$

$$M_r(0) \leq 2\|f'\|\infty \int_{-1}^1 \left( x^{r+1} \arcsin x + x^r \sqrt{1 - x^2} \right)\, dx. \quad (2.25)$$
Remark 2. Let \( f : (-1, 1) \to \mathbb{R} \) be a differentiable mapping whose derivative is bounded and for which the integral \( \int_{-1}^{1} \left( f(s) / (1-s^2) \right) ds \) is finite. Then, for all \( x \in (-1, 1) \) and any even positive integer \( r \)

\[
M_r(0) \leq \frac{-2}{(r+1)\pi} \int_{-1}^{1} \frac{f(s)}{1-s^2} ds + 2\|f'\|_{\infty} \int_{-1}^{1} \left( x^{r+1} \arcsin x + x^r \sqrt{1-x^2} \right) dx. \quad (2.26)
\]

The first four moments about the origin from (2.25) and (2.26) may be evaluated as

\[
M_1(0) = M_3(0) \leq 0,
M_2(0) \leq 0.88357 - 0.21221 \int_{-1}^{1} \frac{f(s)}{1-s^2} ds,
M_4(0) \leq 0.55362 - 0.12732 \int_{-1}^{1} \frac{f(s)}{1-s^2} ds.
\]

3. Applications to the Euler’s Beta Mappings

The Beta mapping for real numbers is

\[
B(m, n) := \int_{0}^{1} s^{m-1}(1-s)^{n-1} ds, \quad m, n > 0 \text{ and } s \in [0, 1].
\]

Set \( h_{m,n}(s) = s^{m-1}(1-s)^{n-1}, s \in [0, 1] \). For \( m, n > 1 \),

\[
h'_{m,n}(s) = h_{m-1,n-1}(s)[m-1-(m+n-2)s].
\]

We note that,

\[
h'_{m,n}(s) = \begin{cases} > 0, & \text{if } s \in \left[0, \frac{m-1}{m+n-2}\right), \\ 0, & \text{if } s = \frac{m-1}{m+n-2}, \\ < 0, & \text{if } s \in \left(\frac{m-1}{m+n-2}, 1\right].
\end{cases}
\]

which shows that \( h_{m,n}(s) \) has a maximum at \( s = (m-1)/(m+n-2) \) and

\[
\sup_{s \in [0,1]} h_{m,n}(s) = \frac{(m-1)^{m-1}n^{n-1}}{(m+n-2)^{m+n-2}}, \quad m, n > 1.
\]

Then, for all \( s \in [0,1] \),

\[
|h'_{m,n}(s)| \leq \max_{s \in [0,1]} |m-1-(m+n-2)s| \frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}}
\]

\[
= \max(m-1,n-1) \frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}}, \quad m, n > 2,
\]

and

\[
||h'_{m,n}||_{\infty} \leq \max(m-1,n-1) \frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}}, \quad m, n > 2.
\]

Consider the Beta probability density function \( f(s) \) with parameters \( m \) and \( n \),

\[
f(s) = \frac{s^{m-1}(1-s)^{n-1}}{B(m,n)}, \quad m, n > 0 \text{ and } s \in [0,1],
\]
where $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. Then, for $m, n > 2$,

$$\|f'\|_\infty = L \leq \max(m - 1, n - 1) \frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}} B(m, n).$$

(3.1)

We can evaluate the upper bounds for the moments of the beta density function by substituting $\|f'\|_\infty = L$ from (3.1) into the inequalities given in (2.1), (2.4), (2.18), (2.19) and (2.21).

As an example, we consider applications of (2.24) and (2.21) where weight function is a Jacobi mapping with $w(s) = 1/\sqrt{s}$ for all $s \in [0, 1]$. Then,

$$\int_0^1 \frac{f(s)}{\sqrt{s}} ds = \int_0^1 \frac{s^{m-1}(1-s)^{n-1}}{\sqrt{s}B(m, n)} ds = \frac{B(m-1/2, n)}{B(m, n)} = \frac{\Gamma(m-1/2)\Gamma(m+n)}{\Gamma(m)\Gamma(m+n-1/2)}.$$

Thus, from (2.21), for $m, n > 2$, any integer $r$ and $L$ given by (3.1),

$$M_r(0) \leq \frac{B(m-1/2, n)}{(r+1)B(m, n)} + \frac{2L}{3} \left( \frac{1}{r+1} - \frac{3}{r+2} + \frac{8}{2r+5} \right),$$

(3.2)

and from (2.4) for $m, n > 2$, any integer $r$ and $L$ given by (3.1),

$$M_r(0) \leq \frac{1}{r+1} \left( 1 + L \left( \frac{1}{2} - \frac{r+1}{(r+2)(r+3)} \right) \right).$$

(3.3)

To get an insight to the behaviour of these bounds, exact values of $M_1, M_2, M_3, M_4$, and their upper bounds from (2.4), (2.21) and from the inequality (7.7) of Kumar [7], for some choices of $\alpha$ and $\beta$ are evaluated in Table 1.

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REFERENCES


