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The Ostrowski Type Moment Integral Inequalities and Moment-Bounds for Continuous Random Variables

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Abstract—We establish Ostrowski type integral inequalities involving moments of a continuous random variable defined on a finite interval. We also derive bounds for moments from these inequalities. Further, we discuss applications of these bounds to the Euler's beta mappings and illustrate their behaviour.

Keywords—Ostrowski's inequality, Grüss inequality, Hölder's inequality, Moments, Beta mappings.

1. INTRODUCTION

Let X be a random variable whose probability density function is $f : [a, b] \to \mathbf{R}$ and $M_r(c)$ represents the r^{th} moment about $c \in \mathbf{R}$ of X defined as $M_r(c) = \int_a^b (x-c)^r f(x)\,dx$, for any positive integer r. It may be noted that for $c = 0, M_r(0)$ produces moments about origin and for $c = M_1(0) = \mu$, $M_r(\mu)$ generates the central moments of X.

Ostrowski [1] proved the following integral inequality which is well known in the literature as the Ostrowski's inequality.

Theorem 1.1. *Let mapping $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbf{R}$ be bounded on (a, b), i.e., $|f'(x)|_\infty := \sup_{t\in(a,b)} |f'(t)\,dt| \le M(< \infty)$. Then, for all $x \in [a, b]$*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{M}{b-a}\left(\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 \right), \tag{1.1}$$

Dragomir *et al.* [2] proved the following version of the Ostrowski's inequality using the Grüss inequality.

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THEOREM 1.2. *Let mapping $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping in the interior of* $\mathbf{I}$ *and let $a, b \in \text{int}(\mathbf{I})$ with $a < b$. If $f' \in L_1[a,b]$ and $\gamma \le f'(x) \le \Gamma$ for all $x \in [(a,b]$, then for all $x \in [a,b]$*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma), \tag{1.2}$$

Now, consider the following function $p(x,t)$ of a variable t for constants A and B and any real numbers $a < b$,

$$p(x,t) = \begin{cases} t - a + A, & \text{if } a \le t < x, \\ t - b + B, & \text{if } x < t \le b, \end{cases}$$

such that

(1) $p(x,t)$ has the jump $[p]_x = (B - A) - (b - a)$ at the point $t = x$ and $(d/dt)p(x,t) = 1 + [p]_x\delta(t-x)$;
(2) let $M_x := \sup_{t\in(a,b)} p(x,t)$ and $m_x := \inf_{t\in(a,b)} p(x,t)$, then
   (a) For $B - A \le 0$, we have $M_x - m_x = -[p]_x$;
   (b) For $B - A > 0$, $M_x - m_x$ can be evaluated as follows:
      (i) If $0 \le B - A \le (b-a)/2$,

$$M_x - m_x = \begin{cases} -x + b, & \text{for } a \le x \le a + (B-A), \\ -[p]_x, & \text{for } a + (B-A) < x \le b - (B-A), \\ x - a, & \text{for } b - (B-A) < x \le b; \end{cases}$$

      (ii) if $((b-a)/2) < B - A \le b - a$,

$$M_x - m_x = \begin{cases} -x + b, & \text{for } a \le x < b - (B-A), \\ B - A, & \text{for } b - (B-A) \le x < a + (B-A), \\ x - a, & \text{for } a + (B-A) \le x \le b; \end{cases}$$

      (iii) if $B - A > b - a$, then $M_x - m_x = B - A$.

Fedotov *et al.* [3] proved the following generalization of the Ostrowski type inequality.

THEOREM 1.3. *Let mapping $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $a < b$, such that $\gamma \le f'(t) \le \Gamma$ for all $t \in [(a,b)$, where γ and Γ are real numbers. Then, for A, B, M_x and m_x as above and for all $x \in [a,b]$,*

$$\begin{aligned} &\left|(C(x) - A)f(a) + (B - C(x))f(b) - (b - a - B + A)f(x) - \int_a^b f(t)\,dt\right| \\ &\qquad \le \frac{1}{4}(b-a)(\Gamma-\gamma)(M_x - m_x), \end{aligned} \tag{1.3}$$

where

$$C(x) = \frac{1}{2(b-a)}[(x-a)(x-a+2A) - (x-b)(x-b+2B)].$$

Dragomir *et al.* [4] established some results on the weighted version of the Ostrowski's inequality for the Hölder type mappings and proved.

THEOREM 1.4. *Let mappings $f, w : (a,b) \subseteq \mathbf{R} \to \mathbf{R}$ be such that $w(s) \ge 0$, w is integrable on (a,b), $\int_a^b w(s)\,ds > 0$, f is of $R-H$ Hölder type, i.e., $|f(x) - f(y)| \le H|x-y|^R$ for all $x \in (a,b)$ where $H > 0$ and $R \in (0,1]$. If $wf \in L_1[a,b]$, then for all $x \in [a,b]$*

$$\left| f(x) - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(s)f(s)\,ds\right| \le H\frac{1}{\int_a^b w(s)\,ds}\int_a^b |x-s|^R w(s)\,ds. \tag{1.4}$$

The constant factor $C = 1$ in the right-hand side is sharp in the sense that this cannot be replaced by a smaller one.

