Multinomial Approximating Models for Options

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Abstract. We ensure non-negative probabilities for the Kamrad and Ritchken (1991) multinomial approximating model by bounding the stretch parameter, which parameterizes the size of the up- and down- jumps in the lattice. Next, we propose the inclusion of an omitted second order term and derive analytical bounds in order to reduce errors. We establish theoretical bounds and mathematical expressions to determine the number of nodes generated by the approximation process. Numerical examples are presented to illustrate our findings.

Key Words: Real Options; Contingent Claims; Option Pricing; Project Evaluation; Multinomial Lattice.

1. Introduction

Contingent claim models whose value depends on multiple sources of uncertainty have been developed for stock options in the finance literature (Kamrad and Ritchken 1991, Boyle 1988, Boyle Evnine and Gibbs 1989, Johnson 1987, and Stulz 1982). These models are useful for valuing real options having multiple sources of uncertainty. Often numerical procedures are used to approximate the stochastic process when there are multiple sources of uncertainties because analytical solutions are unavailable. Numerical procedures can handle early exercise features of American options. Such numerical procedures include finite difference schemes, lattice approaches and simulation.

Several researchers have developed lattice models to value multivariate contingent claims on stock. Boyle (1988) uses a trinomial lattice where he equates the first two moments to obtain jump probabilities. In order to ensure that jump probabilities are non-negative he introduces a stretch parameter, λ which need to be constrained. Boyle considers values of $\lambda \geq 1$ but does not provide ways to select a suitable stretch parameter. Boyle, Evnine and Gibbs (BEG) (1989) consider an alternative approximation procedure that allows them to generalize the model for n state variables. Boyle et al. (1989) uses a binomial lattice that gives a four-jump model when there are two state variables. The problem of negative probability is overcome by selecting a time step that is sufficiently small. Both these models do not however allow for horizontal jumps and do not provide the means to choose the value of time step that is sufficiently small. Nelson and Ramaswamy (1990) show how to construct computationally simple binomial processes that converge weakly to commonly employed diffusions in financial models. The method is based on the volatility stabilization transformation.

Kamrad and Ritchken (1991) propose a general multinomial approximation model for valuing claims on one or more state variables. KR model is mathematically elegant. Their research extends previous literature on multinomial approximating models by allowing for horizontal jumps. The basis of KR model is similar to BEG model in that a multinomial lattice approximates the logarithmic return process. KR model also uses a stretch parameter λ. The authors claim their model with horizontal jumps yields a feasible set of probabilities for any $\lambda \geq 1$. Kamrad and Ritchken (1991) show that when $\lambda = 1$, the binomial model is a special case of their one state model and BEG model is a special case of their two state model. They generalize their model for *k* state variables and illustrate the model using three-state variables.

Kamrad and Ritchken (1991) model has several limitations. KR model ignores higher order terms of time step in the approximation process when calculating jump probabilities. As a result the probability expressions for estimating jump probabilities introduce an error when pricing an option. Furthermore, the stretch parameter λ required to obtain a feasible set of probabilities is chosen arbitrarily. Arbitrary selection may impose an additional problem since the probability values depend on λ. Although Kamrad and Ritchken (1991) argue that any $\lambda \geq 1$ yields a feasible set of probabilities, we find that negative probabilities can occur when $\lambda \geq 1$ thus severely limiting model applicability.

Our work is motivated by an imprecision in the KR model to value a compound option. The imprecision is due to (i) the stretch parameter, which parameterizes the up- and down- jump in the tree and (ii) omission of the second order terms of the time step. We suggest including the omitted second order terms in the probability expressions for the KR model. Then using the new probability expressions we develop bounds which condition the stretch parameter to ensure that probabilities are non-negative. We prove theoretical results for the bounds and provide numerical examples to illustrate model results. The new probability expression are referred to as modified KR (MKR) model in this paper. In order to illustrate the computational advantage of proposed MKR model over KR model we present a general formula for the number of nodes generated by the approximating process for a 2^k +1 jumps and 2^k jumps.

