ON UNIFIED GENERALIZATIONS OF RELATIVE JENSEN–SHANNON AND ARITHMETIC–GEOMETRIC DIVERGENCE MEASURES, AND THEIR PROPERTIES

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Abstract. In this paper we shall consider one parametric generalization of some non-symmetric divergence measures. The *non-symmetric divergence measures* are such as: Kullback-Leibler *relative information*, χ^2-*divergence*, *relative J – divergence*, *relative Jensen – Shannon divergence* and *relative Arithmetic – Geometric divergence*. All the generalizations considered can be written as particular cases of Csiszár's *f-divergence*. By putting some conditions on the probability distribution, the aim here is to develop bounds on these measures and their parametric generalizations.

1. Introduction

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, ..., p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \; n \geqslant 2,$$

be the set of all complete finite discrete probability distributions. There exist many information and divergence measures in the literature on information theory and statistics. Some of them are symmetric with respect to probability distributions, while others are not. Here, in this paper, we shall work only with non-symmetric measures. Through out the paper it is understood that the probability distributions $P, Q \in \Gamma_n$.

1.1. Non-Symmetric Measures.

Here we shall give some non-symmetric measures of information. The well known among them are χ^2-*divergence* and Kullback-Leibler *relative information.*

- χ^2-**Divergence** (Pearson [17])

$$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1. \tag{1.1}$$

- **Relative Information** (Kullback and Leiber [14])

$$K(P||Q) = \sum_{i=1}^{n} p_i \ln(\frac{p_i}{q_i}). \tag{1.2}$$

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• **Relative J–Divergence** (Dragomir et al. [8])

$$D(P||Q) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i + q_i}{2q_i}\right). \tag{1.3}$$

• **Relative JS–Divergence** (Sibson [18])

$$F(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{2p_i}{p_i + q_i}\right). \tag{1.4}$$

• **Relative AG–Divergence** (Taneja [25], [26])

$$G(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i + q_i}{2}\right) \ln\left(\frac{p_i + q_i}{2p_i}\right). \tag{1.5}$$

The symmetric versions of the above measures are given by

$$\Psi(P||Q) = \chi^2(P||Q) + \chi^2(Q||P), \tag{1.6}$$

$$\begin{aligned} J(P||Q) &= K(P||Q) + K(Q||P) \\ &= D(P||Q) + D(Q||P), \end{aligned} \tag{1.7}$$

$$I(P||Q) = \frac{1}{2}\left[F(P||Q) + F(Q||P)\right] \tag{1.8}$$

and

$$T(P||Q) = \frac{1}{2}\left[G(P||Q) + G(Q||P)\right]. \tag{1.9}$$

After simplification, we can write

$$J(P||Q) = 4\left[I(P||Q) + T(P||Q)\right]. \tag{1.10}$$

Dragomir et al. [11] studied the measures (1.6). We call it [24] by *symmetric chi-square divergence*. The measure (1.6) is well known Jeffreys-Kullback-Leiber [13], [14] *J-divergence*. The measure (1.8) is *information radius* studied by Sibson [18]. It is also known by *Jensen Shannon divergence measure* (ref. Burbea and Rao [2]). The measure (1.9) is new in the literature and is studied for the first time by Taneja [21] and is called *arithmetic and geometric mean divergence measure*. Lin [15] studied some interesting properties and applications of the measure 1.4. More details on some of these divergence measures can be seen in Taneja [20], [21] and in on line book by Taneja [22]. An inequality among the measures (1.6)-(1.9) can be seen in Taneja [27].

In this paper our aim is to work with one parametric generalization of non-symmetric divergence measures given by (1.4) and (1.5). A similar kind of study of the measures (1.2) and (1.3) with their one parametric generalizations can be seen in Kumar and Taneja [29].

It is important to note that defining a generalized divergence by introducing a real parameter allows to unify many of the known divergence measures studied individually and also yields a number of new divergences. The properties and bounds established for this family do hold good in particular for member divergences. It provides a vast horizon of divergences for users to choose that deems best for their applications. A few examples of new divergence measures obtained from this generalized divergence are cited in the next section 2.

2. Generalized Non-Symmetric Divergence Measures

A one parametric generalization of the measure (eq1) can be seen in in Liese and Vajda [16]. We refer it here as *relative information of type s.*

- **Relative Information of Type s**

$$\Phi_s(P||Q) = \begin{cases} K_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], & s \neq 0,1 \\ K(Q||P) = \sum_{i=1}^{n} q_i \ln\left(\frac{q_i}{p_i}\right), & s = 0 \\ K(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right), & s = 1 \end{cases}, \tag{2.1}$$

for all $s \in \mathbb{R}$.

