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Relative information of type s, Csiszár's f-divergence, and information inequalities ☆

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Abstract

During past years Dragomir has contributed a lot of work providing different kinds of bounds on the distance, information and divergence measures. In this paper, we have unified some of his results using the *relative information of type s* and relating it with the Csiszár's *f-divergence*.

Keywords: Relative information; Csiszár's f-divergence; χ^2-divergence; Hellinger's discrimination; Relative information of type s; Information inequalities

1. Introduction

Let

$$\Delta_n = \left\{ P = (p_1, p_2, \ldots, p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geqslant 2,$$

be the set of complete finite discrete probability distributions.

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Kullback and Leibler's [12] *relative information* is given by

$$K(P\|Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right), \tag{1.1}$$

for all $P, Q \in \Delta_n$.

In Δ_n, we have taken all $p_i > 0$. If we take $p_i \geqslant 0$, $\forall i = 1, 2, \ldots, n$, then in this case we have to suppose that $0 \ln 0 = 0 \ln(\frac{0}{0}) = 0$. It is generally common to take all the logarithms with base 2, but here we have taken only natural logarithms.

We can observe that the measure (1.1) is not symmetric in P and Q. Its symmetric version famous as *J-divergence* [11,12] is given by

$$J(P\|Q) = K(P\|Q) + K(Q\|P) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i}{q_i}\right). \tag{1.2}$$

In this paper our aim is to present one parametric generalizations of the measure (1.1), calling *relative information of type s* and then to consider it in terms of Csiszár's f-divergence. Aim is also to obtain bounds on these measures using Dragomir's approach. Some particular cases are also studied.

2. Relative information of type s

Rényi (1961) for the first time gave one parametric generalization of the relative information given in (1.1). Later other authors presented alternative ways of generalizing it. These generalizations are as follows:

- *Relative information of order r* [16]

$$K^r(P\|Q) = (r-1)^{-1} \ln\left(\sum_{i=1}^{n} p_i^r q_i^{1-r}\right), \quad r \neq 1, \ r > 0. \tag{2.1}$$

- *Relative information of type s* [17]

$${}^1K_s(P\|Q) = (s-1)^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], \quad s \neq 1, \ s > 0. \tag{2.2}$$

In particular we have

$$\lim_{r\to 1} K^r(P\|Q) = \lim_{s\to 1} {}^1K_s(P\|Q) = K(P\|Q).$$

Let us consider the modified version of the measure (2.2) given by

$${}^2K_s(P\|Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], \quad s \neq 0, 1. \tag{2.3}$$

In this case we have the following limiting cases

$$\lim_{s\to 1} {}^2K_s(P\|Q) = K(P\|Q)$$

and

$$\lim_{s\to 0} {}^2K_s(P\|Q) = K(Q\|P).$$

The expression (2.3) has been studied by Vajda [22].

We have the following particular cases of the measures (2.2) and (2.3).

(i) When $s = 0$, one gets

$$^1K_0(P\|Q) = 0$$

and

$$\lim_{s\to 0} {}^2K_s(P\|Q) = K(Q\|P).$$

(ii) When $s = 1$, we have

$$\lim_{s\to 1} {}^1K_s(P\|Q) = K(P\|Q)$$

and

$$\lim_{s\to 1} {}^2K_s(P\|Q) = K(P\|Q).$$

(iii) When $s = \frac{1}{2}$, we note

$$2\,{}^1K_{1/2}(P\|Q) = {}^2K_{1/2}(P\|Q) = 4[1 - B(P\|Q)] = 4h(P\|Q),$$

where

$$B(P\|Q) = \sum_{i=1}^{n} \sqrt{p_i q_i}, \tag{2.4}$$

is the famous as Bhattacharyya's [1] *distance*, and

$$h(P\|Q) = \frac{1}{2}\sum_{i=1}^{n}(\sqrt{p_i} - \sqrt{q_i})^2, \tag{2.5}$$

is famous as the Hellinger's [10] *discrimination*.

(iv) When $s = 2$, we have

$$^1K_2(P\|Q) = 2\,{}^1K_2(P\|Q) = \chi^2(p,q),$$

where

$$\chi^2(P\|Q) = \sum_{i=1}^{n}\frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n}\frac{p_i^2}{q_i} - 1, \tag{2.6}$$

is the χ^2-*divergence* [14].

For simplicity, let us write the measures (2.2) and (2.3) in the unified way:

$$\Psi_s(P\|Q) = \begin{cases} {}^1K_s(P\|Q), & s \neq 1, \\ K(P\|Q), & s = 1 \end{cases} \tag{2.7}$$

and

$$\Phi_s(P\|Q) = \begin{cases} {}^2K_s(P\|Q), & s \neq 0, 1, \\ K(Q\|P), & s = 0, \\ K(P\|Q), & s = 1, \end{cases} \tag{2.8}$$

respectively.

More details on the generalized information and divergence measures can be seen in Taneja [18–21].

3. Csiszárs f-divergence and information inequalities

In this section we shall present Csiszár's f-divergence and bounds on it in terms of measure (2.3). Some bounds due to Dragomir [5–9] are also specified.

Given a convex function $f : [0, \infty) \to \mathbb{R}$, the f-divergence measure introduced by Csiszár [3] is given by

$$C_f(p, q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{3.1}$$

where $p, q \in \mathbb{R}_+^n$.