If $R = 1$, i.e., the mapping f is Lipschitzian with constant $L > 0$, then from (1.4)

$$\left| f(x) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) f(s)\,ds \right| \le L \frac{1}{\int_a^b w(s)\,ds} \int_a^b |x-s| w(s)\,ds. \tag{1.5}$$

Kumar [5–7] applied integral inequalities of Grüss, Hölder and Hermite-Hadamard and Korkine to establish inequalities involving moments and to evaluate bounds for moments of continuous random variables defined over a finite interval. In what follows now, we prove some results for the Ostrowski type integral inequalities involving moments.

2. OSTROWSKI TYPE INEQUALITIES INVOLVING MOMENTS $M_r(c)$

An inequality which provides estimation of $M_r(c)$ follows from (1.1).

THEOREM 2.1. *Let X be a random variable whose probability density function $f : [a,b] \to \mathbf{R}$ is an absolutely continuous mapping with $c \in \mathbf{R}$ and $|f'(x)| \le M$ for all $x \in [a,b]$, $a < b$. Then, for any positive integer r,*

$$\begin{aligned} M_r(c) \le{}& M\left(\frac{b-a}{2} + \frac{1}{M(b-a)}\right)\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) \\ &-M\left(\frac{(b-c)^{r+2}+(a-c)^{r+2}}{(r+1)(r+2)}\right) + \frac{2M}{(b-a)}\left(\frac{(b-c)^{r+3}-(a-c)^{r+3}}{(r+1)(r+2)(r+3)}\right). \end{aligned} \tag{2.1}$$

PROOF. The reverse inequality from (1.1) provides for all $x \in [a,b]$,

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{M}{b-a}\left(\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right). \tag{2.2}$$

Multiplying both sides of (2.2) by $(x-c)^r$ and integrating and since $\int_a^b f(t)\,dt = 1$, f being the probability density function, we get

$$\begin{aligned} \int_a^b (x-c)^r f(x)\,dx - \frac{1}{b-a}\int_a^b (x-c)^r\,dx \le{}& \frac{M(b-a)}{4}\int_a^b (x-c)^r\,dx \\ &+\frac{M}{b-a}\int_a^b (x-c)^r\left(x-\frac{a+b}{2}\right)^2 dx. \end{aligned} \tag{2.3}$$

Setting

$$I := \int_a^b (x-c)^r \left(x - \frac{a+b}{2}\right)^2 dx,$$

and integrating by parts, we get

$$\begin{aligned} I = \left(\frac{b-a}{2}\right)^2 \left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) - (b-a)\left(\frac{(b-c)^{r+2}+(a-c)^{r+2}}{(r+1)(r+2)}\right) \\ + \frac{(b-c)^{r+3}-(a-c)^{r+3}}{(r+1)(r+2)(r+3)}. \end{aligned}$$

Thus, (2.3) simplifies to

$$\begin{aligned} M_r(c) - \frac{1}{b-a}\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) \le{}& \frac{M(b-a)}{4}\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) \\ &+\frac{M}{(b-a)}\left[\left(\frac{b-a}{2}\right)^2\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right)\right. \\ &\left.-(b-a)\left(\frac{(b-c)^{r+2}+(a-c)^{r+2}}{(r+1)(r+2)}\right) + \frac{(b-c)^{r+3}-(a-c)^{r+3}}{(r+1)(r+2)(r+3)}\right], \end{aligned}$$

which results in (2.1) and proves the theorem. ∎

The r^{th} moment about origin, $M_r(0)$, is obtained by taking $c = 0$ in (2.1) and is given by.

COROLLARY 2.1. *Let X be a random variable whose probability function $f : [a, b] \to \mathbf{R}$ is an absolutely continuous mapping with $|f'(x)| \le M$ for all $x \in [a, b]$, $a < b$. Then, for any positive integer r,*

$$M_r(0) \le M\left(\frac{b-a}{2} + \frac{1}{M(b-a)}\right)\left(\frac{b^{r+1} - a^{r+1}}{r+1}\right) - M\left(\frac{b^{r+2} + a^{r+2}}{(r+1)(r+2)}\right) + \frac{2M}{(b-a)}\left(\frac{b^{r+3} - a^{r+3}}{(r+1)(r+2)(r+3)}\right). \tag{2.4}$$