In the next sections we develop new probability expressions for a single state model, two-state model and a *k*-state model using a three-state model as an illustration. We provide numerical examples under each case that show how negative probabilities may occur when $\lambda \geq 1$ for the KR model and illustrate the relative errors. Next, in order to obtain a feasible set of probabilities for any time step we derive analytical bounds for the stretch parameter and correlation coefficients. We illustrate though an example the gain in accuracy of the MKR model. Finally, we show the computational advantage of the MKR model over the KR model in terms of computational effort measured using the number of nodes generated by the process. Section 6 provides a conclusion.

2. Probability Expressions for One State Model

Kamrad and Ritchken (1991) multinomial approximation approach is as follows. We assume an underlying asset S follows a diffusion process with a drift rate $\mu = r - \sigma^2/2$ where, r is the risk-free rate, and σ is the instantaneous standard deviation. Then for the asset over time interval $[\, t$, $t + \Delta t\,]$ we have:

$$\ln S(t+\Delta t) = \ln S(t) + \zeta(t) \;, \tag{1}$$

where the normal random variable $\zeta(t)$ has mean $\mu\Delta t$ and variance $\sigma^2\,\Delta t$. Since we need to approximate the distribution for $\zeta(t)$ over the period $[\, t\, ,\, t + \Delta t\,]$, we consider a discrete random variable $\zeta^a(t)$ such that

$$\zeta^a(t) = \begin{cases} v & \text{with probability} \quad p_1, \\ 0 & \text{with probability} \quad p_2, \\ -v & \text{with probability} \quad p_3, \end{cases}$$

where $v = \lambda\sigma\sqrt{\Delta t}$ and the stretch parameter λ is to be determined. In order to determine the jump probabilities p_i we equate the mean and variance of $\zeta^a(t)$ to the mean and variance of $\zeta(t)$. More specifically;

$$\text{Mean}: \qquad v(p_1 - p_3) = \mu\Delta t \tag{2a}$$

$$\text{Variance}: \qquad v^2(p_1 + p_3) - (\mu\Delta t)^2 = \sigma^2\Delta t. \tag{2b}$$

If we denote $1/\theta$ as the coefficient of variation of the underlying asset then $\theta = \mu/\sigma$ and since $\sum p_i = 1$, we obtain the following expressions for the MKR probabilities from 2(a) and 2(b):

$$p_1 = \frac{1}{2}\left[\frac{1}{\lambda^2} + \frac{\theta\sqrt{\Delta t}}{\lambda} + \frac{\theta^2\Delta t}{\lambda^2}\right] \tag{3a}$$

$$p_2 = 1 - \left[\frac{1}{\lambda^2} + \frac{\theta^2\Delta t}{\lambda^2}\right] \tag{3b}$$

$$p_3 = \frac{1}{2}\left[\frac{1}{\lambda^2} - \frac{\theta\sqrt{\Delta t}}{\lambda} + \frac{\theta^2\Delta t}{\lambda^2}\right] \tag{3c}$$

KR model does not provide a way of selecting λ which may seriously limit the implementation of their model. Recall that when λ is selected arbitrarily the following problems may arise: (a) inability to guarantee a feasible set of probabilities; and (b) probability values depend on λ which will affect the option values. The following Theorem provides bounds for the stretch parameter, which makes the MKR probability non-negative.

THEOREM 2.1**:** *In the one state model for probabilities to be feasible* ($0 \le p_i \le 1$) *the stretch parameter* λ *must satisfy:* $\sqrt{1+\theta^2\Delta t} \le \lambda \le \dfrac{1+\theta^2\Delta t}{\theta\sqrt{\Delta t}}$

PROOF: For the probabilities to be non-negative from equations (3a) to (3c) we obtain the required result.