The following two theorems are due to Csiszár and Körner [4].

Theorem 3.1 (Joint convexity). *Let $f : [0, \infty) \to \mathbb{R}$ be convex, then $C_f(p, q)$ is jointly convex in p and q, where $p, q \in \mathbb{R}_+^n$.*

Theorem 3.2 (Jensen's inequality). *Let $f : [0, \infty) \to \mathbb{R}$ be a convex function. Then for any $p, q \in \mathbb{R}_+^n$, with $P_n = \sum_{i=1}^n p_i > 0$, $Q_n = \sum_{i=1}^n p_i > 0$, we have the inequality*

$$C_f(p, q) \geqslant Q_n f\left(\frac{P_n}{Q_n}\right).$$

The equality sign holds iff

$$\frac{p_1}{q_1} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}.$$

In particular, for all $P, Q \in \Delta_n$, we have

$$C_f(P\|Q) \geqslant f(1),$$

with equality iff $P = Q$.

In view of Theorems 3.1 and 3.2, we can state the following results.

Result 3.1. *For all $P, Q \in \Delta_n$, we note that*

(i) $\Psi_s(P\|Q) \geqslant 0$, $s \geqslant 0$ *with equality iff* $P = Q$;
(ii) $\Phi_s(P\|Q) \geqslant 0$ *for any* $s \in \mathbb{R}$, *with equality iff* $P = Q$;
(iii) $\Psi_s(P\|Q)$, $s \geqslant 0$ *a convex function of the pair of distributions* $(P, Q) \in \Delta_n \times \Delta_n$;
(iv) $\Phi_s(P\|Q)$ *a convex function of the pair of distributions* $(P, Q) \in \Delta_n \times \Delta_n$ *and for any* $s \in \mathbb{R}$.

Proof. Take

$$\psi_s(u) = \begin{cases} (s-1)^{-1}(u^s - u), & s \neq 1, \\ u \ln u, & s = 1, \end{cases} \tag{3.2}$$

for all $u > 0$ in (3.1). Then

$$C_f(P\|Q) = \Psi_s(P\|Q) = \begin{cases} {}^1K_s(P\|Q), & s \neq 1, \\ K(P\|Q), & s = 1. \end{cases}$$

Moreover,

$$\psi_s'(u) = \begin{cases} (s-1)^{-1}(su^{s-1} - 1), & s \neq 1, \\ 1 + \ln u, & s = 1 \end{cases} \tag{3.3}$$

and

$$\psi_s''(u) = \begin{cases} su^{s-2}, & s \neq 1, \\ u^{-1}, & s = 1. \end{cases} \tag{3.4}$$

Thus we have $\psi_s''(u) > 0$ for all $u > 0$ and $s \geqslant 0$, and hence, $\psi_s(u)$ is convex for all $u > 0$. Also, we have $\psi_s(1) = 0$. In view of Theorems 3.1 and 3.2 we have the proof of parts (i) and (iii), respectively.

Again take

$$\phi_s(u) = \begin{cases} [s(s-1)]^{-1}[u^s - 1 - s(u-1)], & s \neq 0, 1, \\ u - 1 - \ln u, & s = 0, \\ 1 - u + u \ln u, & s = 1, \end{cases} \tag{3.5}$$

then for all $u > 0$ in (3.1). Then we get

$$C_f(P\|Q) = \Phi_s(P\|Q) = \begin{cases} {}^2K_s(P\|Q), & s \neq 0, 1, \\ K(Q\|P), & s = 0, \\ K(P\|Q), & s = 1. \end{cases}$$

Moreover,

$$\phi'_s(u) = \begin{cases} (s-1)^{-1}(u^{s-1} - 1), & s \neq 0, 1, \\ 1 - u^{-1}, & s = 0, \\ \ln u, & s = 1 \end{cases} \tag{3.6}$$

and

$$\phi''_s(u) = \begin{cases} u^{s-2}, & s \neq 0, 1, \\ u^{-2}, & s = 0, \\ u^{-1}, & s = 1. \end{cases} \tag{3.7}$$

Thus we have $\phi''_s(u) > 0$ for all $u > 0$, and any $s \in \mathbb{R}$, and hence, $\phi_s(u)$ is convex for all $u > 0$. Also, we have $\phi_s(1) = 0$. In view of Theorems 3.1 and 3.2 we have the proof of parts (ii) and (iv), respectively. □

For some studies on the measure (3.5) refer to Liese and Vajda [13], Österreicher [15] and Cerone et al. [2].

Since the measure (2.3) gives more particular cases rather than measure (2.2) and is also nonnegative for all $s \in \mathbb{R}$, from now onward, we shall consider only the measure (2.3).

The following theorem is due to Dragomir [5,6].

Theorem 3.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a differentiable convex function. Then for all $p, q \in \mathbb{R}^n_+$, we have the inequalities*:

$$f'(1)(P_n - Q_n) \leqslant C_f(p, q) - Q_n f(1) \leqslant C_{f'}\left(\frac{p^2}{q}, p\right) - C_{f'}(p, q) \tag{3.8}$$

and

$$0 \leqslant C_f(p, q) - Q_n f\left(\frac{P_n}{Q_n}\right) \leqslant C_{f'}\left(\frac{p^2}{q}, p\right) - \frac{P_n}{Q_n} C_{f'}(p, q), \tag{3.9}$$

where $f' : \mathbb{R}_+ \to \mathbb{R}$ is the derivative of f.

