On Simple Binomial Approximations for Two Variable Functions in Finance Applications

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Abstract

We extend the volatility stabilization transformation technique to two correlated Brownian motions. This technique allows to construct a computationally simple binomial tree and to obtain the probabilities for the up- and down- movements. We derive the expressions for correlated Geometric Brownian Motions by considering two variable functions. We discuss particular functions of two variables, which are commonly employed in finance. Further, we simulate results for the numerical accuracy of the approximations using an exchange option.

Keywords: Contingent claims, option pricing, numerical approximations, volatility stabilization transformation

1. Introduction

Nelson and Ramaswamy (1990) used an elegant instantaneous volatility stabilization transformation to approximate diffusions commonly used in finance such as the Ornstein-Uhlenbeck (OU or mean reversion) process and the Constant Elasticity of Variance (CEV) to a computationally simple binomial lattice. Although, binomial approximations for these types of diffusions may exist, the binomial tree structures may not necessarily recombine. Such binomial tree structures are computationally complex because the number of nodes in the tree doubles at each time step. The idea is to obtain a computationally simple binomial tree structure where an up move followed by a down move causes a displacement which is equal to a displacement caused by a down move that is followed by an up move. This objective is achieved by employing a transformation that makes the heteroskedastic process a homoskedastic process. In other words, employing a transformation that makes the instantaneous volatility of the transformed process constant.

In this paper, we extend the volatility stabilization transformation technique for two variable functions. There are numerous situations where two variable functions are commonly encountered when pricing options. We derive general expressions for correlated Geometric Brownian Motions. Then we consider some cases, which are commonly employed in finance applications. The paper is organized as follows: Section 2 includes the transformation technique applied by Nelson and Ramaswamy (1990) to a single asset that follows a diffusion process. We extend the transformation technique to two correlated Brownian motions in Section 3. Log transformed variables are presented in Section 4. Section 5 discuses the numerical accuracy of the approximations using an exchange option. A summary of findings and conclusions are included in Section 6.

2. Nelson-Ramaswamy Instantaneous Volatility Stabilization Transformation

The basic intuition of the instantaneous volatility stabilization transformation is as follows. Consider the stochastic differential equation

$$dy = \mu(y,t)dt + \sigma(y,t)dW \tag{1}$$

where W is a standard Brownian motion, $\mu(y,t)$, $\sigma(y,t) \geq 0$, are the instantaneous drift and standard deviation of y at time t and the initial value y_0 is a constant. The time interval $[0, T]$ is divided into n equal time steps of size $\Delta t = T/n$. The objective is to find a sequence of binomial processes that converge in probability to the process (1) on $[0, T]$.

Nelson and Ramaswamy (1990) consider a transform $X(y,t)$ which is twice differentiable in y and once in t. By Ito's Lemma,

$$dX(y,t) = \left(\mu(y,t)\frac{\partial X(y,t)}{\partial y} + \frac{1}{2}\sigma^2(y,t)\frac{\partial^2 X(y,t)}{\partial^2 y} + \frac{\partial X(y,t)}{\partial t} \right)dt + \left(\sigma(y,t)\frac{\partial X(y,t)}{\partial y} \right)dW \tag{2}$$

Now make the term

$$\frac{\partial X(y,t)}{\partial y}\sigma(y,t)dW = dW$$

in (2) so that the instantaneous volatility of the transformed process $x = X(y,t)$ is constant by taking

$$\sigma(y,t)\frac{\partial X(y,t)}{\partial y} = 1$$

Then by integrating the above term

$$\int \partial X(y,t) = \int \frac{\partial y}{\sigma(y,t)}$$

and substituting y by z we get

$$X(y,t) = \int^{y} \frac{dZ}{\sigma(z,t)}, \tag{3}$$

on the support of y. The above transformation allows one to construct a computationally simple binomial tree for the transformed process x where the variance of local change in x is constant at each node. The binomial lattice for the X process can be obtained by defining $X_0 = X(y_0)$ and drawing the X tree as shown in Figure 1.