The measure (2.1) admits the following particular cases:

(i) $\Phi_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P)$.
(ii) $\Phi_0(P||Q) = K(Q||P)$.
(iii) $\Phi_{1/2}(P||Q) = 4\left[1 - B(P||Q)\right] = 4h(P||Q)$.
(iv) $\Phi_1(P||Q) = K(P||Q)$.
(v) $\Phi_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.

Thus we observe that $\Phi_2(P||Q) = \Phi_{-1}(Q||P)$and $\Phi_1(P||Q) = \Phi_0(Q||P)$.

The measures $B(P||Q)$ and $h(P||Q)$ appearing in part (iii) are given by

$$B(P||Q) = \sum_{i=1}^{n} \sqrt{p_i q_i} \tag{2.2}$$

and

$$h(P||Q) = 1 - B(P||Q) = \frac{1}{2}\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2 \tag{2.3}$$

respectively.

The measure $B(P||Q)$ is famous as Bhattacharyya [1] *coefficient* and the measure $h(P||Q)$is known as Hellinger [12] *discrimination.*

Now we shall give one parametric generalization of the measures given by (1.3) and (1.4). These generalizations are based on the measure (2.1).

- **Unified Relative AG and JS – Divergence of Type s**

Let us consider the following unified one parametric generalizations of the measures (1.4) and (1.5) simultaneously.

$$\Omega_s(P||Q) = \begin{cases} FG_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i \left(\frac{p_i+q_i}{2p_i}\right)^s - 1\right], & s \neq 0,1 \\ F(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{2p_i}{p_i+q_i}\right), & s=0 \\ G(P||Q) = \sum_{i=1}^{n} \left(\frac{p_i+q_i}{2}\right) \ln\left(\frac{p_i+q_i}{2p_i}\right), & s=1 \end{cases}. \tag{2.4}$$

The measure (2.4) admits the following particular cases:

(i) $\Omega_{-1}(P||Q) = \frac{1}{4}\Delta(P||Q)$

(ii) $\Omega_0(P||Q) = F(P||Q)$.

(iii) $\Omega_{1/2}(P||Q) = 4\left[1 - B\left(P||\frac{P+Q}{2}\right)\right] = 4\,h\left(P||\frac{P+Q}{2}\right)$.

(iv) $\Omega_1(P||Q) = G(P||Q)$.

(v) $\Omega_2(P||Q) = \frac{1}{8}\chi^2(Q||P)$.

The expression $\Delta(P||Q)$ appearing in part (i) is the well known *triangular discrimination*, and is given by

$$\Delta(P||Q) = \sum_{i=1}^{n} \frac{(p_i-q_i)^2}{p_i+q_i}. \tag{2.5}$$

The new divergences can be obtained from $\Omega_s(P||Q)$ by considering different choices of the real parameter s. For example, $s = -\frac{1}{2}$ and $s = -2$ in (2.4) respectively result the following new divergence measures:

$$\Omega_{-1/2}(P||Q) = \frac{4}{3}\left[\sum_{i=1}^{n} p_i\sqrt{\frac{2p_i}{p_i+q_i}} - 1\right] \tag{2.6}$$

and

$$\Omega_{-2}(P||Q) = \frac{1}{6}\left[\sum_{i=1}^{n} p_i\left(\frac{2p_i}{p_i+q_i}\right)^2 - 1\right]. \tag{2.7}$$

In this paper we shall study some properties of the *unified generalized measure* (2.4). Some properties and application of the this measure following the lines of Lin [15] shall be dealt elsewhere. Some applications of this new generalized measure (2.4) towards, *pattern recognition*, *statistics*, *minimization problem*, etc. are also under study.

3. Csiszár's $f-$Divergence and its Properties

Given a function$f : [0,\infty) \to \mathbb{R}$, the *f-divergence* measure introduced by Csiszár's [4] is given by

$$C_f(P||Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{3.1}$$

for all $P, Q \in \Gamma_n$.

Here below are some theorems giving properties of the measure (3.1).

Theorem 3.1. (*Csiszár's* [4, 5]) *If the function f is convex and normalized, i.e., $f(1)=0$, then the f-divergence, $C_f(P||Q)$ is nonnegative and convex in the pair of probability distribution $(P,Q)\in\Gamma_n\times\Gamma_n$.*

Theorem 3.2. (*Dragomir* [6, 7]). *Let $f:\mathbb{R}_+\to\mathbb{R}$ be differentiable convex and normalized i.e., $f(1)=0$. Then*

$$0\leqslant C_f(P||Q)\leqslant E_{C_f}(P||Q) \tag{3.2}$$

where

$$E_{C_f}(P||Q)=\sum_{i=1}^{n}(p_i-q_i)f'(\frac{p_i}{q_i}), \tag{3.3}$$

for all $P,Q\in\Gamma_n$.