If f is strictly convex then the equalities in (3.8) *and* (3.9) *hold iff $p = q$.*

We can also write

$$\rho_f(p, q) = C_{f'}\left(\frac{p^2}{q}, p\right) - C_{f'}(p, q) = \sum_{i=1}^{n} (p_i - q_i) f'\left(\frac{p_i}{q_i}\right). \tag{3.10}$$

From the *information-theoretic* point of view we shall use the following proposition.

Proposition 3.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex. If $P, Q \in \Delta_n$, then we can state*

$$0 \leqslant C_f(P\|Q) - f(1) \leqslant C_{f'}\left(\frac{P^2}{Q}\|P\right) - C_{f'}(P\|Q), \tag{3.11}$$

with equalities iff $P = Q$.

In view of Proposition 3.1, we have the following result.

Result 3.2. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Then*

$$0 \leqslant \Phi_s(P\|Q) \leqslant \eta_s(P\|Q), \tag{3.12}$$

where

$$\begin{aligned} \eta_s(P\|Q) &= C_{\phi'_s}\left(\frac{P^2}{Q}\|P\right) - C_{\phi'_s}(P\|Q) \\ &= \begin{cases} (s-1)^{-1}\sum_{i=1}^n (p_i - q_i)\left(\frac{p_i}{q_i}\right)^{s-1}, & s \neq 1, \\ \sum_{i=1}^n (p_i - q_i)\ln\left(\frac{p_i}{q_i}\right), & s = 1. \end{cases} \end{aligned} \tag{3.13}$$

The proof is an immediate consequence of the Proposition 3.1 by substituting $f(\cdot)$ by $\phi_s(\cdot)$, where $\phi_s(\cdot)$ is given by (3.5).

The measure (3.13) admits the following particular cases:

(i) $\eta_0(P\|Q) = \chi^2(Q\|P)$,
(ii) $\eta_1(P\|Q) = J(P\|Q)$,
(iii) $\eta_2(P\|Q) = \chi^2(P\|Q)$.

We state the following corollaries as particular cases of Result 3.2.

Corollary 3.1. *We have*

$$0 \leqslant K(Q\|P) \leqslant \chi^2(Q\|P), \tag{3.14}$$

$$0 \leqslant K(P\|Q) \leqslant J(P\|Q), \tag{3.15}$$

$$0 \leqslant 4h(P\|Q) \leqslant \eta_{1/2}(P\|Q), \tag{3.16}$$

$$0 \leqslant \frac{1}{2}\chi^2(P\|Q) \leqslant \chi^2(P\|Q). \tag{3.17}$$

Proof. (3.14) follows by taking $s = 0$, (3.15) follows by taking $s = 1$, (3.16) follows by taking $s = \frac{1}{2}$ and (3.17) follows by taking $s = 2$ in (3.12). □

The measure $\eta_{1/2}(P\|Q)$ appearing in (3.16) is given by

$$\eta_{1/2}(P\|Q) = \frac{1}{2}\sum_{i=1}^{n}(q_i - p_i)\sqrt{\frac{q_i}{p_i}}. \tag{3.18}$$

The expression (3.18) is the same as (3.13) for $s = \frac{1}{2}$.

We observe that the inequalities (3.15) and (3.17) of the above corollary are quite obvious.

Now, we shall present a theorem that generalizes the one studied by Dragomir [7–9]. The theorem studied here cover three theorems studied in each of the papers [7–9] separately. Its particular cases are given in Section 4.

Theorem 3.4. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a mapping which is normalized, i.e., $f(1) = 0$ and suppose the assumptions*:

(i) *f is twice differentiable on (r, R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M such that $m < M$ and*

$$m \leqslant x^{2-s} f''(x) \leqslant M, \quad \forall x \in (r, R),\ s \in \mathbb{R}. \tag{3.19}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$m\Phi_s(P\|Q) \leqslant C_f(P\|Q) \leqslant M\Phi_s(P\|Q) \tag{3.20}$$

and

$$m(\eta_s(P\|Q) - \Phi_s(P\|Q)) \leqslant \rho_f(P\|Q) - C_f(P\|Q) \leqslant M(\eta_s(P\|Q) - \Phi_s(P\|Q)), \tag{3.21}$$

where $\Phi_s(P\|Q)$, $\rho_f(P\|Q)$ and $\eta_s(P\|Q)$ are as given by (2.8), (3.10) *and* (3.13), *respectively*.

Proof. Let us consider the functions $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ given by

$$F_{m,s}(u) = f(u) - m\phi_s(u) \tag{3.22}$$

and

$$F_{M,s}(u) = M\phi_s(u) - f(u), \tag{3.23}$$

respectively, where m and M are as given by (3.19) and function $\phi_s(\cdot)$ is as given by (3.5).