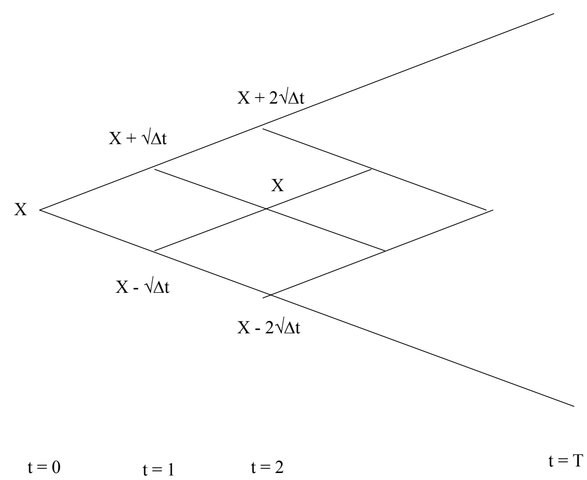

Figure 1: Simple Binomial Tree for X

In order to arrive at the binomial process for y one has to transform from x back to y. Using an inverse transformation defined as

$$Y(x,t) = \{y : X(y,t) = x\} \tag{4}$$

does this. Substituting equation (4) in equation (3) we get

$$x = \int^{Y} \frac{dZ}{\sigma(z,t)}$$

and taking the partial derivative we obtain $\partial y/\partial x = \sigma(y,t)$ which implies that $Y(x,t)$ is weakly monotone in x for a fixed value of t. The inverse transform in equation (4) can be used to construct the lattice for y such that the up- movement $Y^{+}(x,t)$ and a down- movement $Y^{-}(x,t)$ are given by

$$Y^{+}(x,t) = Y\left(x + \sqrt{\Delta t}, t + \Delta t\right) \tag{5}$$

$$Y^{-}(x,t) = Y\left(x - \sqrt{\Delta t}, t + \Delta t\right) \tag{6}$$

and the up- movement probability

$$p = \frac{\Delta t \mu\left(Y(x,t),t\right) + Y(x,t) - Y^{-}(x,t)}{Y^{+}(x,t) - Y^{-}(x,t)} \tag{7}$$

The use of the transform, inverse transform and a feasible probability enables one to construct computationally simple binomial approximation for y. The binomial tree for y is shown in Figure 2.

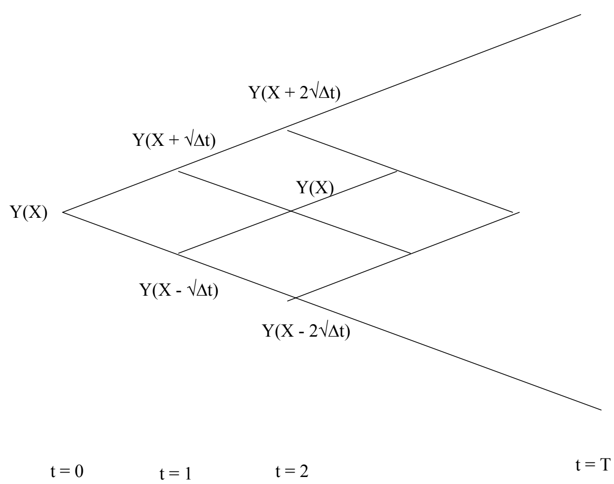

Figure 2: A Simple Binomial Tree for $y = \mathrm{Y}(X)$

3. Transform for Two Variables

We extend the transformation technique for two correlated Brownian motions. Consider a function of two variables S_1, S_2 each following a Geometric Brownian Motion where

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW_1 \tag{8a}$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW_2 \tag{8b}$$

with $\varepsilon\left[dW_1 dW_2\right] = \rho$, the correlation between S_1 and S_2.

Consider a general functional form as a power function of S_1 and S_2 given by

$F(S_1 , S_2, t) = S_1^{\mathrm{a}} S_2^{\mathrm{b}}$, where constants a and b are real numbers