It is apparent from Theorem 2.1 that the choice of stretch parameter λ is not arbitrary. It is a function of the coefficient of variation of the asset and time step. In order to compare the KR and the MKR models we set the stretch parameter to its lower bound $\lambda = \sqrt{1+\theta^2 \Delta t}$. Note that when $\lambda = \sqrt{1+\theta^2 \Delta t}$ then $p_2 = 0$ and MKR model collapse to a two jump model and a binomial model as the KR model does when Δt is made sufficiently small.

REMARK 2.1. Kamrad and Ritchken (1991) claim that any $\lambda \geq 1$ yields a feasible set of probabilities. Contrary to this we find that negative probabilities can occur as illustrated in the example below. However, the MKR model always provides a feasible set of probabilities for the feasible bounds of λ.

EXAMPLE 2.1. Our first example deals with the situation when KR model gives negative probabilities. Specifically we select the following model parameters. Let r = 7%, σ = 3%, μ = 0.0695 and θ = 2.3183. For Δt = 0.25 (n = 4) the probabilities from KR model with λ = 1 are (1.08, 0, -0.08) and with λ = 1.2247 as considered by KR in their paper are (0.81, 0.33, -0.14). In the proposed model with λ = 1.5309 (obtained by setting $\lambda = \sqrt{(1 + \theta^2 \Delta t)}$) yield the probabilities (0.88, 0, 0.12).

REMARK 2.2. It may be noted that assuming Δt to be sufficiently small, Kamrad and Ritchken ignore the order term $(\mu\Delta t)^2$ in (2b). The resulting effect is not equating the variance of the two distributions, but considering the second moment of $\zeta(t)$ as equal to the variance of $\zeta^a(t)$. These omissions introduce errors in determining the jump probability p_i as given in Table 1.

Insert Table 1 here

As shown below the extent of this omission may be significant even for small values of Δt.

EXAMPLE 2.2. In this example we illustrate the size of the error from ignoring the order terms. Let r = 7%, σ = 8%, μ = 0.0668 and θ = 0.835. For Δt = 0.25 (n = 4) the probabilities from KR model with λ = 1 are (0.71, 0, 0.29) and from MKR model λ = 1.0836 are (0.69, 0, 0.31). Similarly when σ = 5%, Δt = 0.1 (n = 10) the probabilities from KR model with λ = 1 are (0.72, 0, 0.28) and from the proposed model λ = 1.0904 are (0.70, 0, 0.30). For σ = 5%, Δt = 0.5 (n = 2) the probabilities from KR model with λ = 1 are (0.75, 0, 0.25) and from the proposed MKR model with λ = 1.1613 are (0.80, 0, 0.20). The percent absolute relative errors in estimating p_i, defined as = $100|p_i(\mathrm{KR}) - p_i(\mathrm{MKR})| / p_i(\mathrm{MKR})$, in these cases are given in Table 2.

Insert Table 2 here

3. Probability Expressions for Two State Model

We define the asset pair $\{S_1(t), S_2(t)\}$ over time t with joint density of the two underlying assets as bi-variate lognormal. Assume the drift rate is $\mu_i = r - \sigma_i^2/2$ where σ_i is the instantaneous standard deviation of the i^{th} asset. As in the one state model for each underlying asset over time interval $[\,t\,,\,t + \Delta t\,]$ we have:

$$\ln S_i(t + \Delta t) = \ln S_i(t) + \zeta_i\ (t) \tag{3}$$

where $\zeta_i(t)$ is a normal random variable with mean $\mu_i\,\Delta t$ and variance $\sigma_i^2\,\Delta t$. Let the instantaneous correlation between $\zeta_1(t)$ and $\zeta_2(t)$ be ρ.

A pair of multinomial discrete random variables having the following distribution is used to approximate the joint normal random variable $\{\zeta_1(t), \zeta_2(t)\}$;

$\zeta_1^a(t)$	$\zeta_2^a(t)$	**Probability**
ν_1	ν_2	p_1
ν_1	$-\nu_2$	p_2
$-\nu_1$	$-\nu_2$	p_3
$-\nu_1$	ν_2	p_4
0	0	p_5

where $v_i = \lambda_i \sigma_i \sqrt{\Delta t}$ (i= 1,2), and as before the stretch parameter λ_i is to be determined.