Taking $r = 1$ in (2.4), the mean μ on the random variable X has an upper bound

$$M_1(0) = \mu \le \left(\frac{a+b}{2}\right)\left(1 + \frac{M(b-a)^2}{3}\right), \tag{2.5a}$$

and $r = 2$, $c = \mu$ in (2.1), the variance σ^2 has the upper bound

$$M_2(\mu) = \sigma^2 \le M\left(\frac{b-a}{2} + \frac{1}{M(b-a)}\right)\left(\frac{(b-\mu)^3 - (a-\mu)^3}{3}\right) - M\left(\frac{(b-\mu)^4 + (a-\mu)^4}{4}\right) + M\left(\frac{(b-\mu)^5 - (a-\mu)^5}{30(b-a)}\right). \tag{2.5b}$$

Note that, the following inequality which provides the lower bound for $M_r(c)$ follows immediately from inequality (1.1) and (2.1).

THEOREM 2.2. *Let X be a random variable whose probability density function $f : [a, b] \to \mathbf{R}$ is an absolutely continuous mapping with $c \in \mathbf{R}$ and $|f'(x)| \le M$ for all $x \in [a, b]$, $a < b$. Then, for any positive integer r,*

$$M_r(c) \ge M\left(\frac{(b-c)^{r+2} + (a-c)^{r+2}}{(r+1)(r+2)}\right) - M\left(\frac{b-a}{2} + \frac{1}{M(b-a)}\right)\left(\frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1}\right) - \frac{2M}{(b-a)}\left(\frac{(b-c)^{r+3} - (a-c)^{r+3}}{(r+1)(r+2)(r+3)}\right). \tag{2.6}$$

Now, we present an inequality for moments $M_r(c)$ by using the Ostrowski and Grüss inequalities (1.2).

THEOREM 2.3. *Let X be a random variable whose probability density function $f : [a, b] \to \mathbf{R}$ is an absolutely continuous mapping with $c \in \mathbf{R}$ and $\gamma \le f'(x) \le \Gamma$ for all $x \in [a, b]$, $a < b$. Then, for any positive integer r,*

$$M_r(c) \le \left[\frac{1}{b-a} + \frac{(b-a)(\Gamma-\gamma)}{4} - \left(\frac{a+b}{2}\right)\left(\frac{f(b)-f(a)}{(b-a)}\right)\right]\left(\frac{(b-c)^{r+1} - (a-c)^{r+1}}{r+1}\right) \left(\frac{f(b)-f(a)}{(b-a)}\right)\left(\frac{b(b-c)^{r+1} - a(a-c)^{r+1}}{(r+1)} - \frac{(b-c)^{r+2} - (a-c)^{r+2}}{(r+1)(r+2)}\right). \tag{2.7}$$

PROOF. From (1.2), we have for all $x \in [a, b]$,

$$f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) \le \frac{1}{4}(b-a)(\Gamma-\gamma),$$

or

$$f(x) \le \left[\frac{1}{b-a} + \frac{1}{4}(b-a)(\Gamma-\gamma) - \frac{f(b)-f(a)}{b-a}\left(\frac{a+b}{2}\right)\right] + \frac{f(b)-f(a)}{b-a}x. \tag{2.8}$$

Multiplying both sides of (2.8) by $(x-c)^r$ and integrating and since $\int_a^b f(t)\,dt = 1$, we get

$$\int_a^b (x-c)^r f(x)\,dx \le \left[\frac{1}{b-a} + \frac{1}{4}(b-a)(\Gamma-\gamma) - \frac{f(b)-f(a)}{b-a}\left(\frac{a+b}{2}\right)\right]\int_a^b (x-c)^r\,dx + \frac{f(b)-f(a)}{b-a}\int_a^b x(x-c)^r\,dx.$$

or,

$$M_r(c) \le \left[\frac{1}{b-a} + \frac{1}{4}(b-a)(\Gamma-\gamma) - \frac{f(b)-f(a)}{b-a}\left(\frac{a+b}{2}\right)\right]\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) + \frac{f(b)-f(a)}{b-a}\left[\left(\frac{b(b-c)^{r+1}-a(a-c)^{r+1}}{r+1}\right) - \left(\frac{(b-c)^{r+2}+(a-c)^{r+2}}{(r+1)(r+2)}\right)\right],$$

which proves the theorem. ∎

The r^{th} moment about origin, $M_r(0)$, is obtained in the following corollary by taking $c = 0$ in (2.7).