Since

$$\partial F/\partial S_1 = aS_1^{a-1}S_2^b \ , \qquad \partial F/\partial S_2 = bS_1^a S_2^{b-1} \ , \qquad \partial F/\partial t = 0$$

$$\partial^2 F/\partial S_1^2 = a(a-1)S_1^{a-2}S_2^b \qquad \partial^2 F/\partial S_2^2 = b(b-1)S_1^a S_2^{b-2} \quad , \ \partial^2 F/\partial S_1 \partial S_2 = abS_1^{a-1}S_2^{b-1}$$

from the Ito lemma

$$dF = aS_1^{a-1}S_2^b dS_1 + bS_1^a S_2^{b-1} dS_2 + \frac{1}{2}a(a-1)S_1^{a-2}S_2^b\left(dS_1\right)^2 + \frac{1}{2}b(b-1)S_1^a S_2^{b-2}\left(dS_2\right)^2 + abS_1^{a-1}S_2^{b-1}dS_1 dS_2$$

8(c)

Substituting for dS_1 and dS_2 and rearranging terms we obtain

$$dF = \left[(a\mu_1 + b\mu_2) + \frac{1}{2}\left(a(a-1)\sigma_1^2 + b(b-1)\sigma_2^2 + \rho ab\sigma_1\sigma_2\right)\right]Fdt + \left(a\sigma_1 dW_1 + b\sigma_2 dW_2\right)F$$

8(d)

Notice that F follows a Geometric Brownian Motion with

$$\mu\left(F,t\right) = \left[(a\mu_1 + b\mu_2) + \frac{1}{2}\left(a(a-1)\sigma_1^2 + b(b-1)\sigma_2^2 + \rho ab\sigma_1\sigma_2\right)\right],$$

$\left(a\sigma_1 dW_1 + b\sigma_2 dW_2\right) \geq 0$ where W_i are Wiener processes. Now define

$dW = a\sigma_1 dW_1 + b\sigma_2 dW_2$, then since $dW_i \sim \mathrm{N}(0,1)$ and σ_i are constant, for i = 1, 2 we have

$$dW \sim \mathrm{N}\left(0, \mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2\right) \tag{8e}$$

The standardized value is

$$dW_z = \frac{dW}{\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2}} \tag{8f}$$

Substituting for dW in Equation (8d) we have

$$dF = \left[(a\mu_1 + b\mu_2) + \frac{1}{2}\left(a(a-1)\sigma_1^2 + b(b-1)\sigma_2^2 + \rho ab\sigma_1\sigma_2\right)\right]Fdt + F\left(\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2}\right)dW_z \tag{8g}$$

In order to obtain a computationally simple binomial approximation we need to make the volatility term constant in Equation (8g). The transform is

$$H(F,t) = \int^F \frac{dZ}{\sigma(Z,t)} = \frac{\ln F}{\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma}} \tag{8h}$$

The inverse transformation provides $F(H) = e^{H\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma}}$ and defining

$H(F_0) = \dfrac{\ln F_0}{\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma}}$, we obtain the H – tree and F tree as previously. From the

F –tree in Figure 3 and equation (7), we can obtain the expressions for up, down movements and the probability on an up- movement.