A necessary condition for the convergence of true joint normal distribution and the approximate multinomial distribution is equality of mean, variance and covariance terms. Specifically this ensures that

$$\text{Mean:} \quad \nu_1(p_1 + p_2 - p_3 - p_4) = \mu_1 \Delta t \tag{3.1a}$$

$$\nu_2(p_1 - p_2 - p_3 + p_4) = \mu_2 \Delta t \tag{3.1b}$$

$$\text{Variance:} \quad \nu_1^2(p_1 + p_2 + p_3 + p_4) - (\mu_1 \Delta t)^2 = \sigma_1^2 \Delta t \tag{3.1c}$$

$$\nu_2^2(p_1 + p_2 + p_3 + p_4) - (\mu_2 \Delta t)^2 = \sigma_2^2 \Delta t \tag{3.1d}$$

$$\text{Covariance:} \quad \nu_1 \nu_2 (p_1 - p_2 + p_3 - p_4) - \mu_1 \mu_2 \Delta t^2 = \sigma_1 \sigma_2 \rho \Delta t \tag{3.1e}$$

REMARK 3.1. We now examine the implications of ignoring the O(Δt^2) terms in the two-state KR model. The corresponding equations obtained by equating the variance of the true and approximating distributions in the KR model (as per equation 5(c) and 5(d) page 1647 Kamrad and Ritchken (1991)) are

$$p_1 + p_2 + p_3 + p_4 = \frac{1}{\lambda_1^2}$$

$$p_1 + p_2 + p_3 + p_4 = \frac{1}{\lambda_2^2}$$

implying that $\lambda_1 = \lambda_2$. The corresponding equations from the MKR model as per equations 3.1(c) and 3.1(d) are

$$p_1 + p_2 + p_3 + p_4 = \frac{1+\theta_1^2 \Delta t}{\lambda_1^2}$$

$$p_1 + p_2 + p_3 + p_4 = \frac{1+\theta_2^2 \Delta t}{\lambda_2^2}$$

$$\frac{\lambda_1}{\sqrt{1+\theta_1{}^2 \Delta t}} = \frac{\lambda_2}{\sqrt{1+\theta_2{}^2 \Delta t}}.$$

where, $1/\theta_i$ is the coefficient of variation for asset *i*.

It is therefore evident that λ_i is a function of θ_i and Δt and λ_1, λ_2 will be equal if and only if $\theta_1 = \theta_2$. Thus the error in the KR model by ignoring the $O(\Delta t)$ terms, which means equating the variance to the second moment has resulted in making $\lambda_1 = \lambda_2$.

Next we obtain the estimations for jump probabilities from equations 3.1(1) through 3.1(e). Since $\sum p_i = 1$, the expressions for jump probabilities are given by:

$$p_1 = \frac{1}{4}\left[\frac{\theta_1\sqrt{\Delta t}}{\lambda_1} + \frac{\theta_2\sqrt{\Delta t}}{\lambda_2} + \frac{1+\theta_1^2\Delta t}{\lambda_1{}^2} + \frac{\rho+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}\right] \tag{3.2a}$$

$$p_2 = \frac{1}{4}\left[\frac{\theta_1\sqrt{\Delta t}}{\lambda_1} - \frac{\theta_2\sqrt{\Delta t}}{\lambda_2} + \frac{1+\theta_1^2\Delta t}{\lambda_1{}^2} - \frac{\rho+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}\right] \tag{3.2b}$$

$$p_3 = \frac{1}{4}\left[-\frac{\theta_1\sqrt{\Delta t}}{\lambda_1} - \frac{\theta_2\sqrt{\Delta t}}{\lambda_2} + \frac{1+\theta_1^2\Delta t}{\lambda_1{}^2} + \frac{\rho+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}\right] \tag{3.2c}$$

$$p_4 = \frac{1}{4}\left[-\frac{\theta_1\sqrt{\Delta t}}{\lambda_1} + \frac{\theta_2\sqrt{\Delta t}}{\lambda_2} + \frac{1+\theta_1^2\Delta t}{\lambda_1{}^2} - \frac{\rho+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2} \right] \tag{3.2d}$$

$$p_5 = 1 - \left[\frac{1+\theta_1^2\Delta t}{\lambda_1{}^2} \right] \tag{3.2e}$$