Corollary 2.2. *Let X be a random variable whose probability function $f : [a,b] \to \mathbf{R}$ is an absolutely continuous mapping with $\gamma \le f'(x) \le \Gamma$ for all $x \in [a,b]$, $a < b$. Then, for any positive integer r,*

$$\begin{aligned} M_r(0) \le &\left(\frac{1}{b-a} + \frac{(b-a)(\Gamma-\gamma)}{4}\right)\left(\frac{b^{r+1}-a^{r+1}}{r+1}\right) \\ &-\frac{f(b)-f(a)}{(b-a)}\left(\frac{(a+b)(b^{r+1}-a^{r+1})}{2(r+1)} - \frac{b^{r+2}-a^{r+2}}{r+2}\right). \end{aligned} \tag{2.9}$$

Taking $r = 1$ in (2.9), the mean μ of X follows

$$\mu \le \frac{a+b}{2} + \frac{(b-a)^2(a+b)(\Gamma-\gamma)}{8} + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^3-a^3}{3} - \frac{(b-a)(a+b)^2}{4}\right), \tag{2.10a}$$

and $r = 2$, $c = \mu$ in (2.7), the variance σ^2

$$\begin{aligned} \sigma^2 \le &\left[\frac{1}{b-a} + \frac{(b-a)(\Gamma-\gamma)}{4} - \left(\frac{a+b}{2}\right)\left(\frac{f(b)-f(a)}{(b-a)}\right)\right]\left(\frac{(b-\mu)^3-(a-\mu)^3}{3}\right) \\ &+\left(\frac{f(b)-f(a)}{(b-a)}\right)\left(\frac{b(b-\mu)^3-a(a-\mu)^3}{3} - \frac{(b-\mu)^4-(a-\mu)^4}{12}\right). \end{aligned} \tag{2.10b}$$

The following inequality which provides the lower bound for $M_r(c)$ follows immediately from inequality (1.2) and (2.7).

Theorem 2.4. *Let X be a random variable whose probability density function $f : [a,b] \to \mathbf{R}$ is an absolutely continuous mapping with $c \in \mathbf{R}$ and $\gamma \le f'(x) \le \Gamma$ for all $x \in [a,b]$, $a < b$. Then, for any positive integer r,*

$$\begin{aligned} M_r(c) \ge &\left[\left(\frac{a+b}{2}\right)\left(\frac{f(b)-f(a)}{(b-a)}\right) - \frac{1}{b-a} - \frac{(b-a)(\Gamma-\gamma)}{4}\right]\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) \\ &+\left(\frac{f(b)-f(a)}{(b-a)}\right)\left(\frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+1)(r+2)} - \frac{b(b-c)^{r+1}-a(a-c)^{r+1}}{(r+1)}\right). \end{aligned} \tag{2.11}$$

Now, using the generalized Ostrowski type inequality (1.3), we have the following results.

THEOREM 2.5. *Let X be a random variable whose probability density function $f : [a,b] \to \mathbf{R}$ is an absolutely continuous mapping with $c \in \mathbf{R}$ and $\gamma \le f'(x) \le \Gamma$ for all $x \in [a,b]$, $a < b$. Then, for any positive integer r,*

$$\begin{aligned}(b-a-B+A)M_r(c) &\le R_1\left(\frac{(b-c)^{r+1}+(a-c)^{r+1}}{r+1}\right)\\ +R_2\Bigg[(b-a)^2&\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right)-2(b-a)\left(\frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+1)(r+2)}\right)\\ -2(B-A)&\left\{(b-a)\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right)-\frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+1)(r+2)}\right\}\Bigg],\end{aligned} \tag{2.12}$$

where

$$R_1 = 1+\frac{(M_x-m_x)(b-a)(\Gamma-\gamma)}{4} - Bf(b)+Af(a), \qquad R_2 = \frac{f(b)-f(a)}{2(b-a)},$$

and A, B, M_x and m_x are as above.

PROOF. From (1.3) and since $\int_a^b f(t)\,dt = 1$, we have

$$\begin{aligned}(b-a-B+A)f(x) \le \left[1+Af(a)-Bf(b)+\frac{1}{4}(b-a)(\Gamma-\gamma)(M_x-m_x)\right]\\ +[f(b)-f(a)]C(x).\end{aligned} \tag{2.13}$$

Multiplying (2.13) by $(x-c)^r$ and integrating, we get

$$\begin{aligned}(b-a-B+A)\int_a^b (x-c)^r f(x)\,dx \le \left[1+Af(a)-Bf(b)+\frac{1}{4}(b-a)(\Gamma-\gamma)(M_x-m_x)\right]\\ \times\int_a^b (x-c)^r\,dx + [f(b)-f(a)]\int_a^b C(x)(x-c)^r\,dx,\end{aligned}$$

or

$$\begin{aligned}(b-a-B+A)M_r(c) \le \left[1+Af(a)-Bf(b)+\frac{1}{4}(b-a)(\Gamma-\gamma)(M_x-m_x)\right]\\ \times\left(\frac{(b-c)^{r+1}+(a-c)^{r+1}}{r+1}\right)+[f(b)-f(a)]I,\end{aligned}$$

where

$$\begin{aligned}I &:= \int_a^b C(x)(x-c)^r\,dx\\ &= \int_a^b\left[\frac{1}{2(b-a)}[(x-a)(x-a+2A)-(x-b)(x-b+2B)]\right](x-c)^r\,dx\\ &= \frac{1}{2(b-a)}\left[\int_a^b (x-a)^2(x-c)^r\,dx - \int_a^b (x-b)^2(x-c)^r\,dx\right.\\ &\quad\left. +2A\int_a^b (x-a)(x-c)^r\,dx + 2B\int_a^b (x-b)(x-c)^r\,dx\right].\end{aligned} \tag{2.14}$$