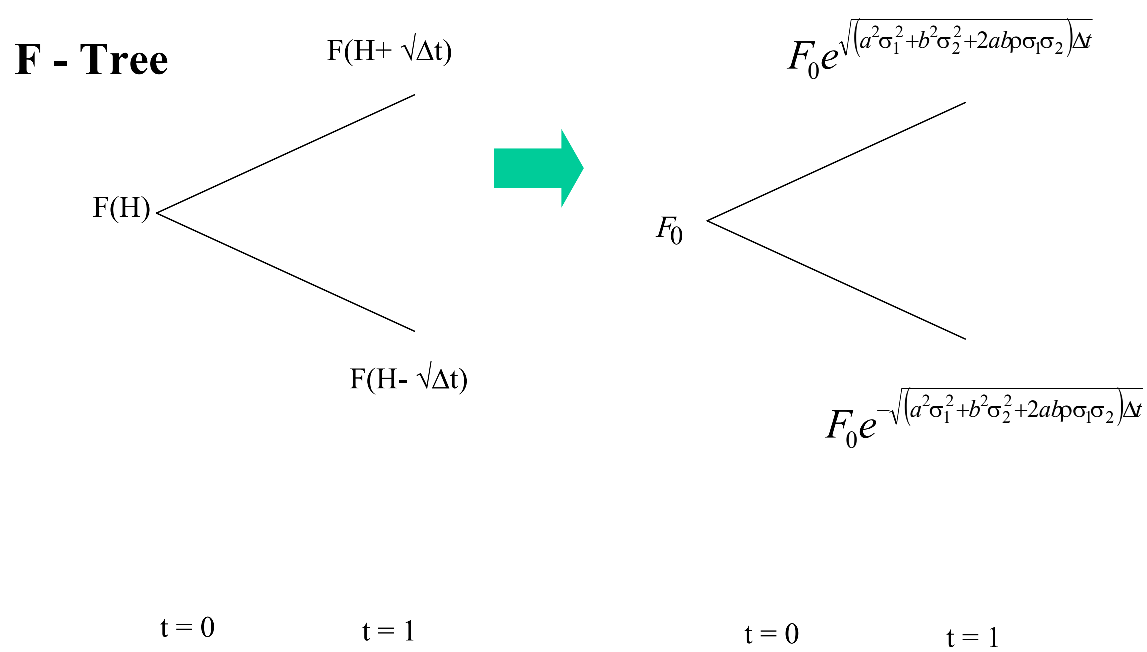

Figure 3: One Period Binomial Tree F = $S_1^{a} S_2^{b}$

In what follows now, we present some commonly used functions in finance applications. The multiplicative function form can be used when an option on an underlying asset value has two sources of uncertainty. For example, when valuing a forest concession, where the value of standing timber is a function of price and inventories each following a diffusion process.

3.1: Product of the underlying variables: a =b =1, i.e., F = F(S_1, S_2, t) = S_1S_2

From equation (8d) we have

$$dF = \left(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2\right)Fdt + \left(\sigma_1 dW_1 + \sigma_2 dW_2\right)F \tag{9a}$$

Hence F follows a Geometric Brownian Motion with $\mu(F,t) = \left(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2\right)$, $\left(\sigma_1 dW_1 + \sigma_2 dW_2\right) \geq 0$ and W_i are Wiener processes. Define $dW = \sigma_1 dW_1 + \sigma_2 dW_2$, then since $dW_i \sim \mathrm{N}(0,1)$ and σ_i are constant for i = 1, 2 we have

$$dW \sim \mathrm{N}\left(0, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right) \tag{9b}$$

The standardized value is

$$dW_z = \frac{dW}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}} \tag{9c}$$

Substituting for dW in Equation (9a) we have

$$dF = \left(\mu_1 + \mu_2 + \rho\sigma_1\sigma_2\right)Fdt + F\left(\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\right)dW_z \tag{9d}$$

To make the volatility term constant in Equation (9d) the transformation is

$$H(F,t) = \int^F \frac{dZ}{\sigma(Z,t)} = \frac{\ln F}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}} \tag{9e}$$

The inverse transformation provides $F(H) = e^{H\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}$ and defining

$H(F_0) = \dfrac{\ln F_0}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}$, we obtain the $H-$ tree and F tree as previously.

Now we consider the ratio functional form, which is typically, encountered among others in exchange options and real options to abandon a project for its salvage value. For example, an opportunity to exchange one company's securities for those of another within a stated time period Margrabe (1978).

3.2: Relative value of the underlying variables a=1, b= -1, i.e., F = F(S_1, S_2, t) = S_1 / S_2

From equation (8d) we have

$$dF = \left(\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2\right)Fdt + \left(\sigma_1 dW_1 - \sigma_2 dW_2\right)F \tag{10a}$$

Therefore F follows a Geometric Brownian Motion with $\mu(F,t) = \left(\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2\right)$,

and $\left(\sigma_1 dW_1 - \sigma_2 dW_2\right) \geq 0$. Define $dW = \sigma_1 dW_1 - \sigma_2 dW_2$, then since $dW_i \sim \mathrm{N}(0,1)$ and σ_i are constant for i = 1, 2 we have

$$dW \sim \mathrm{N}\left(0, \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) \tag{10b}$$

The standardized value is

$$dW_z = \frac{dW}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \tag{10c}$$

Substituting for dW in Equation (10a) we have

$$dF = \left(\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2\right)Fdt + F\left(\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\right)dW_z \tag{10d}$$

Making the volatility term constant in Equation (10d) gives us a computationally simple binomial approximation. The transform is

$$H(F,t) = \int^{F} \frac{dZ}{\sigma(Z,t)} = \frac{\ln F}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \tag{10e}$$

The inverse transformation provides $F(H) = e^{H\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}$ and defining

$H(F_0) = \frac{\ln F_0}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}$, we obtain the $H-$ tree and F tree as in the previous cases.