To obtain a feasible set of jump probabilities we follow the same approach as in the one state model. We present the results in the form of theorems with proofs in Appendix. We first derive the bound for the stretch parameters as per Theorem 3.1.

THEOREM 3.1: *In the two state MKR model for probabilities to be feasible* ($0 \le p_i \le 1$) *the stretch parameters* λ_1 *and* λ_2 *must satisfy*

$$\lambda_1 \ge \sqrt{1+\theta_1{}^2\Delta t} \tag{3.3(a)}$$

$$\lambda_2 = \lambda_1 \frac{\sqrt{1+\theta_2{}^2\Delta t}}{\sqrt{1+\theta_1{}^2\Delta t}} \;. \tag{3.3(b)}$$

We find that constraining the stretch parameter does not ensure positive probabilities for any Δt since probability values are also a function of the correlation coefficient. In order to ensure feasible probabilities we establish bounds for the instantaneous correlation coefficient ρ that makes the probabilities non-negative for any value of Δt using $\lambda_1 = \sqrt{1+\theta_1{}^2\Delta t}$ from (3.3a) and λ_2 from equation (3.3b). Notice that when $\lambda_1 = \sqrt{1+\theta_1{}^2\Delta t}$ and λ_2 from equation 3.3(b) MKR model collapses to a four jump model. On the other hand, KR model with $\lambda_1 = \lambda_2 = \lambda = \sqrt{1+\theta_1{}^2\Delta t}$ gives a five jump model. When $\lambda_1 = \lambda_2 = 1$, then KR model collapses to the BEG model. Theorem 3.2 provides bounds for correlation coefficient.

THEOREM 3.2: *For a two-asset MKR model the probabilities will always be non-negative when*

$$\max[L_1 \quad, L_2 \quad] \le \rho \le \min[U_1 \quad, \quad U_2] \tag{3.4}$$

where

$$L_1 = \left[-\lambda_1\left(\lambda^*\theta_1 + \theta_2\right)\sqrt{\Delta t} - \lambda^*\left(1+\theta_1^2\Delta t\right) - \theta_1\theta_2\Delta t\right] \tag{3.5a}$$

$$L_2 = \left[\lambda_1\left(-\lambda^*\theta_1 + \theta_2\right)\sqrt{\Delta t} + \lambda^*\left(1+\theta_1^2\Delta t\right) - \theta_1\theta_2\Delta t\right] \tag{3.5b}$$

$$U_1 = \left[\lambda_1\left(\lambda^*\theta_1 - \theta_2\right)\sqrt{\Delta t} + \lambda^*\left(1+\theta_1^2\Delta t\right) - \theta_1\theta_2\Delta t\right] \tag{3.5c}$$

$$U_2 = \left[\lambda_1\left(\lambda^*\theta_1 + \theta_2\right)\sqrt{\Delta t} - \lambda^*\left(1+\theta_1^2\Delta t\right) - \theta_1\theta_2\Delta t\right] \tag{3.5d}$$

and $\lambda^* = \lambda_2/\lambda_1 = \sqrt{(1 + \theta_2{}^2\Delta t\,)}/\sqrt{(1 + \theta_1{}^2\Delta t\,)}$.

The following example illustrate that a feasible set of bounds obtained from Theorem 3.2 provide non negative probabilities for any Δt.