Let $P,Q\in\Gamma_n$ be such that there exists r,R with $0<r\leqslant\frac{p_i}{q_i}\leqslant R<\infty$, $\forall i\in\{1,2,...,n\}$, then

$$0\leqslant C_f(P||Q)\leqslant A_{C_f}(r,R), \tag{3.4}$$

where

$$A_{C_f}(r,R)=\frac{1}{4}(R-r)\left[f'(R)-f'(r)\right]. \tag{3.5}$$

Further, if we suppose that $0<r\leqslant 1\leqslant R<\infty$, $r\neq R$, then

$$0\leqslant C_f(P||Q)\leqslant B_{C_f}(r,R), \tag{3.6}$$

where

$$B_{C_f}(r,R)=\frac{(R-1)f(r)+(1-r)f(R)}{R-r}. \tag{3.7}$$

Moreover, the following inequalities hold:

$$E_{C_f}(P||Q)\leqslant A_{C_f}(r,R), \tag{3.8}$$

$$B_{C_f}(r,R)\leqslant A_{C_f}(r,R) \tag{3.9}$$

and

$$0\leqslant B_{C_f}(r,R)-C_f(P||Q)\leqslant A_{C_f}(r,R). \tag{3.10}$$

The inequalities (3.8) and (3.10) can be seen in Dragomir [7], while the inequality (3.9) can be proved easily.

Theorem 3.3. *Let $P,Q\in\Gamma_n$ be such that $0<r\leqslant\frac{p_i}{q_i}\leqslant R<\infty$, $\forall i\in\{1,2,...,n\}$, for some r and R with $0<r<1<R<\infty$. Let $f:\mathbb{R}_+\to\mathbb{R}$ be differentiable convex, normalized, of bounded variation, and second derivative is monotonic with f''' absolutely* *continuous on* $[r,R]$ *and* $f''' \in L_\infty[r,R]$, *then*

$$\left|C_f(P||Q) - \frac{1}{2}E_{C_f}(P||Q)\right| \leqslant \min\left\{\frac{1}{8}k(f)\left[f''(R)-f''(r)\right]\chi^2(P||Q), \frac{1}{12}\left\|f'''\right\|_\infty |\chi|^3(P||Q),\, \left[f'(R)-f'(r)\right]V(P||Q)\right\}, \tag{3.11}$$

and

$$\left|C_f(P||Q) - E^*_{C_f}(P||Q)\right| \leqslant \min\left\{\frac{1}{8}k(f)\left[f''(R)-f''(r)\right]\chi^2(P||Q), \frac{1}{24}\left\|f'''\right\|_\infty |\chi|^3(P||Q),\, \frac{1}{2}\left[f'(R)-f'(r)\right]V(P||Q)\right\}, \tag{3.12}$$

where

$$E^*_{C_f}(P||Q) = \sum_{i=1}^{n}(p_i-q_i)f'\left(\frac{p_i+q_i}{2q_i}\right), \tag{3.13}$$

$$|\chi|^3(P||Q) = \sum_{i=1}^{n}\frac{|p_i-q_i|^3}{q_i^2}, \tag{3.14}$$

$$V(P||Q) = \sum_{i=1}^{n}|p_i-q_i|, \tag{3.15}$$

$$\left\|f'''\right\|_\infty = ess \sup_{x\in[r,R]}\left|f'''\right|, \tag{3.16}$$

and

$$k(f) = \begin{cases} -1, & \textit{if } f'' \textit{ is monotonically decreasing} \\ 1, & \textit{if } f'' \textit{ is monotonically increasing} \end{cases}. \tag{3.17}$$

The above theorem is a combination of different papers due to Dragomir et al. [8, 9, 10].