The case discussed in subsection 3.3 has applications for example, in the valuation of basket options (Rubinstein 1994) where the distribution of the weighted forward price of all assets in the basket is approximated by the geometric average.

3.3: Geometric average of underlying variables a = b = 0.5, i.e. F = F(S_1, S_2, t) = (S_1 S_2)$^{1/2}$

Substituting a = b = 0.5 in equation (8d), we have

$$dF = \frac{1}{2}\left((\mu_1 + \mu_2) - \frac{1}{4}(\sigma_1^2 + \sigma_2^2 - \rho\sigma_1\sigma_2)\right)Fdt + \frac{1}{2}\left(\sigma_1 dW_1 + \sigma_2 dW_2\right)F \tag{11a}$$

In the above equation F follows a Geometric Brownian Motion with

$\mu(F,t)=\frac{1}{2}\left((\mu_1+\mu_2)-\frac{1}{4}(\sigma_1^2+\sigma_2^2-\rho\sigma_1\sigma_2)\right)$, and $(\sigma_1 dW_1+\sigma_2 dW_2)\geq 0$. Define

$dW=\frac{1}{2}(\sigma_1 dW_1+\sigma_2 dW_2)$, then since $dW_i \sim \mathrm{N}(0,1)$ and σ_i are constant for i = 1, 2, we have

$$dW \sim \mathrm{N}\left(0,\frac{1}{4}(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)\right) \tag{11b}$$

The standardized value is

$$dW_z=\frac{2dW}{\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}} \tag{11c}$$

Substituting for dW in Equation (11a) we have

$$dF=\frac{1}{2}\left((\mu_1+\mu_2)-\frac{1}{4}(\sigma_1^2+\sigma_2^2-\rho\sigma_1\sigma_2)\right)Fdt+F\left(\frac{1}{2}\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}\right)dW_z \tag{11d}$$

Making the volatility term constant in Equation (11d) gives us a computationally simple binomial approximation. The transform is

$$H(F,t)=\int^F\frac{dZ}{\sigma(Z,t)}=\frac{2\ln F}{\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}} \tag{11e}$$

The inverse transformation provides $F(H)=e^{\frac{1}{2}H\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}}$ and defining

$H(F_0)=\frac{2\ln F_0}{\sqrt{\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2}}$, we obtain the $H-$ tree and F tree as in the previous cases.

We summarize the parameters, mean and variance of the processes for two state variables in Table 1.

Table 1: Mean and variances of the commonly used functions.

(a,b)	F(S_1, S_2, t)	Mean	Variance
(1,1)	S_1S_2	$\mu_1+\mu_2+\rho\sigma_1\sigma_2$	$\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$
(1,-1)	S_1/S_2	$\mu_1-\mu_2+\sigma_2^2-\rho\sigma_1\sigma_2$	$\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$
(0.5,0.5)	$(S_1 S_2)^{1/2}$	$\frac{1}{2}\left((\mu_1+\mu_2)-\frac{1}{4}(\sigma_1^2+\sigma_2^2-\rho\sigma_1\sigma_2)\right)$	$\frac{1}{4}(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)$

In the next section, we consider $F(S_1, S_2, t)$ as a function of log transformed variables. The log transformed variables are useful in valuing complex investments with multiple interactive options, options with non-proportional dividends and compound options (with a series of exercise prices) Trigeorgis (1991).

4 Log-Transformed Variables

In general, let $F(S_1, S_2, t) = \ln(S_1^a S_2^b)$ where constants a and b are real numbers and ln is the natural logarithm.

$$\text{Since} \qquad \partial F/\partial S_1 = \frac{a}{S_1}, \qquad \partial F/\partial S_2 = \frac{b}{S_2}, \qquad \partial F/\partial t = 0$$

$$\partial^2 F/\partial S_1^2 = \frac{-a}{S_1^2} \qquad \partial^2 F/\partial S_2^2 = \frac{-b}{S_2^2}, \quad \partial^2 F/\partial S_1 \partial S_2 = 0$$

from the Ito lemma

$$dF = \frac{a}{S_1} dS_1 + \frac{b}{S_2} dS_2 + \frac{1}{2}\left(\frac{-a}{S_1^2}\right)(dS_1)^2 + \frac{1}{2}\left(\frac{-b}{S_2^2}\right)(dS_2)^2 \tag{12a}$$