Integrating by parts, we evaluate integrals in (2.14) as

$$I_1 := \int_a^b (x-a)^2 (x-c)^r \, dx = \frac{(b-a)^2 (b-c)^{r+1}}{r+1} - \frac{2}{r+1}\left[\frac{(b-a)(b-c)^{r+2}}{r+2} - \left(\frac{(b-c)^{r+3}-(a-c)^{r+3}}{(r+2)(r+3)}\right)\right]$$

$$I_2 := \int_a^b (x-b)^2 (x-c)^r \, dx = -\frac{(b-a)^2 (a-c)^{r+1}}{r+1} - \frac{2}{r+1}\left[\frac{(b-a)(a-c)^{r+2}}{r+2} - \left(\frac{(b-c)^{r+3}-(a-c)^{r+3}}{(r+2)(r+3)}\right)\right]$$

$$I_3 := 2A\int_a^b (x-a)(x-c)^r \, dx = 2A\left[\frac{(b-a)(b-c)^{r+1}}{r+1} - \left(\frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+1)(r+2)}\right)\right]$$

$$I_4 := 2B\int_a^b (x-b)(x-c)^r \, dx = 2B\left[\frac{(b-a)(a-c)^{r+1}}{r+1} - \left(\frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+1)(r+2)}\right)\right].$$

Substituting the values of the above integrals in (2.14), we arrive at (2.12), and hence, the theorem. ∎

The inequality involving the r^{th} moment about origin, $M_r(0)$, follows from Theorem 2.3 by setting $c = 0$.

COROLLARY 2.3. *Let X be a random variable whose probability density function $f : [a,b] \to \mathbf{R}$ is an absolutely continuous mapping $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a,b]$, $a < b$. Then, for any positive integer r,*

$$\begin{aligned}(b-a-B+A)M_r(0) \leq R_1\left(\frac{b^{r+1}+a^{r+1}}{r+1}\right) \\ +R_2\left[\left(\frac{b^{r+1}-a^{r+1}}{r+1}\bullet(b-a)^2\right) - 2\left(\frac{b^{r+2}-a^{r+2}}{(r+1)(r+2)}\bullet(b-a)\right)\right. \\ \left.-2(B-A)\left\{\frac{b^{r+1}-a^{r+1}}{r+1}\bullet(b-a) - \frac{b^{r+2}-a^{r+2}}{(r+1)(r+2)}\right\}\right],\end{aligned} \tag{2.15}$$

where R_1, R_2, A, B, M_x and m_x are defined above.

Setting $r = 1$ in (2.15), the mean μ of X has the upper bound

$$\begin{aligned}\mu \leq \left(\frac{a^2+b^2}{2}\right)R_1 + \left[\frac{(b-a)^3(a+b)}{2} - \frac{(b-a)(b^3-a^3)}{3}\right. \\ \left.-2(B-A)\left(\frac{(b-a)^2(a+b)}{2} - \frac{(b^3-a^3)}{6}\right)\right]R_2,\end{aligned} \tag{2.16a}$$

and $r = 2$, $c = \mu$ in (2.12), the upper bound for the variance σ^2 is

$$\begin{aligned}(b-a-B+A)\sigma^2 \leq R_1\left(\frac{(b-\mu)^3+(a-\mu)^3}{3}\right) \\ +R_2\left[(b-a)^2\left(\frac{(b-\mu)^3-(a-\mu)^3}{3}\right) - 2(b-a)\left(\frac{(b-\mu)^4-(a-\mu)^4}{12}\right)\right. \\ \left.-2(B-A)\left\{(b-a)\left(\frac{(b-\mu)^3-(a-\mu)^3}{3}\right) - \frac{(b-\mu)^4-(a-\mu)^4}{12}\right\}\right].\end{aligned} \tag{2.16b}$$

Note that, the following inequality which provides the lower bound for $M_r(c)$ follows immediately from inequality (1.3) and (2.12).