The measures (1.1), (3.14) and (3.15) are the particular cases of Vajda [30] $|\chi|^m-$*divergence* given by

$$|\chi|^m(P||Q) = \sum_{i=1}^{n}\frac{|p_i-q_i|^m}{q_i^{m-1}},\ m \geqslant 1. \tag{3.18}$$

The measure (3.18) satisfy the following [3, 8] properties

$$|\chi|^m(P||Q) \leqslant \frac{(1-r)(R-1)}{(R-r)}\left[(1-r)^{m-1}+(R-1)^{m-1}\right] \leqslant \left(\frac{R-r}{2}\right)^m,\ m \geqslant 1. \tag{3.19}$$

and

$$\left(\frac{1-r^m}{1-r}\right)V(P||Q)\leqslant|\chi|^m(P||Q)\leqslant\left(\frac{R^m-1}{R-1}\right)V(P||Q),\ m\geqslant 1. \tag{3.20}$$

For $m=2$, $m=3$ and $m=1$ in (3.19), we have

$$\chi^2(P||Q)\leqslant(R-1)(1-r)\leqslant\frac{(R-r)^2}{4}, \tag{3.21}$$

$$|\chi|^3(P||Q)\leqslant\frac{(R-1)(1-r)}{R-r}\left[(1-r)^2+(R-1)^2\right]\leqslant\frac{1}{8}(R-r)^3 \tag{3.22}$$

and

$$V(P||Q)\leqslant\frac{2(R-1)(1-r)}{(R-r)}\leqslant\frac{1}{2}(R-r). \tag{3.23}$$

respectively.

In view of the last inequalities given in (3.21), (3.22) and (3.23) the bounds given in (3.11) and (3.12) can be written in terms of r,R as

$$\begin{aligned}&\left|C_f(P||Q)-\frac{1}{2}E_{C_f}(P||Q)\right| \\ &\leqslant\frac{(R-r)^2}{4}\min\left\{\frac{1}{8}k(f)\left[f''(R)-f''(r)\right],\ \frac{R-r}{24}\left\|f'''\right\|_\infty,\ \frac{2\left[f'(R)-f'(r)\right]}{R-r}\right\}\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}&\left|C_f(P||Q)-E^*_{C_f}(P||Q)\right| \\ &\leqslant\frac{(R-r)^2}{4}\min\left\{\frac{1}{8}k(f)\left[f''(R)-f''(r)\right],\ \frac{R-r}{48}\left\|f'''\right\|_\infty,\ \frac{f'(R)-f'(r)}{R-r}\right\},\end{aligned} \tag{3.25}$$

respectively.

¿From now onwards, unless otherwise specified, it is understood that, if there are r,R, then $0<r\leqslant\frac{p_i}{q_i}\leqslant R<\infty$, $\forall i\in\{1,2,...,n\}$, with $0<r<1<R<\infty$, $P=(p_1,p_2,....,p_n)\in\Gamma_n$ and $Q=(q_1,q_2,....,q_n)\in\Gamma_n$. Also, throughout the paper we shall make use of the *p-logarithmic power mean* [19] given by

$$L_p(a,b)=\begin{cases}\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p\neq-1,0\\ \frac{b-a}{\ln b-\ln a}, & p=-1\\ \frac{1}{e}\left[\frac{b^b}{a^a}\right]^{\frac{1}{b-a}}, & p=0\end{cases}, \tag{3.26}$$

for all $p\in\mathbb{R}$, $a\neq b$. In particular we shall use the following notation

$$L_p^p(a,b)=\begin{cases}\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}, & p\neq-1\\ \frac{\ln b-\ln a}{b-a} & p=-1\\ 1 & p=0\end{cases}, \tag{3.27}$$

for all $p\in\mathbb{R}$, $a\neq b$.

4. Relative AG and JS – Divergence of Type s

In this section we shall consider the measures given by (2.4) and shall give some properties.

Let us consider

$$\psi_s(x) = \begin{cases} [s(s-1)]^{-1}\left[x\left(\frac{x+1}{2x}\right)^s - x - s\left(\frac{1-x}{2}\right)\right], & s \neq 0,1 \\ \frac{1-x}{2} - x\ln\left(\frac{x+1}{2x}\right), & s = 0 \\ \frac{x-1}{2} + \left(\frac{x+1}{2}\right)\ln\left(\frac{x+1}{2x}\right), & s = 1 \end{cases}, \tag{4.1}$$

for all $x > 0$ in (3.1), then $C_f(P||Q) = \Omega_s(P||Q)$, where $\Omega_s(P||Q)$ is as given by (2.4).