Substituting for dS_1 and dS_2 and rearranging terms we obtain

$$dF = \left[(a\mu_1 + b\mu_2) - \frac{a\sigma_1^2}{2} - \frac{b\sigma_2^2}{2} \right] dt + \left(a\sigma_1 dW_1 + b\sigma_2 dW_2 \right) \tag{12b}$$

Here, F follows a Geometric Brownian Motion with $\mu(F,t) = \left[(a\mu_1 + b\mu_2) - \frac{a\sigma_1^2}{2} - \frac{b\sigma_2^2}{2} \right]$,

$\left(a\sigma_1 dW_1 + b\sigma_2 dW_2 \right) \geq 0$ where W_i are Wiener processes. Define $dW = a\sigma_1 dW_1 + b\sigma_2 dW_2$,

then since $dW_i \sim \mathrm{N}(0,1)$ and σ_i are constant for $i = 1, 2$ we have

$$dW \sim \mathrm{N}\left(0, \mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2\right) \tag{12c}$$

The standardized value is

$$dW_z = \frac{dW}{\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2}} \tag{12d}$$

Substituting for dW in Equation (12b) we have

$$dF = \left[(a\mu_1 + b\mu_2) - \frac{a\sigma_1^2}{2} - \frac{b\sigma_2^2}{2} \right] F dt + F\left(\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2} \right) dW_z \tag{12e}$$

In order to obtain a computationally simple binomial approximation for the log variables we need to make the volatility term constant in Equation (12e). The transform is

$$H(F,t) = \int^F \frac{dZ}{\sigma(Z,t)} = \frac{\ln F}{\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2}} \tag{12f}$$

The inverse transformation provides $F(H) = e^{H\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2}}$ and defining

$H(F_0) = \dfrac{\ln F_0}{\sqrt{\mathrm{a}^2\sigma_1^2 + b^2\sigma_2^2 + 2ab\rho\sigma_1\sigma_2}}$, we obtain the H – tree and F tree as previously.

We discuss special processes which include the sum, and difference of two log transformed variables.

4.1: Sum of Log Transformed Variables: a =b =1, i.e., F = F(S_1, S_2, t) = ln(S_1S_2) = ln(S_1)+ ln(S_2)

Substituting a = b = 1in equation (12e), we have

$$dF = \left(\mu_1 + \mu_2 - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} \right) F dt + (\sigma_1 dW_1 + \sigma_2 dW_2) F \tag{13a}$$

where F follows a Geometric Brownian Motion with $\mu(F,t) = \left(\mu_1 + \mu_2 - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} \right)$,

$(\sigma_1 dW_1 + \sigma_2 dW_2) \geq 0$ and W_i Wiener processes. Define $dW = \sigma_1 dW_1 + \sigma_2 dW_2$, then since $dW_i \sim \mathrm{N}(0,1)$ and σ_i are constant for i = 1, 2 we have

$$dW \sim \mathrm{N}\left(0, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\right) \tag{13b}$$

The standardized value is given by

$$dW_z = \frac{dW}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}} \tag{13c}$$

Substituting for dW in Equation (13a) we have

$$dF = \left(\mu_1 + \mu_2 - \frac{\sigma_1^2}{2} - \frac{\sigma_2^2}{2} \right) F dt + F\left(\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\right) dW_z \tag{13d}$$

In order to obtain a computationally simple binomial approximation we need to make the volatility term constant in Equation (13d). The transform is

$$H(F,t) = \int^F \frac{dZ}{\sigma(Z,t)} = \frac{\ln F}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}} \tag{13e}$$

The inverse transformation provides $F(H) = e^{H\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}$ and defining

$H(F_0) = \frac{\ln F_0}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}}$, we obtain the $H-$ tree and F tree as previously.