Since $f(u)$ and $\phi_s(u)$ are normalized, then $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ are also normalized, i.e., $F_{m.s}(1) = 0$ and $F_{M.s}(1) = 0$. Moreover, the functions $f(u)$ and $\phi_s(u)$ are twice differentiable. Then in view of (3.7), we have

$$F''_{m,s}(u) = f''(u) - mu^{s-2} = u^{s-2}(u^{2-s}f''(u) - m) \geqslant 0$$

and

$$F''_{M,s}(u) = Mu^{s-2} - f''(u) = u^{s-2}(M - u^{2-s}f''(u)) \geqslant 0,$$

for all $u \in (r, R)$ and $s \in \mathbb{R}$. Thus the functions $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ are convex on (r, R).

According to Proposition 3.1, we have

$$C_{F_{m,s}}(P\|Q) = C_f(P\|Q) - m\Phi_s(P\|Q) \geqslant 0 \tag{3.24}$$

and

$$C_{F_{M,s}}(P\|Q) = M\Phi_s(P\|Q) - C_f(P\|Q) \geqslant 0. \tag{3.25}$$

Combining (3.24) and (3.25) we have the proof of (3.20).

We shall now prove the validity of inequalities (3.21). We have seen above that the real mappings $F_{m.s}(\cdot)$ and $F_{M.s}(\cdot)$ defined over $\mathbb{R}_+$ given by (3.22) and (3.23), respectively are normalized, twice differentiable and convex related to (r, R). Applying the r.h.s. of the inequalities (3.11), we have

$$C_{F_{m,s}}(P\|Q) \leqslant C_{F'_{m,s}}\left(\frac{P^2}{Q}\|P\right) - C_{F'_{m,s}}(P\|Q) \tag{3.26}$$

and

$$C_{F_{M,s}}(P\|Q) \leqslant C_{F'_{M,s}}\left(\frac{P^2}{Q}\|P\right) - C_{F'_{M,s}}(P\|Q), \tag{3.27}$$

respectively.

Moreover,

$$C_{F_{m,s}}(P\|Q) = C_f(P\|Q) - m\Phi_s(P\|Q) \tag{3.28}$$

and

$$C_{F_{M,s}}(P\|Q) = M\Phi_s(P\|Q) - C_f(P\|Q). \tag{3.29}$$

In view of (3.26) and (3.28), we have

$$C_f(P\|Q) - m\Phi_s(P\|Q) \leqslant C_{f'-m\phi'_s}\left(\frac{P^2}{Q}\|P\right) - C_{f'-m\phi'_s}(P\|Q).$$

Thus,

$$C_f(P\|Q) - m\Phi_s(P\|Q) \leqslant C_{f'}\left(\frac{P^2}{Q}\|P\right) - mC_{\phi_s'}\left(\frac{P^2}{Q}\|P\right) - C_{f'}(P\|Q) + mC_{\phi_s'}(P\|Q).$$

Equivalently,

$$m\left[C_{\phi_s'}\left(\frac{P^2}{Q}\|P\right) - C_{\phi_s'}(P\|Q) - \Phi_s(P\|Q)\right] \leqslant C_{f'}\left(\frac{P^2}{Q}\|P\right) - C_{f'}(P\|Q) - C_f(P\|Q).$$

This gives,

$$m(\eta_s(P\|Q) - \Phi_s(P\|Q)) \leqslant \rho_f(P\|Q) - C_f(P\|Q).$$

Thus, we have the l.h.s. of the inequalities (3.23).

Again in view of (3.27) and (3.29), we have

$$M\Phi_s(P\|Q) - C_f(P\|Q) \leqslant C_{M\phi_s'-f'}\left(\frac{P^2}{Q}\|P\right) - C_{M\phi_s'-f'}(P\|Q).$$

Thus,

$$M\Phi_s(P\|Q) - C_f(P\|Q) \leqslant MC_{\phi_s'}\left(\frac{P^2}{Q}\|P\right) - C_{f'}\left(\frac{P^2}{Q}\|P\right) - MC_{\phi_s'}(P\|Q) + C_{f'}(P\|Q).$$

This gives,

$$C_{f'}\left(\frac{P^2}{Q}\|P\right) - C_{f'}(P\|Q) - C_f(P\|Q) \leqslant M\left[C_{\phi_s'}\left(\frac{P^2}{Q}\|P\right) - C_{\phi_s'}(P\|Q) - \Phi_s(P\|Q)\right].$$

Finally,

$$\rho_f(P\|Q) - C_f(P\|Q) \leqslant M(\eta_s(P\|Q) - \Phi_s(P\|Q)).$$

Thus we have the r.h.s. of the inequalities (3.22) □

Remark 3.1. The above theorem unifies and generalizes the three different theorems studied by Dragomir in three different papers [7] (for $s = 2$), [8] (for $s = 1$) and [9] (for $s = \frac{1}{2}$). These particular cases will appear in the next section. Moreover, we have one more particular case for $s = 0$ which was not studied before. The above theorem also admits one more interesting case for $s = 3$, and it shall be studied elsewhere.

4. Information inequalities

In this section, we shall present particular cases of Theorem 3.4. Some of these particular cases include the results due to Dragomir [7–9].