Moreover,

$$\psi_s'(x) = \begin{cases} (s-1)^{-1}\left\{\frac{1}{s}\left[\left(\frac{x+1}{2x}\right)^s - 1\right] + \frac{1}{2}\left[1 - \frac{1}{x}\left(\frac{x+1}{2x}\right)^{s-1}\right]\right\}, & s \neq 0,1 \\ \frac{1-x}{2(1+x)} - \ln\left(\frac{x+1}{2x}\right), & s = 0 \\ \frac{1}{2}\left[1 - x^{-1} + \ln\left(\frac{x+1}{2x}\right)\right], & s = 1 \end{cases} \tag{4.2}$$

and

$$\psi_s''(x) = \begin{cases} \frac{1}{4x^3}\left(\frac{x+1}{2x}\right)^{s-2}, & s \neq 0,1 \\ \frac{1}{x(x+1)^2}, & s = 0 \\ \frac{1}{2x^2(x+1)}, & s = 1 \end{cases} \tag{4.3}$$

Thus we have $\psi_s''(x) > 0$ for all $x > 0$, and hence, $\psi_s(x)$ is convex for all $x > 0$. Also, we have $\psi_s(1) = 0$. In view of this we can say that *relative AG and JS – divergence of type s* is *nonnegative* and *convex* in the pair of probability distributions $(P, Q) \in \Gamma_n \times \Gamma_n$.

Based on Theorem 3.2, we have the following theorem.

Theorem 4.1. *The following bounds on $\Omega_s(P||Q)$ hold:*

$$\Omega_s(P||Q) \leqslant E_{\Omega_s(P||Q)}(P||Q) \leqslant A_{\Omega_s(P||Q)}(r,R) \tag{4.4}$$

and

$$\Omega_s(P||Q) \leqslant B_{\Omega_s(P||Q)}(r,R) \leqslant A_{\Omega_s(P||Q)}(r,R), \tag{4.5}$$

where

$$\begin{aligned} &E_{\Omega_s(P||Q)}(P||Q) \\ &= \begin{cases} [s(s-1)]^{-1}\sum_{i=1}^n \left(\frac{p_i-q_i}{p_i+q_i}\right)\left(\frac{p_i+q_i}{2p_i}\right)^s [p_i + (1-s)q_i], & s \neq 0,1 \\ D(Q||P) - \frac{1}{2}\Delta(P||Q), & s = 0 \\ \frac{1}{2}\left[\chi^2(P||Q) - D(Q||P)\right], & s = 1 \end{cases}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &A_{\Omega_s(P||Q)}(r,R) \\ &= \frac{(R-r)^2}{4rR}2^{-s}\left\{L_{s-1}^{s-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right) - L_{s-2}^{s-2}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right\} \end{aligned} \tag{4.7}$$

and

$$B_{\Omega_s(P||Q)}(r,R) \tag{4.8}$$
$$= \begin{cases} \frac{1}{2(s-1)} L_{s-1}^{s-1}\left(\frac{r+1}{2r}, \frac{R+1}{2R}\right) \\ \quad + \frac{1}{s(s-1)(R-r)}\left\{R\left[\left(\frac{R+1}{2R}\right)^s - 1\right] - r\left[\left(\frac{r+1}{2r}\right)^s - 1\right]\right\}, & s \neq 0,1 \\ \frac{1}{(R-r)}\left[r\ln\left(\frac{r+1}{2r}\right) - R\ln\left(\frac{R+1}{2R}\right)\right] - \frac{1}{2}L_{-1}^{-1}\left(\frac{r+1}{2r}, \frac{R+1}{2R}\right), & s = 0 \\ \frac{rR-1}{4rR} L_{-1}^{-1}\left(\frac{r+1}{2r}, \frac{R+1}{2R}\right) + \frac{1}{2}\ln\left(\frac{(R+1)(r+1)}{4rR}\right), & s = 1 \end{cases}$$

Corollary 4.1. *The following inequality holds:*

$$\frac{1}{2}\Delta(P||Q) \leqslant D(Q||P) \leqslant \chi^2(Q||P). \tag{4.9}$$

Proof. It follows from (4.4), where we take $s = 0$ and $s = 1$ in (4.6). □

Theorem 4.1 admits some particular cases. These are summarized in the following two corollaries.

Corollary 4.2. *The following bounds hold:*

$$\begin{aligned} F(P||Q) &\leqslant D(Q||P) - \frac{1}{2}\Delta(P||Q) \\ &\leqslant \frac{(R-r)^2}{4rR}\left[L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right) - L_{-2}^{-2}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right] \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} G(P||Q) &\leqslant \frac{1}{2}\left[\chi^2(Q||P) - D(Q||P)\right] \\ &\leqslant \frac{(R-r)^2}{8rR}\left[1 - L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right] \end{aligned} \tag{4.11}$$

Proof. In inequalities (4.4), take $s = 0$ and $s = 1$ we get (4.10) and (4.11) respectively. □

For $s = -1$ and $s = 2$ the results are obvious.