EXAMPLE 3.1. For this example we take r = 7%, σ_1 = 50%, σ_2 = 100%, Δt = 0.25 (n = 4) and λ_1 = 1.0015 and λ_2 = 1.0228 (values obtained from equation 3.3a and 3.3b). From the MKR model the feasible bounds for ρ are max[-0.765, -1.308] and min[1.172, 0.854] given by -0.765≤ ρ ≤ 0.854. Any values of correlation coefficient between the two assets within this interval will always result in a feasible set of probabilities in MKR model; for instance ρ = 0.85 gives probabilities as (0.3949, 0.0776, 0.5275, 0, 0). However, there can be instances when KR model can give an infeasible set of probabilities, for example when ρ = 0.85 and λ = 1.0015 probabilities are (0.395, 0.076, 0.529, -0.003, 0.003).

COROLLARY 3.3: *When* $\theta_1 = \theta_2 = \theta$ *then it implies that* $\lambda_1 = \lambda_2 = \lambda$ *and in order to obtain the feasible probabilities we must have* $\lambda = \sqrt{1+\theta^2\Delta t}$ *and*

$$\max\left[2\lambda\theta\sqrt{\Delta t}-\left(1+2\theta^2\Delta t\right),-2\lambda\theta\sqrt{\Delta t}-\left(1+2\theta^2\Delta t\right)\right]\leq\rho\leq 1. \tag{3.6}$$

EXAMPLE 3.2. In our second example we set $\lambda_1 = \lambda_2 = \lambda$ by selecting equal volatility for the two assets. More specifically, letting r = 7%, $\sigma_1 = \sigma_2$ = 5%, Δt = 0.1 (n = 10) and λ = 1.0904 (value obtained from equation 3.3a). The feasible bounds for ρ from the Corollary 3.3 are max[-0.4298, -2.326] and 1, that means $-0.43 \leq \rho \leq 1$. For ρ = -0.30 the MKR model gives probabilities as (0.43, 0.27, 0.03, 0.27, 0) and KR probabilities are (0.35, 0.27, -0.05, 0.27, 0.16). The percent absolute relative error for the probabilities are (18.60, 0, 266, 0, ∞).

We next analyze how KR model behaves when time step Δt varies. If the time step Δt is made sufficiently small, then λ_1 and λ_2 converge to 1 for any values of asset volatility σ_i and MKR model converges to the KR model. As Δt is made sufficiently small the bounds for the correlation coefficient become $-1 \leq \rho \leq 1$. Consequently, both the models will work for any values of ρ. For example for both models with λ = 1, and when ρ = -1 the probability values are ($p_1 = p_3 = p_5 = 0$) and ($p_2 = p_4 = 0.5$). In the case when ρ = +1 the probability values are ($p_2 = p_4 = p_5 = 0$) and ($p_1 = p_3 = 0.5$). This does not hold however, if λ is arbitrarily selected to be greater than 1.

3.1 Comparison of the Accuracy on Option Prices with the KR and the MKR Models

We compare the accuracy of the computed option prices using transition probabilities calculated from the KR and MKR models with true value on the maximum of two underlying assets. We consider the data in Johnson (1987) with $S_1(0) = S_2(0)$ = exercise price (X) = 40, $\sigma_1 = \sigma_2$ = 30%, r = 10%, ρ = 0.5, and λ = 1. The computed option prices for T= 1, and 10 years are given in Table 3. The figures in parenthesis give the percentage errors.

Insert Table 3 here

4. Probability Expressions for *k*- State Model

The approach for determining jump probabilities for two state variable model can be extended for k states. We assume that joint density of k underlying assets follow a multi-variate lognormal distribution with instantaneous mean $\mu_i = r - \sigma_i^2/2$, and instantaneous standard deviation σ_i. We denote instantaneous correlation between assets i and j as ρ_{ij} and as before define $1/\theta_i$ as the coefficient of variation for asset i. The total number of jump probabilities in the model are $2^k + 1$ such that