THEOREM 2.6. *Let X be a random variable whose probability density function $f : [a,b] \to \mathbf{R}$ is an absolutely continuous mapping with $c \in \mathbf{R}$ and $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a,b]$, $a < b$. Then, for any positive integer r,*

$$\begin{aligned} &(b-a-B+A)M_r(c) \\ \geq R_2 &\left[2(b-a)\left(\frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+1)(r+2)}\right) - (b-a)^2\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right)\right. \\ &\left. +2(B-A)\left\{(b-a)\left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{r+1}\right) - \frac{(b-c)^{r+2}-(a-c)^{r+2}}{(r+1)(r+2)}\right\}\right] \\ &-R_1\left(\frac{(b-c)^{r+1}+(a-c)^{r+1}}{r+1}\right), \end{aligned} \tag{2.17}$$

where R_1, R_2, A, B, M_x and m_x are defined above.

We apply the weighted Ostrowski inequality for the Lipschitzian mappings of Hölder type (1.5) to prove the following theorem.

THEOREM 2.7. *Let mapping f be Lipschitzian with constant $L > 0$ and $f, w : (a,b) \subseteq \mathbf{R} \to \mathbf{R}$ be such that $w(s) \geq 0$, w is integrable on (a,b), $\int_a^b w(s)\,ds > 0$. If $wf \in L_1[a,b]$, then for all $x \in [a,b]$ and for any positive integer r*

$$M_r(c) \leq \left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{(r+1)}\right)\frac{\int_a^b w(s)f(s)\,ds}{\int_a^b w(s)\,ds} + \frac{L\int_a^b |x-s|(x-c)^r w(s)\,ds}{\int_a^b w(s)\,ds}. \tag{2.18}$$

PROOF. From (1.5), we can write the inequality

$$f(x) \leq \frac{\int_a^b w(s)f(s)\,ds}{\int_a^b w(s)\,ds} + \frac{L\int_a^b |x-s|w(s)\,ds}{\int_a^b w(s)\,ds}.$$

Multiplying it by $(x-c)^r$ and integrating we obtain (2.18). ∎

COROLLARY 2.4. *If f is differentiable on (a,b) and its derivative f' is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then $L = \|f'\|_\infty$ and for any positive integer r*

$$M_r(c) \leq \left(\frac{(b-c)^{r+1}-(a-c)^{r+1}}{(r+1)}\right)\frac{\int_a^b w(s)f(s)\,ds}{\int_a^b w(s)\,ds} + \|f'\|_\infty\frac{\int_a^b |x-s|(x-c)^r w(s)\,ds}{\int_a^b w(s)\,ds}. \tag{2.19}$$

The inequality for the r^{th} moment about origin, $M_r(0)$, follows by setting $c = 0$ in (2.19).

COROLLARY 2.5. *If f is differentiable on (a,b) and its derivative f' is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then $L = \|f'\|_\infty$ and for any positive integer r*

$$M_r(0) \leq \left(\frac{b^{r+1}-a^{r+1}}{(r+1)}\right)\frac{\int_a^b w(s)f(s)\,ds}{\int_a^b w(s)\,ds} + \|f'\|_\infty\frac{\int\left(\int_a^b |x-s|w(s)\,ds\right)x^r\,dx}{\int_a^b w(s)\,ds}. \tag{2.20}$$

The following inequality which provides the lower bound for $M_r(c)$ follows immediately from inequality (1.5) and (2.18).

THEOREM 2.8. *Let mapping f be Lipschitzian with constant $L > 0$ and $f, w : (a,b) \subseteq \mathbf{R} \to \mathbf{R}$ be such that $w(s) \geq 0$, w is integrable on (a,b), $\int_a^b w(s)\,ds > 0$. If $wf \in L_1[a,b]$, then for all $x \in [a,b]$ and for any positive integer r*

$$M_r(c) \geq \left(\frac{(a-c)^{r+1}-(b-c)^{r+1}}{(r+1)}\right)\frac{\int_a^b w(s)f(s)\,ds}{\int_a^b w(s)\,ds} - \frac{L\int_a^b |x-s|(x-c)^r w(s)\,ds}{\int_a^b w(s)\,ds}. \tag{2.21}$$

In what follows now, we provide results for some commonly employed weight functions.

2.1.1. Mapping $w(s) = 1$

Corollary 2.1.1. *We obtain inequality in Theorem 2.1.*

2.1.2. Logarithmic mapping $w(s) = \ln(1/s)$

Corollary 2.1.2. *Let $f : (0,1) \to \mathbf{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_0^1 \ln(1/s) f(s)\, ds$ is finite. Then, from for all $x \in (0,1)$ and for any positive integer r*

$$M_r(0) \le \frac{\int_0^1 \ln(1/s) f(s)\, ds}{(r+1)} + \|f'\|_\infty \left(\frac{1}{4(r+1)} - \frac{1}{r+2} + \frac{3}{2(r+3)} \right). \tag{2.22}$$

Proof. We have $w(s) = \ln(1/s)$, $a = 0$, $b = 1$. Thus, $\int_0^1 \ln(1/s)\, ds = 1$, and for all $x \in (0,1)$

$$\int_0^1 |x-s| \ln\left(\frac{1}{s}\right) ds = \int_0^x (s-x) \ln s\, ds + \int_x^1 (x-s) \ln s\, ds = x^2 \left(\frac{3}{2} - \ln x\right) - x + \frac{1}{4}.$$