Corollary 4.3. *The following bounds hold:*

$$\Delta(P||Q) \leqslant \frac{2(R-1)(1-r)}{(R+1)(r+1)} \leqslant \frac{(R-r)^2(R+r+2)}{(R+1)^2(r+1)^2}, \tag{4.12}$$

$$\begin{aligned} F(P||Q) &\leqslant \frac{1}{R-r}\left[r\ln\left(\frac{r+1}{2r}\right) - R\ln\left(\frac{R+1}{2R}\right)\right] - L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right) \\ &\leqslant \frac{(R-r)^2}{4rR}\left[L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right) - L_{-2}^{-2}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right] \end{aligned} \tag{4.13}$$

$$G(P||Q) \leqslant \frac{rR-1}{4rR} L_{-1}^{-1}\left(\frac{r+1}{2r}, \frac{R+1}{2R}\right) + \frac{1}{2}\ln\left(\frac{(R+1)(r+1)}{4rR}\right) \leqslant \frac{(R-r)^2}{8rR}\left[1 - L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right] \tag{4.14}$$

and

$$\chi^2(Q||P) \leqslant \frac{(R-1)(1-r)}{rR} \leqslant \frac{(R-r)^2(R+r)}{4r^2R^2}. \tag{4.15}$$

Proof. In inequalities (4.5), take $s=-1$, $s=0$, $s=1$ and $s=2$ we get respectively (4.12), (4.13), (4.14) and (4.15). □

Based on Theorem 3.3, we have the following result.

Theorem 4.2. *The following bounds hold:*

$$\begin{aligned} &\left|\Omega_s(P||Q) - \frac{1}{2}E_{\Omega_s}(P||Q)\right| \\ &\leqslant \min\left\{\frac{1}{8}\delta_{\Omega_s}(r,R)\chi^2(P||Q),\ \frac{1}{12}\left\|\psi_s'''\right\|_\infty |\chi|^3(P||Q),\ [f'(R)-f'(r)]\right\} \\ &\leqslant \frac{(R-r)^2}{32}\min\left\{\delta_{\Omega_s}(r,R),\ \frac{R-r}{3}\left\|\psi_s'''\right\|_\infty,\ \frac{R-r}{2}[f'(R)-f'(r)]\,V(P||Q)\right\} \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &\left|\Omega_s(P||Q) - E^*_{\Omega_s}(P||Q)\right| \\ &\leqslant \min\left\{\frac{1}{8}\delta_{\Omega_s}(r,R)\chi^2(P||Q),\ \frac{1}{24}\left\|\psi_s'''\right\|_\infty |\chi|^3(P||Q),\ \frac{1}{2}[f'(R)-f'(r)]\,V(P||Q)\right\} \\ &\leqslant \frac{(R-r)^2}{32}\min\left\{\delta_{\Omega_s}(r,R),\ \frac{R-r}{6}\left\|\psi_s'''\right\|_\infty,\ \frac{R-r}{4}[f'(R)-f'(r)]\right\}, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} &E^*_{\Omega_s}(P||Q) \\ &= \begin{cases} [s(s-1)]^{-1}\sum_{i=1}^n (p_i-q_i)\left(\frac{p_i+3q_i}{2(p_i+q_i)}\right)^s\left(\frac{p_i+(3-2s)q_i}{p_i+3q_i}\right), & s\neq 0,1 \\ \sum_{i=1}^n (q_i-p_i)\ln\left(\frac{p_i+3q_i}{2(p_i+q_i)}\right) - \frac{1}{2}\sum_{i=1}^n \frac{(p_i-q_i)^2}{p_i+3q_i}, & s=0 \\ \frac{1}{2}\Delta(P||Q) + \frac{1}{2}\sum_{i=1}^n (p_i-q_i)\ln\left(\frac{p_i+3q_i}{2(p_i+q_i)}\right), & s=1 \end{cases}, \end{aligned} \tag{4.18}$$

$$\delta_{\Omega_s}(r,R) = \frac{1}{4}\left[\frac{1}{r^3}\left(\frac{r+1}{2r}\right)^{s-2} - \frac{1}{R^3}\left(\frac{R+1}{2R}\right)^{s-2}\right],\ s \geqslant -1 \tag{4.19}$$

and

$$\left\|\psi_s'''\right\|_\infty = \frac{(s+1+3r)}{r^2(r+1)^3}\left(\frac{r+1}{2r}\right)^s,\ s \geqslant -1. \tag{4.20}$$

Proof. The third order derivative of the function $\psi_s(x)$ is given by

$$\psi_s'''(x) = -\frac{(s+1+3x)}{x^2(x+1)^3}\left(\frac{x+1}{2x}\right)^s, \; x \in (0,\infty) \tag{4.21}$$