4.1. Information bounds in terms of χ^2-divergence

The case $s = 2$ of Theorem 3.4 gives:

Proposition 4.1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be a mapping which is normalized, i.e., $f(1) = 0$ and suppose the assumptions:*

(i) *f is twice differentiable on (r, R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M such that $m < M$ and*

$$m \leqslant f''(x) \leqslant M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$\frac{m}{2}\chi^2(P\|Q) \leqslant C_f(P\|Q) \leqslant \frac{M}{2}\chi^2(P\|Q) \tag{4.1}$$

and

$$\frac{m}{2}\chi^2(P\|Q) \leqslant \rho_f(P\|Q) - C_f(P\|Q) \leqslant \frac{M}{2}\chi^2(P\|Q), \tag{4.2}$$

where $\rho_f(P\|Q)$ and $\chi^2(P\|Q)$ are as given by (3.10) *and* (2.6), *respectively.*

In view of Proposition 4.1 we can state the following result.

Result 4.1. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exists r, R such that $r < R$ and*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then Proposition 4.1 *yields*

$$\frac{R^{s-2}}{2}\chi^2(P\|Q) \leqslant \Phi_s(P\|Q) \leqslant \frac{r^{s-2}}{2}\chi^2(P\|Q), s \leqslant 2, \tag{4.3}$$

$$\frac{r^{s-2}}{2}\chi^2(P\|Q) \leqslant \Phi_s(P\|Q) \leqslant \frac{R^{s-2}}{2}\chi^2(P\|Q), s \geqslant 2, \tag{4.4}$$

$$\frac{R^{s-2}}{2}\chi^2(P\|Q) \leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \leqslant \frac{r^{s-2}}{2}\chi^2(P\|Q), s \leqslant 2, \tag{4.5}$$

$$\frac{r^{s-2}}{2}\chi^2(P\|Q) \leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \leqslant \frac{R^{s-2}}{2}\chi^2(P\|Q), s \geqslant 2. \tag{4.6}$$

Proof. According to expression (3.7), we have

$$\phi_s''(u) = u^{s-2}.$$

Now if $u \in [a,b] \subset (0,\infty)$, then we have

$$b^{s-2} \leqslant \phi_s''(u) \leqslant a^{s-2}, \quad s \leqslant 2,$$

or accordingly,

$$\phi_s''(u)\begin{cases} \leqslant r^{s-2}, & s \leqslant 2, \\ \geqslant r^{s-2}, & s \geqslant 2 \end{cases} \tag{4.7}$$

and

$$\phi_s''(u)\begin{cases} \leqslant R^{s-2}, & s \geqslant 2, \\ \geqslant R^{s-2}, & s \leqslant 2, \end{cases} \tag{4.8}$$

where r and R are as defined above.

Thus in view of (4.7), (4.8) and (4.1), we get the inequalities (4.3) and (4.4). Again, in view of (4.7), (4.8) and (4.2), we get the inequalities (4.5) and (4.6). □

In view of Result 4.1, we obtain the following corollary.

Corollary 4.1. *Under the conditions of Result* 4.1, *we have*

$$\frac{1}{2R^2}\chi^2(P\|Q) \leqslant K(Q\|P) \leqslant \frac{1}{2r^2}\chi^2(P\|Q), \tag{4.9}$$

$$\frac{1}{2R}\chi^2(P\|Q) \leqslant K(P\|Q) \leqslant \frac{1}{2r}\chi^2(P\|Q), \tag{4.10}$$

$$\frac{1}{8\sqrt{R^2}}\chi^2(P\|Q) \leqslant h(P\|Q) \leqslant \frac{1}{8\sqrt{r^3}}\chi^2(P\|Q), \tag{4.11}$$

$$\frac{R+1}{2R^2}\chi^2(P\|Q) \leqslant J(P\|Q) \leqslant \frac{r+1}{2r^2}\chi^2(P\|Q). \tag{4.12}$$

Proof. (4.9) follows by taking $s = 0$, (4.10) follows by taking $s = 1$ and (4.11) follows by taking $s = \frac{1}{2}$ in (4.3). (4.12) follows by adding (4.9) and (4.10). While for $s = 2$, we have equality sign. □

Corollary 4.2. *Under the conditions of Result* 4.1, *one gets*

$$\frac{1}{2R^2}\chi^2(P\|Q) \leqslant \chi^2(Q\|P) - K(Q\|P) \leqslant \frac{1}{2r^2}\chi^2(P\|Q), \tag{4.13}$$

$$\frac{1}{2R}\chi^2(P\|Q) \leqslant K(Q\|P) \leqslant \frac{1}{2r}\chi^2(P\|Q), \tag{4.14}$$

$$\frac{1}{8\sqrt{R^3}}\chi^2(P\|Q) \leqslant \frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q) \leqslant \frac{1}{8\sqrt{r^3}}\chi^2(P\|Q). \tag{4.15}$$

Proof. (4.13) follows by taking $s = 0$, (4.14) follows by taking $s = 1$ and (4.15) follows by taking $s = \frac{1}{2}$ in (4.5). While for $s = 2$, we have equality sign. $\square$

4.2. Information bounds in terms of Kullback–Leibler relative information

The case $s = 1$ of Theorem 3.4 gives:

Proposition 4.2. *Let* $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ *be a mapping which is normalized, i.e.,* $f(1) = 0$ *and suppose the assumptions*:

(i) f *is twice differentiable on* (r, R), *where* $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;
(ii) *there exists the real constants* m, M *such that* $m < M$ *and*

$$m \leqslant xf''(x) \leqslant M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ *are discrete probability distributions satisfying the assumption*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$mK(P\|Q) \leqslant C_f(P\|Q) \leqslant MK(P\|Q) \tag{4.16}$$

and

$$mK(Q\|P) \leqslant \rho_f(P\|Q) - C_f(P\|Q) \leqslant MK(Q\|P), \tag{4.17}$$

where $\rho_f(P\|Q)$ *and* $K(P\|Q)$ *are as given by* (3.10) *and* (1.1), *respectively*.