Substituting these values in (2.19), we get (2.22). ∎

2.1.3. Jacobi mapping $w(s) = 1/\sqrt{s}$

Corollary 2.1.3. *Let $f : (0,1) \to \mathbf{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_0^1 (f(s)/\sqrt{s})\, ds$ is finite. Then, for all $x \in (0,1)$ and for any positive integer r*

$$M_r(0) \le \frac{\int_0^1 (f(s)/\sqrt{s})\, ds}{(r+1)} + \frac{2\|f'\|_\infty}{3} \left(\frac{1}{r+1} - \frac{3}{r+2} + \frac{8}{2r+5} \right). \tag{2.23}$$

Proof. We are given $w(s) = 1/\sqrt{s}$, $a = 0$, $b = 1$. Thus, $\int_0^1 (1/\sqrt{s})\, ds = 1$, and for all $x \in (0,1)$

$$\int_0^1 \frac{|x-s|}{\sqrt{s}}\, ds = \frac{8x^{3/2} - 6x + 2}{3}.$$

Substituting these values in (2.19) provides (2.23) and hence the corollary. ∎

2.1.4. Chebyshev mapping $w(s) = 1/\sqrt{1-s^2}$

Corollary 2.1.4. *Let $f : (-1,1) \to \mathbf{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_{-1}^1 (f(s))/\sqrt{1-s^2}\, ds$ is finite. Then, for all $x \in (-1,1)$ and for any positive integer r*

$$M_r(0) \le \frac{((-1)^{r+1} - 1)}{(r+1)\pi} \int_{-1}^1 \frac{f(s)}{\sqrt{1-s^2}}\, ds + 2\|f'\|_\infty \int_{-1}^1 \left(x^{r+1} \arcsin x + x^r \sqrt{1-x^2} \right) dx. \tag{2.24}$$

Proof. We have $w(s) = 1/\sqrt{1-s^2}$, $a = 0$, $b = 1$. Thus, $\int_{-1}^1 (1/\sqrt{1-s^2})\, ds = \pi$, and for all $x \in (-1,1)$

$$\int_{-1}^1 \frac{|x-s|}{\sqrt{1-s^2}}\, ds = 2\left(\arcsin x + \sqrt{1-x^2} \right).$$

Substituting these values in (2.19) proves the corollary. ∎

Further, note the following from (2.23).

Remark 1. Let $f : (-1,1) \to \mathbf{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_{-1}^1 ((f(s)/\sqrt{1-s^2})\, ds$ is finite. Then, for all $x \in (-1,1)$ and any odd positive integer r

$$M_r(0) \le 2\|f'\|_\infty \int_{-1}^1 \left(x^{r+1} \arcsin x + x^r \sqrt{1-x^2} \right) dx. \tag{2.25}$$

REMARK 2. Let $f:(-1,1)\to \mathbf{R}$ be a differentiable mapping whose derivative is bounded and for which the integral $\int_{-1}^{1}((f(s)/\sqrt{1-s^2})\,ds$ is finite. Then, for all $x\in(-1,1)$ and any even positive integer r

$$M_r(0) \le \frac{-2}{(r+1)\pi}\int_{-1}^{1}\frac{f(s)}{\sqrt{1-s^2}}\,ds + 2\|f'\|_\infty \int_{-1}^{1}\left(x^{r+1}\arcsin x + x^r\sqrt{1-x^2}\right)dx. \qquad (2.26)$$

The first four moments about the origin from (2.25) and (2.26) may be evaluated as

$$\begin{aligned} &M_1(0) = M_3(0) \le 0,\\ &M_2(0) \le 0.88357 - 0.21221\int_{-1}^{1}\frac{f(s)}{\sqrt{1-s^2}}\,ds,\\ &M_4(0) \le 0.55362 - 0.12732\int_{-1}^{1}\frac{f(s)}{\sqrt{1-s^2}}\,ds. \end{aligned}$$

3. APPLICATIONS TO THE EULER'S BETA MAPPINGS

The Beta mapping for real numbers is

$$B(m,n) := \int_0^1 s^{m-1}(1-s)^{n-1}\,ds, \qquad m,n>0 \text{ and } s\in[0,1].$$