4.2: Difference of Log Transformed Variables: a = 1, b =-1, i.e., F = F(S_1, S_2, t) = ln(S_1/$S2$) = ln(S_1)- ln(S_2)

Substituting a = 1, b = -1 in equation (12e), we have

$$dF = \left(\mu_1 - \mu_2 - \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\right)Fdt + \left(\sigma_1 dW_1 - \sigma_2 dW_2\right)F \tag{14a}$$

Hence F follows a Geometric Brownian Motion with $\mu(F,t) = \left(\mu_1 - \mu_2 - \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\right)$, $\left(\sigma_1 dW_1 - \sigma_2 dW_2\right) \geq 0$ and W_i Wiener processes. Define $dW = \sigma_1 dW_1 - \sigma_2 dW_2$, then since $dW_i \sim \mathrm{N}(0,1)$ and σ_i are constant for i = 1, 2 we have

$$dW \sim \mathrm{N}\left(0, \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right) \tag{14b}$$

The standardized value is

$$dW_z = \frac{dW}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \tag{14c}$$

Substituting for dW in Equation (14a) we have

$$dF = \left(\mu_1 - \mu_2 - \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\right)Fdt + F\left(\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\right)dW_z \tag{14d}$$

In order to obtain a computationally simple binomial approximation we need to make the volatility term constant in Equation (14d). The transform is

$$H(F,t) = \int^F \frac{dZ}{\sigma(Z,t)} = \frac{\ln F}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}} \tag{14e}$$

The inverse transformation provides $F(H) = e^{H\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}$ and defining

$H(F_0) = \dfrac{\ln F_0}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}$, we obtain the H – tree and F tree as previously.

The mean and variances of the processes with the log transformed variables discussed above are given in Table 2.

Table 2: Mean and variances of the commonly used functions.

(a,b)	F(S_1, S_2, t)	Mean	Variance
(1,1)	ln(S_1)+ ln(S_2)	$\mu_1+\mu_2-\frac{\sigma_1^2}{2}-\frac{\sigma_2^2}{2}$	$\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$
(1,-1)	ln(S_1)- ln(S_2)	$\mu_1-\mu_2-\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}$	$\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2$
(1,0)	ln(S_1)	$\mu_1-\frac{\sigma_1^2}{2}$	σ_1^2

5. Numerical Accuracy

In order to study the numerical accuracy of the binomial approximations with the volatility transformation for two variable functions, we consider the option to exchange one asset for another. For this purpose, we use the relative value of underlying assets discussed in section 3.2. We compare the exchange option values obtained from a one period and a two period binomial approximations (Rubinstein 1992b) with Margrabe's (1978) continuous time exchange option model. We choose the following parameters: asset values (in \$) $S_1 = S_1 = 10, 20, 30, 40, 50$; volatility $\sigma_1 = \sigma_2 = 5\%, 20\%$; correlation $\rho = 0$; and time to expiration T = 1 and 10 weeks. The percentage relative error in estimates is calculated as

$$\% \text{ Relative Error } = 100\left(\frac{\text{Estimate - Margrabe}}{\text{Margrabe}}\right)$$

The results for exchange option values with respect to different periods, and the percentage relative errors (in parenthesis) are given in Table 3. For the given parameters in Table 3, we have the following indicative observations:

(1) When the volatility is low, one period binomial approximations overestimate the exchange option values and the relative error is 25.3%, while the two period binomial approximations underestimates the exchange option values and the relative error is -9.52%

(2) When the volatility is high then both the one period and two period binomial approximations overestimate the exchange option values and the relative errors are 25.25% and 15.3% respectively.

(3) For both time periods, the two period binomial approximations provide better exchange option values (smaller relative error) than the one period binomial approximations.