In view of Proposition 4.2 we have the following result.

Result 4.2. *Let* $P, Q \in \Delta_n$ *and* $s \in \mathbb{R}$. *Let there exists* r, R *such that* $r < R$ *and*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then Proposition 4.2 *yields*

$$r^{s-1}K(P\|Q) \leqslant \Phi_s(P\|Q) \leqslant R^{s-1}K(P\|Q), \quad s \geqslant 1, \tag{4.18}$$

$$R^{s-1}K(P\|Q) \leqslant \Phi_s(P\|Q) \leqslant r^{s-1}K(P\|Q), \quad s \leqslant 1, \tag{4.19}$$

$$r^{s-1}K(Q\|P) \leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \leqslant R^{s-1}K(Q\|P), \quad s \geqslant 1, \tag{4.20}$$

$$R^{s-1}K(Q\|P) \leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \leqslant r^{s-1}K(Q\|P), \quad s \leqslant 1. \tag{4.21}$$

Proof. According to expression (3.7), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r,R] \to \mathbb{R}$ such that $g(u) = u\phi_s''(u) = u^{s-1}$, then

$$\sup_{u\in[r,R]} g(u) = \begin{cases} R^{s-1}, & s \geqslant 1, \\ r^{s-1}, & s \leqslant 1 \end{cases} \tag{4.22}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{s-1}, & s \geqslant 1, \\ R^{s-1}, & s \leqslant 1. \end{cases} \tag{4.23}$$

In view of (4.22), (4.23) and (4.16), we have the proof of the inequalities (4.18) and 4.19. Again in view of (4.22), (4.23) and (4.17) we have the proof of the inequalities (4.20) and (4.21). □

In view of Result 4.2, we state the following corollaries.

Corollary 4.3. *Under the conditions of Result* 4.2, *one gets*

$$\frac{1}{R}K(P\|Q) \leqslant K(Q\|P) \leqslant \frac{1}{r}K(P\|Q), \tag{4.24}$$

$$\frac{1}{4\sqrt{R}}K(P\|Q) \leqslant h(P\|Q) \leqslant \frac{1}{4\sqrt{r}}K(P\|Q), \tag{4.25}$$

$$2rK(P\|Q) \leqslant \chi^2(P\|Q) \leqslant 2RK(P\|Q). \tag{4.26}$$

Proof. (4.24) follows by taking $s = 0$, (4.25) follows by taking $s = \frac{1}{2}$ in (4.19) and (4.26) follows by taking $s = 2$ in (4.18). For $s = 1$, we have equality sign. □

Corollary 4.4. *Under the conditions of Result* 4.2, *we obtain*

$$\frac{1}{R}K(Q\|P) \leqslant \chi^2(P\|Q) - K(Q\|P) \leqslant \frac{1}{r}K(Q\|P), \tag{4.27}$$

$$\frac{1}{4\sqrt{R}}K(P\|Q) \leqslant \frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q) \leqslant \frac{1}{4\sqrt{r}}K(Q\|P), \tag{4.28}$$

$$2rK(Q\|P) \leqslant \chi^2(P\|Q) \leqslant 2RK(Q\|P). \tag{4.29}$$

Proof. (4.27) follows by taking $s=0$, (4.28) follows by taking $s=\frac{1}{2}$ in (4.21) and (4.29) follows by taking $s=2$ in (4.20). For $s=1$, we have equality sign. □

The following corollary is a consequence of the Corollaries 4.3 and 4.4.

Corollary 4.5. *Under the conditions of Result* 4.2, *one gets*

$$r \leqslant \frac{K(P\|Q)}{K(Q\|P)} \leqslant R, \tag{4.30}$$

$$4\sqrt{r} \leqslant \frac{K(P\|Q)}{h(P\|Q)} \leqslant 4\sqrt{R}, \tag{4.31}$$

$$2r \leqslant \frac{\chi^2(P\|Q)}{K(P\|Q)} \leqslant 2R. \tag{4.32}$$

The inequalities given in Corollary 4.5 can also be written in different forms given below.

Corollary 4.6. *Under the conditions of Result* 4.2, *we obtain*

$$\frac{1+R}{R}K(P\|Q) \leqslant J(P\|Q) \leqslant \frac{1+r}{r}K(P\|Q), \tag{4.33}$$

$$4\sqrt{r}(1-B(P\|Q)) \leqslant K(P\|Q) \leqslant 4\sqrt{R}(1-B(P\|Q)), \tag{4.34}$$

$$1-\frac{1}{4\sqrt{r}}K(P\|Q) \leqslant B(P\|Q) \leqslant 1-\frac{1}{4\sqrt{R}}K(P\|Q), \tag{4.35}$$

$$\frac{1}{R}\chi^2(P\|Q) \leqslant J(P\|Q) \leqslant \frac{1}{r}\chi^2(P\|Q), \tag{4.36}$$

$$\frac{1}{2R}\chi^2(P\|Q) \leqslant K(P\|Q) \leqslant \frac{1}{2r}\chi^2(P\|Q). \tag{4.37}$$

Inequalities (4.34) and (4.35) are equivalent but are written in different forms.