This gives

$$\psi_s'''(x) \leqslant 0, \; \forall \; s \geqslant -1. \tag{4.22}$$

¿From (4.22), we can say that the function $\psi''(x)$ is monotonically decreasing function in $x \in (0,\infty)$, and hence, for all $x \in [r,R]$, we have

$$\begin{aligned} \delta_{\Omega_s}(r,R) &= \psi''(r) - \psi''(R) \\ &= \frac{1}{4}\left[\frac{1}{r^3}\left(\frac{r+1}{2r}\right)^{s-2} - \frac{1}{R^3}\left(\frac{R+1}{2R}\right)^{s-2}\right], \; s \geqslant -1. \end{aligned} \tag{4.23}$$

¿From (4.21), we have

$$|\psi_s'''|\,' = -\left(\frac{(s+1)(s+2)+8(s+1)x+12x^2}{x^3(x+1)^4}\right)\left(\frac{x+1}{2x}\right)^s. \tag{4.24}$$

¿From (4.24), we can say that the function $|\psi_s'''|$ is monotonically decreasing function in $x \in (0,\infty)$ for $s \geqslant -1$, and hence, for all $x \in [r,R]$, we have

$$\|\psi_s'''\|_\infty = \sup_{x\in[r,R]} |\psi_s'''| = \frac{(s+1+3r)}{r^2(r+1)^3}\left(\frac{r+1}{2r}\right)^s, \; s \geqslant -1 \tag{4.25}$$

Applying Theorem 3.3 for the measure (2.4) along with (4.23) and (4.25) we get the required proof. □

In view of the inequalities (4.16) and (4.17) we have some particular cases given in the following corollary.

Corollary 4.4. *The following bounds hold:*

$$\begin{aligned} &\left|\Delta(P||Q) - 2\sum_{i=1}^n q_i\left(\frac{p_i-q_i}{p_i+q_i}\right)^2\right| \\ &\leqslant \min\left\{2\left[\frac{1}{(r+1)^3} - \frac{1}{(R+1)^3}\right]\chi^2(P||Q)\,,\right. \\ &\qquad \left.\frac{4}{(r+1)^4}|\chi|^3(P||Q), \; \frac{8(R-r)(R+r+2)}{(r+1)^2(R+1)^2}V(P||Q)\right\} \\ &\leqslant \min\left\{2(R-r)^2\left[\frac{1}{(r+1)^3} - \frac{1}{(R+1)^3}\right],\right. \\ &\qquad \left.\frac{(R-r)^3}{2(r+1)^4}, \; \frac{4(R-r)^2(R+r+2)}{(r+1)^2(R+1)^2}\right\} \end{aligned} \tag{4.26}$$

$$
\begin{aligned}
(4.27)\qquad & \left|F(P||Q)-\frac{1}{2}D(Q||P)+\frac{1}{4}\Delta(P||Q)\right| \\
& \leqslant \min\left\{\frac{1}{8}\left[\frac{1}{r(r+1)^2}-\frac{1}{R(R+1)^2}\right]\chi^2(P||Q)\ ,\ \frac{3r+1}{12r^2(r+1)^3}|\chi|^3(P||Q),\right. \\
& \qquad \left.\frac{R-r}{rR}\left[L_{-1}^{-1}\left(\frac{r+1}{r},\frac{R+1}{R}\right)-L_{-2}^{-2}\left(\frac{r+1}{r},\frac{R+1}{R}\right)\right]V(P||Q)\right\} \\
& \leqslant \min\left\{\frac{(R-r)^2}{32}\left[\frac{1}{r(r+1)^2}-\frac{1}{R(R+1)^2}\right],\frac{(3r+1)(R-r)^3}{96r^2(r+1)^3},\right. \\
& \qquad \left.\frac{(R-r)^2}{2rR}\left[L_{-1}^{-1}\left(\frac{r+1}{r},\frac{R+1}{R}\right)-L_{-2}^{-2}\left(\frac{r+1}{r},\frac{R+1}{R}\right)\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.28)\qquad & \left|G(P||Q)-\frac{1}{4}\left[\chi^2(Q||P)-D(Q||P)\right]\right| \\
& \leqslant \min\left\{\frac{1}{16}\left[\frac{1}{r^2(r+1)}-\frac{1}{R^2(R+1)}\right]\chi^2(P||Q)\ ,\frac{3r+2}{24r^3(r+1)^2}|\chi|^3(P||Q),\right. \\
& \qquad \left.\frac{(R-r)}{2rR}\left[1-L_{-1}^{-1}\left(\frac{r+1}{r},\frac{R+1}{R}\right)\right]V(P||Q)\right\}. \\
& \leqslant \min\left\{\frac{(R-r)^2}{64}\left[\frac{1}{r^2(r+1)}-\frac{1}{R^2(R+1)}\right],\frac{(3r+2)(R-r)^2}{192r^3(r+1)^2},\right. \\
& \qquad \left.\frac{(R-r)^2}{4rR}\left[1-L_{-1}^{-1}\left(\frac{r+1}{r},\frac{R+1}{R}\right)\right]\right\}.
\end{aligned}
$$