Set $h_{m,n}(s) = s^{m-1}(1-s)^{n-1}$, $s\in[0,1]$. For $m,n>1$,

$$h'_{m,n}(s) = h_{m-1,n-1}(s)[m-1-(m+n-2)s].$$

We note that,

$$h'_{m,n}(s)\begin{cases} >0, & \text{if } s\in\left[0,\dfrac{m-1}{m+n-2}\right),\\ =0, & \text{if } s=\dfrac{m-1}{m+n-2},\\ <0, & \text{if } s\in\left(\dfrac{m-1}{m+n-2},1\right], \end{cases}$$

which shows that $h_{m,n}(s)$ has a maximum at $s=(m-1)/(m+n-2)$ and

$$\sup_{s\in[0,1]} h_{m,n}(s) = \frac{(m-1)^{m-1}(n-1)^{n-1}}{(m+n-2)^{m+n-2}}, \qquad m,n>1.$$

Then, for all $s\in[0,1]$,

$$\begin{aligned} |h'_{m,n}(s)| &\le \max_{s\in[0,1]}|m-1-(m+n-2)s|\frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}}\\ &= \max(m-1,n-1)\frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}}, \qquad m,n>2, \end{aligned}$$

and

$$\|h'_{m,n}\|_\infty \le \max(m-1,n-1)\frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}}, \qquad m,n>2.$$

Consider the Beta probability density function $f(s)$ with parameters m and n,

$$f(s) = \frac{s^{m-1}(1-s)^{n-1}}{B(m,n)}, \qquad m,n>0 \text{ and } s\in[0,1],$$

where $B(m,n) = (\Gamma(m)\Gamma(n))/(\Gamma(m+n))$. Then, for $m,n > 2$,

$$\|f'\|_\infty = L \leq \max(m-1,n-1)\frac{(m-2)^{m-2}(n-2)^{n-2}}{(m+n-4)^{m+n-4}B(m,n)}. \tag{3.1}$$

We can evaluate the upper bounds for the moments of the beta density function by substituting for $\|f'\|_\infty = L$ from (3.1) into the inequalities given in (2.1), (2.4), (2.18), (2.19) and (2.21).

As an example, we consider applications of (2.24) and (2.21) where weight function is a Jacobi mapping with $w(s) = 1/\sqrt{s}$ for all $s \in [0,1]$. Then,

$$\begin{aligned}\int_0^1 \frac{f(s)}{\sqrt{s}}\,ds &= \int_0^1 \frac{s^{m-1}(1-s)^{n-1}}{\sqrt{s}B(m,n)}\,ds = \int_0^1 \frac{s^{(m-1/2)-1}(1-s)^{n-1}}{B(m,n)}\,ds \\ &= \frac{B(m-1/2,n)}{B(m,n)} = \frac{\Gamma(m-1/2)\Gamma(m+n)}{\Gamma(m)\Gamma(m+n-1/2)}.\end{aligned}$$

Thus, from (2.21), for $m,n > 2$, any integer r and L given by (3.1),

$$M_r(0) \leq \frac{B(m-1/2,n)}{(r+1)B(m,n)} + \frac{2L}{3}\left(\frac{1}{r+1} - \frac{3}{r+2} + \frac{8}{2r+5}\right), \tag{3.2}$$

and from (2.4) for $m,n > 2$, any integer r and L given by (3.1),

$$M_r(0) \leq \frac{1}{r+1}\left(1 + L\left(\frac{1}{2} - \frac{r+1}{(r+2)(r+3)}\right)\right). \tag{3.3}$$

To get an insight to the behaviour of these bounds, exact values of M_1, M_2, M_3, M_4, and their upper bounds from (2.4), (2.21) and from the inequality (7.7) of Kumar [7], for some choices of α and β are evaluated in Table 1.

Table 1. Exact values of M_1, M_2, M_3, M_4 and upper bounds ($m,n = 3,4,5$).

		M_1	$\hat{M}_1$	$\hat{M}_1$	$\hat{M}_1$	M_2	$\hat{M}_2$	$\hat{M}_2$	$\hat{M}_2$
m	n		(2.4)	(2.21)	(7.7)		(2.4)	(2.21)	(7.7)
3	3	0.50	0.5028	0.77	0.57	0.29	0.3353	0.51	0.41
4	4	0.50	0.5002	0.75	0.57	0.28	0.3335	0.50	0.41
3	4	0.43	0.5012	0.83	0.56	0.21	0.3342	0.56	0.39
4	5	0.44	0.5001	0.80	0.56	0.22	0.3334	0.53	0.40

		M_3	$\hat{M}_3$	$\hat{M}_3$	$\hat{M}_3$	M_4	$\hat{M}_4$	$\hat{M}_4$	$\hat{M}_4$
m	n		(2.4)	(2.21)	(7.7)		(2.4)	(2.21)	(7.7)
3	3	0.18	0.2515	0.39	0.32	0.12	0.2013	0.31	0.27
4	4	0.17	0.2501	0.37	0.32	0.11	0.2001	0.30	0.27
3	4	0.12	0.2507	0.42	0.31	0.07	0.2006	0.33	0.25
4	5	0.12	0.2500	0.40	0.31	0.07	0.2000	0.32	0.26

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