In particular for $s=0$ in Theorem 3.4, we can state the following proposition.

Proposition 4.3. *Let $f: I \subset \mathbb{R}_+ \to \mathbb{R}$ be a mapping which is normalized, i.e., $f(1)=0$ and suppose the assumptions*:

(i) *f is twice differentiable on (r,R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M such that $m<M$ and*

$$m \leqslant x^2 f''(x) \leqslant M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$mK(Q\|P) \leqslant C_f(P\|Q) \leqslant MK(Q\|P) \tag{4.38}$$

and

$$\begin{aligned} m(\chi^2(Q\|P) - K(Q\|P)) &\leqslant \rho_f(P\|Q) - C_f(P\|Q) \\ &\leqslant M(\chi^2(Q\|P) - K(Q\|P)), \end{aligned} \tag{4.39}$$

where $\rho_f(P\|Q)$, $\chi^2(P\|Q)$ and $K(P\|Q)$ as given by (3.10), (2.6) *and* (1.1), *respectively*.

Inequalities (4.38) and (4.39) are new and were not studied before.

In view of Proposition 4.3, we obtain the following result.

Result 4.3. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exists r, R such that $r < R$ and*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then Proposition 4.3 *yields*

$$r^s K(Q\|P) \leqslant \Phi_s(P\|Q) \leqslant R^s K(Q\|P), \quad s \geqslant 0, \tag{4.40}$$

$$R^s K(Q\|P) \leqslant \Phi_s(P\|Q) \leqslant r^s K(Q\|P), \quad s \leqslant 0, \tag{4.41}$$

$$\begin{aligned} r^s(\chi^2(Q\|P) - K(Q\|P)) &\leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \\ &\leqslant R^s(\chi^2(Q\|P) - K(Q\|P)), \quad s \geqslant 0, \end{aligned} \tag{4.42}$$

$$\begin{aligned} R^s(\chi^2(Q\|P) - K(Q\|P)) &\leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \\ &\leqslant r^s(\chi^2(Q\|P) - K(Q\|P)), \quad s \leqslant 0. \end{aligned} \tag{4.43}$$

Proof. According to expression (3.7), we can write

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \to \mathbb{R}$ such that $g(u) = u^2\phi_s''(u) = u^s$, then

$$\sup_{u \in [r,R]} g(u) = \begin{cases} R^s, & s \geqslant 0, \\ r^s, & s \leqslant 0 \end{cases} \tag{4.44}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^s, & s \geqslant 0, \\ R^s, & s \leqslant 0. \end{cases} \tag{4.45}$$

In view of (4.44), (4.45) and (4.38) we have the inequalities (4.40) and (4.41). Again in view of (4.44), (4.45) and (4.39) we have the inequalities (4.42) and (4.43). □

In view of Result 4.3, we get the following corollaries.

Corollary 4.7. *Under the conditions of Result* 4.3, *we have*

$$rK(Q\|P) \leqslant K(P\|Q) \leqslant RK(Q\|P), \tag{4.46}$$

$$\frac{1}{4}\sqrt{r}K(Q\|P) \leqslant h(P\|Q) \leqslant \frac{1}{4}\sqrt{R}K(Q\|P), \tag{4.47}$$

$$2r^2K(Q\|P) \leqslant \chi^2(P\|Q) \leqslant 2R^2K(Q\|P). \tag{4.48}$$

Proof. (4.46) follows by taking $s = 1$, (4.47) follows by taking $s = \frac{1}{2}$ and (4.48) follows by taking $s = 2$ in (4.40). For $s = 0$, we have equality sign. □

Corollary 4.8. *Under the conditions of Result* 4.3, *we obtain*

$$\frac{1+R}{R}K(Q\|P) \leqslant \chi^2(Q\|P) \leqslant \frac{1+r}{r}K(Q\|P), \tag{4.49}$$

$$\begin{aligned} \frac{1}{4}\sqrt{r}(\chi^2(Q\|P) - K(Q\|P)) &\leqslant \frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q) \\ &\leqslant \frac{1}{4}\sqrt{R}(\chi^2(Q\|P) - K(Q\|P)), \end{aligned} \tag{4.50}$$

$$2r^2(\chi^2(Q\|P) - K(Q\|P)) \leqslant \chi^2(P\|Q) \leqslant 2R^2(\chi^2(Q\|P) - K(Q\|P)). \tag{4.51}$$

Proof. (4.49) follows by taking $s = 1$, (4.50) follows by taking $s = \frac{1}{2}$ and (4.51) follows by taking $s = 2$ in (4.42). For $s = 0$, we have equality sign. □

4.3. Information bounds in terms of Hellinger's discrimination

The case $s = \frac{1}{2}$ of Theorem 3.4 gives:

Proposition 4.4. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping is normalized, i.e., $f(1) = 0$ and satisfy the assumptions*:

(i) *f is twice differentiable on (r, R), where $0 \leqslant r \leqslant 1 \leqslant R \leqslant \infty$;*
(ii) *there exists the real constants m, M such that $m < M$ and*

$$m \leqslant x^{3/2} f''(x) \leqslant M, \quad \forall x \in (r, R).$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty,$$

then we have the inequalities:

$$4mh(P\|Q) \leqslant C_f(P\|Q) \leqslant 4Mh(P\|Q) \tag{4.52}$$

and

$$4m\left(\frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q)\right) \leqslant \rho_f(P\|Q) - C_f(P\|Q) \leqslant 4M\left(\frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q)\right), \tag{4.53}$$

where $h(P\|Q)$ is the Hellinger's divergence given by (2.5), *$\rho_f(P\|Q)$ is as given by* (3.10) *and $\eta_{1/2}(P\|Q)$ is as given by* (3.18).