Proof. Letting in (4.16), $s=-1$, $s=0$ and $s=1$ we get the inequalities (4.26), (4.27) and (4.28) respectively. □

In view of the inequalities (4.17) we have some particular cases given in the following corollary.

Corollary 4.5. *The following bounds hold:*

$$
\begin{aligned}
(4.29)\qquad & \left|\Delta(P||Q)-\sum_{i=1}^{n}(p_i+7q_i)\left(\frac{p_i-q_i}{p_i+3q_i}\right)^2\right| \\
& \leqslant \min\left\{\left[\frac{1}{(r+1)^3}-\frac{1}{(R+1)^3}\right]\chi^2(P||Q)\ ,\ \frac{1}{(r+1)^4}\left|\chi\right|^3(P||Q),\right. \\
& \qquad \left.\frac{2(R-r)(R+r+2)}{(r+1)^2(R+1)^2}V(P||Q)\right\}. \\
& \leqslant \min\left\{\frac{(R-r)^2}{4}\left[\frac{1}{(r+1)^3}-\frac{1}{(R+1)^3}\right],\frac{(R-r)^3}{8(r+1)^4},\right. \\
& \qquad \left.\frac{(R-r)^2(R+r+2)}{(r+1)^2(R+1)^2}\right\}.
\end{aligned}
$$

$$
\begin{aligned}
(4.30)\qquad & \left|F(P||Q) - \sum_{i=1}^{n}(q_i - p_i)\ln\left(\frac{p_i - q_i}{p_i + 3q_i}\right) + \frac{1}{2}\sum_{i=1}^{n}\frac{(p_i - q_i)^2}{p_i + 3q_i}\right| \\
& \leqslant \min\left\{\frac{1}{8}\left[\frac{1}{r(r+1)^2} - \frac{1}{R(R+1)^2}\right]\chi^2(P||Q)\,, \frac{3r+1}{24r^2(r+1)^3}\left|\chi\right|^3(P||Q),\right. \\
& \qquad \left.\frac{R-r}{2rR}\left[L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right) - L_{-2}^{-2}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right]V(P||Q)\right\}. \\
& \leqslant \min\left\{\frac{(R-r)^2}{32}\left[\frac{1}{r(r+1)^2} - \frac{1}{R(R+1)^2}\right], \frac{(3r+1)(R-r)^3}{192r^2(r+1)^3},\right. \\
& \qquad \left.\frac{(R-r)^2}{4rR}\left[L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right) - L_{-2}^{-2}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right]\right\}.
\end{aligned}
$$

and

$$
\begin{aligned}
(4.31)\qquad & \left|G(P||Q) - \frac{1}{2}\Delta(P||Q) - \frac{1}{2}\sum_{i=1}^{n}(p_i - q_i)\ln\left(\frac{p_i - q_i}{p_i + 3q_i}\right)\right| \\
& \leqslant \min\left\{\frac{1}{16}\left[\frac{1}{r^2(r+1)} - \frac{1}{R^2(R+1)}\right]\chi^2(P||Q)\,, \frac{3r+2}{48r^3(r+1)^2}\left|\chi\right|^3(P||Q),\right. \\
& \qquad \left.\frac{(R-r)}{4rR}\left[1 - L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right]V(P||Q)\right\} \\
& \leqslant \min\left\{\frac{(R-r)^2}{64}\left[\frac{1}{r^2(r+1)} - \frac{1}{R^2(R+1)}\right], \frac{(3r+2)(R-r)^2}{384r^3(r+1)^2},\right. \\
& \qquad \left.\frac{(R-r)^2}{8rR}\left[1 - L_{-1}^{-1}\left(\frac{r+1}{r}, \frac{R+1}{R}\right)\right]\right\}.
\end{aligned}
$$

Proof. Letting in (4.17), $s = -1$, $s = 0$ and $s = 1$ we get the inequalities (4.29), (4.30) and (4.31) respectively. □

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