$$\sum_{m=1}^{2^k+1} p_m = 1 . \tag{4.1}$$

Let $v_i = \lambda_i \sigma_i \sqrt{\Delta t}$ (i= 1,2,…..,k) and as before the stretch parameter λ_i is to be determined. Then by equating the mean, variance and pair-wise covariance terms of the true and approximating distributions we obtain the following exact expressions for the MKR jump probabilities for m = 1, 2, …..,2^k, $k \geq 2$

$$p_m = \frac{1}{2^k}\left[\frac{1+\theta_1^2\Delta t}{\lambda_1^{\,2}} + \sqrt{\Delta t}\sum_{i=1}^{k} x_{im}\left(\frac{\theta_i}{\lambda_i}\right) + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} x_{ij}^m\left(\frac{\rho_{ij}+\theta_i\theta_j\Delta t}{\lambda_i\lambda_j}\right)\right] , \tag{4.2a}$$

where $x_{im} = \begin{cases} 1 & \text{if asset } i \text{ has an up jump in state } m, \\ -1 & \text{if asset } i \text{ has a down jump in state } m \end{cases}$,

and $x_{ij}^m = \begin{cases} 1 & \text{if asset } i \text{ and } j \text{ have jumps in the same direction in state } m, \\ -1 & \text{if asset } i \text{ and } j \text{ have jumps in the opposite direction in state } m \end{cases}$

Further

$$p_{2^k+1} = 1 - \left[\frac{1+\theta_1^2\Delta t}{\lambda_1^{\,2}}\right] . \tag{4.2b}$$

Notice that as in the previous cases when $\lambda_1 = \sqrt{1+\theta_1^{\,2}\Delta t}$, the MKR model collapses to a 2^k jump model.

To illustrate the *k* state MKR model we consider the following three assets $\{S_1(t), S_2(t), S_3(t)\}$. The approximating multivariate distribution $\{\zeta_1^a(t), \zeta_2^a(t), \zeta_3^a(t)\}$ is now given by

$\zeta_1^a(t)$	$\zeta_2^a(t)$	$\zeta_3^a(t)$	**Probability**
ν_1	ν_2	ν_3	p_1
ν_1	ν_2	$-\nu_3$	p_2
ν_1	$-\nu_2$	ν_3	p_3
ν_1	$-\nu_2$	$-\nu_3$	p_4
$-\nu_1$	ν_2	ν_3	p_5
$-\nu_1$	ν_2	$-\nu_3$	p_6
$-\nu_1$	$-\nu_2$	ν_3	p_7
$-\nu_1$	$-\nu_2$	$-\nu_3$	p_8
0	0	0	p_9

REMARK 4.1. Here we investigate the implications of ignoring the $O(\Delta t^2)$ terms in the KR model. We first equate the variance terms of the true and approximate distributions to obtain the following set of equations from KR and MKR models. The KR model results in

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1}{\lambda_1^2}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1}{\lambda_2^2}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1}{\lambda_3^2}$$

implying that $\lambda_1 = \lambda_2 = \lambda_3$. And from the MKR model we get

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1+\theta_1^2\Delta t}{\lambda_1^2}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1+\theta_2^2\Delta t}{\lambda_2^2}$$

$$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 = \frac{1+\theta_3^2\Delta t}{\lambda_3^2}$$

which imply that

$$\frac{\lambda_1}{\sqrt{1+\theta_1{}^2\Delta t}} = \frac{\lambda_2}{\sqrt{1+\theta_2{}^2\Delta t}} = \frac{\lambda_3}{\sqrt{1+\theta_3{}^2\Delta t}}.$$