In view of Proposition 4.4, we state the following result.

Result 4.4. *Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exists r, R such that $r < R$ and*

$$0 < r \leqslant \frac{p_i}{q_i} \leqslant R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then Proposition 4.4 *yields*

$$4r^{\frac{2s-1}{2}} h(P\|Q) \leqslant \Phi_s(P\|Q) \leqslant 4R^{\frac{2s-1}{2}} h(P\|Q), s \geqslant \frac{1}{2}, \tag{4.54}$$

$$4R^{\frac{2s-1}{2}} h(P\|Q) \leqslant \Phi_s(P\|Q) \leqslant 4r^{\frac{2s-1}{2}} h(P\|Q), s \leqslant \frac{1}{2}, \tag{4.55}$$

$$4r^{\frac{2s-1}{2}}\left(\frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q)\right) \leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \leqslant 4R^{\frac{2s-1}{2}}\left(\frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q)\right), \quad s \geqslant \frac{1}{2}, \tag{4.56}$$

$$4R^{\frac{2s-1}{2}}\left(\frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q)\right) \leqslant \eta_s(P\|Q) - \Phi_s(P\|Q) \leqslant 4r^{\frac{2s-1}{2}}\left(\frac{1}{4}\eta_{1/2}(P\|Q) - h(P\|Q)\right), \quad s \leqslant \frac{1}{2}. \tag{4.57}$$

Proof. Let the function $\phi_s(u)$ given by (3.5) is defined over $[r, R]$. Defining $g(u) = u^{3/2}\phi_s''(u) = u^{3/2}u^{s-2} = u^{\frac{2s-1}{2}}$, we get

$$\sup_{u\in[r,R]} g(u) = \begin{cases} R^{\frac{2s-1}{2}}, & s \geqslant \frac{1}{2}, \\ r^{\frac{2s-1}{2}}, & s \leqslant \frac{1}{2} \end{cases} \tag{4.58}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{\frac{2s-1}{2}}, & s \geqslant \frac{1}{2}, \\ R^{\frac{2s-1}{2}}, & s \leqslant \frac{1}{2}. \end{cases} \tag{4.59}$$

In view of (4.58), (4.59) and (4.52) we get the proof of the inequalities (4.54) and (4.55). Again in view of (4.58), (4.59) and (4.53) we get the proof of the inequalities (4.56) and (4.57). □

In view of Result 4.4, we obtain the following corollary.

Corollary 4.9. *Under the conditions of Result* 4.4, *one gets*

$$\frac{4}{\sqrt{R}} h(P\|Q) \leqslant K(Q\|P) \leqslant \frac{4}{\sqrt{r}} h(P\|Q), \tag{4.60}$$

$$4\sqrt{r} h(P\|Q) \leqslant K(P\|Q) \leqslant 4\sqrt{R} h(P\|Q), \tag{4.61}$$

$$8\sqrt{r^3} h(P\|Q) \leqslant \chi^2(P\|Q) \leqslant 8\sqrt{R^3} h(P\|Q). \tag{4.62}$$

Proof. (4.60) follows by taking $s = 0$ in (4.55). (4.61) follows by taking $s = 1$ and (4.62) follows by taking $s = 2$ in (4.57). For $s = \frac{1}{2}$, we have equality sign. □

Corollary 4.10. *Under the conditions of Result* 4.4, *we have*

$$\begin{aligned} \frac{4}{\sqrt{R}} \left(\frac{1}{4} \eta_{1/2}(P\|Q) - h(P\|Q) \right) &\leqslant \chi^2(Q\|P) - K(Q\|P) \\ &\leqslant \frac{4}{\sqrt{r}} \left(\frac{1}{4} \eta_{1/2}(P\|Q) - h(P\|Q) \right), \end{aligned} \tag{4.63}$$

$$4\sqrt{r} \left(\frac{1}{4} \eta_{1/2}(P\|Q) - h(P\|Q) \right) \leqslant K(P\|Q) \leqslant 4\sqrt{R} \left(\frac{1}{4} \eta_{1/2}(P\|Q) - h(P\|Q) \right), \tag{4.64}$$

$$8\sqrt{r^3} \left(\frac{1}{4} \eta_{1/2}(P\|Q) - h(P\|Q) \right) \leqslant \chi^2(P\|Q) \leqslant 8\sqrt{R^3} \left(\frac{1}{4} \eta_{1/2}(P\|Q) - h(P\|Q) \right). \tag{4.65}$$

Proof. (4.63) follows by taking $s = 0$, (4.64) follows by taking $s = 1$ and (4.65) follows by taking $s = 2$ in Result 4.4. For $s = \frac{1}{2}$, we have equality sign. □

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