Table 3: Exchange option values and percentage relative errors

$S_1 = S_2$	$\sigma_1 = \sigma_2 = 5\%;\ \rho = 0\ ;\ T = 1$			$\sigma_1 = \sigma_2 = 20\%;\ \rho = 0\ ;\ T = 10$		
	Margrabe	One period	Two period	Margrabe	One period	Two period
10	0.0391196	0.0490286 (25.3%)	0.0353973 (-9.52%)	0.4945114	0.6193798 (25.25%)	0.5701518 (15.3%)
20	0.0782392	0.0980573 (25.3%)	0.0707947 (-9.52%)	0.9890228	1.2387596 (25.25%)	1.1403036 (15.3%)
30	0.1173588	0.1470859 (25.3%)	0.106192 (-9.52%)	1.4835342	1.8581394 (25.25%)	1.7104554 (15.3%)
40	0.1564783	0.1961146 (25.3%)	0.1415893 (-9.52%)	1.9780456	2.4775192 (25.25%)	2.2806072 (15.3%)
50	0.1955979	0.2451432 (25.3%)	0.1769866 (-9.52%)	2.4725569	3.096899 (25.25%)	2.850759 (15.3%)

The one period binomial approximations provide option values accurate to within (.009 to .049) for $\sigma_1 = \sigma_2 = 5\%$; $\rho = 0$; T = 1 and (-0.003 to -0.018) for $\sigma_1 = \sigma_2 = 20\%$; $\rho = 0$; T = 10. For the two period binomial approximations, estimates are accurate within (.125 to .624) for $\sigma_1 = \sigma_2 = 5\%$; $\rho = 0$; T = 1 and (.075 to .378) for $\sigma_1 = \sigma_2 = 20\%$; $\rho = 0$; T = 10. The binomial approximations deteriorate as the option life is lengthened consistent with Nelson and Ramaswamy 1990.

Next by varying values of parameters of the exchange option, we simulated the option values presented in Table 4. We observe the following from numerical results in Table 4:

(1) Given $S_1 = S_2 = 10$, $\sigma_1 = \sigma_2 = 5\%$, 20%, and $S_1 = S_1 = 10$, $\sigma_1 = 5\%$, $\sigma_2 = 20\%$, and T = 1. With increasing ρ, it is observed that for the one period binomial approximation the percent relative error remains constant, while relative error is reduced in the two period binomial approximations.

(2) Given $S_1 = 20$, $S_2 = 10$, $\sigma_1 = \sigma_2 = 5\%$, 20%, and T = 1. When ρ is increased, the exchange option values for both the one and two period binomial approximations and Margrabe models are very close.

Table 4: Percentage Relative Errors

S_1	S_2	T	σ_1	σ_2	ρ	Margrabe	% Relative Error	
							One period	Two period
10	10	0.0192	0.05	0.05	-1	0.0553	25.33	-8.73
10	10	0.0192	0.05	0.05	0	0.0391	25.33	-9.52
10	10	0.0192	0.05	0.05	0.5	0.0277	25.33	-10.06
10	10	0.0192	0.05	0.05	0.95	0.0087	25.33	-10.96
10	10	0.0192	0.2	0.2	-1	0.2213	25.31	-0.31
10	10	0.0192	0.2	0.2	0	0.1565	25.32	-3.69
10	10	0.0192	0.2	0.2	0.5	0.1106	25.33	-6.01
10	10	0.0192	0.2	0.2	0.95	0.0350	25.33	-9.71
10	10	0.0192	0.05	0.2	-1	0.1383	25.32	-4.61
10	10	0.0192	0.05	0.2	0	0.1140	25.33	-5.84
10	10	0.0192	0.05	0.2	0.5	0.0997	25.33	-6.55
10	10	0.0192	0.05	0.2	0.95	0.0848	25.33	-7.29
20	10	0.0192	0.05	0.05	-1	10	0	0.07
20	10	0.0192	0.05	0.05	0	10	0	0.03
20	10	0.0192	0.05	0.05	0.5	10	0	0.02
20	10	0.0192	0.05	0.05	0.95	10	0	0
20	10	0.0192	0.2	0.2	-1	10	0	1.08
20	10	0.0192	0.2	0.2	0	10	0	0.54
20	10	0.0192	0.2	0.2	0.5	10	0	0.27
20	10	0.0192	0.2	0.2	0.95	10	0	0.03

6. Conclusions

To construct a computationally simple binomial approximation for diffusions, we considered a family of two correlated variables and of two log transformed variables. In particular, we showed how one could obtain the transforms for functions of two variables in multiplicative and ratio forms. We simulated exchange option values using one period and two period binomial approximations and compared with the Margrabe's model. Our numerical results indicate that the approximations work well for options with short maturity. Further as also noted by Nelson and Ramaswamy (1991), the error in estimate increases when option life is lengthened.

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