From the above expressions it evident that λ_i are equal if and only if $\theta_i = \theta$. Ignoring O(Δt) terms in the KR model as we have shown introduces an error in the probability expressions. Next the resulting expressions for MKR probabilities p_m, for m = 1, 2, …..8 from equation (4.2a) are

$$p_1 = \frac{1}{8}\left[\sqrt{\Delta t}\left(\frac{\theta_1}{\lambda_1}+\frac{\theta_2}{\lambda_2}+\frac{\theta_3}{\lambda_3}\right)+\frac{1+\theta_1^2\Delta t}{\lambda_1{}^2}+\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ a})$$

$$p_2 = \frac{1}{8}\left[\sqrt{\Delta t}\left(\frac{\theta_1}{\lambda_1}+\frac{\theta_2}{\lambda_2}-\frac{\theta_3}{\lambda_3}\right)+\frac{1+\theta_1^2\Delta t}{\lambda_1{}^2}+\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}-\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}-\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ b})$$

$$p_3 = \frac{1}{8}\left[\sqrt{\Delta t}\left(\frac{\theta_1}{\lambda_1}-\frac{\theta_2}{\lambda_2}+\frac{\theta_3}{\lambda_3}\right)+\frac{1+\theta_1^2\Delta t}{\lambda_1{}^2}-\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}-\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ c})$$

$$p_4 = \frac{1}{8}\left[\sqrt{\Delta t}\left(\frac{\theta_1}{\lambda_1}-\frac{\theta_2}{\lambda_2}-\frac{\theta_3}{\lambda_3}\right)+\frac{1+\theta_1^2\Delta t}{\lambda_1{}^2}-\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}-\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ d})$$

$$p_5 = \frac{1}{8}\left[\sqrt{\Delta t}\left(-\frac{\theta_1}{\lambda_1}+\frac{\theta_2}{\lambda_2}+\frac{\theta_3}{\lambda_3}\right)+\frac{1+\theta_1^2\Delta t}{\lambda_1{}^2}-\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}-\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ e})$$

$$p_6 = \frac{1}{8}\left[\sqrt{\Delta t}\left(-\frac{\theta_1}{\lambda_1}+\frac{\theta_2}{\lambda_2}-\frac{\theta_3}{\lambda_3}\right)+\frac{1+\theta_1^2\Delta t}{\lambda_1{}^2}-\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}-\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ f})$$

$$p_7 = \frac{1}{8}\left[\sqrt{\Delta t}\left(-\frac{\theta_1}{\lambda_1} - \frac{\theta_2}{\lambda_2} + \frac{\theta_3}{\lambda_3}\right) + \frac{1+\theta_1^2\Delta t}{\lambda_1^{\ 2}} + \frac{\rho_{12} + \theta_1\theta_2\Delta t}{\lambda_1\lambda_2} - \frac{\rho_{13} + \theta_1\theta_3\Delta t}{\lambda_1\lambda_3} - \frac{\rho_{23} + \theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ g})$$

$$p_8 = \frac{1}{8}\left[\sqrt{\Delta t}\left(-\frac{\theta_1}{\lambda_1} - \frac{\theta_2}{\lambda_2} - \frac{\theta_3}{\lambda_3}\right) + \frac{1+\theta_1^2\Delta t}{\lambda_1^{\ 2}} + \frac{\rho_{12} + \theta_1\theta_2\Delta t}{\lambda_1\lambda_2} + \frac{\rho_{13} + \theta_1\theta_3\Delta t}{\lambda_1\lambda_3} + \frac{\rho_{23} + \theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right] \quad (4.3\text{ h})$$

and from equation (4.2b) we get

$$p_9 = 1 - \left[\frac{1+\theta_1^2\Delta t}{\lambda_1^{\ 2}}\right] \quad . \qquad (4.3\text{ i})$$

To obtain a feasible set of jump probabilities we follow the same approach as in the two-state model. We present results in the form of theorems with proofs in Appendix. The following theorem provides bounds for the stretch parameters.

THEOREM 4.1: *In the three state MKR model for probabilities to be feasible* ($0 \le p_i \le 1$) *the stretch parameters* λ_1 λ_2 *and* λ_3 *must satisfy*

$$\lambda_1 \ge \sqrt{1+\theta_1^{\ 2}\Delta t} \qquad (4.4\text{ a})$$

$$\lambda_2 = \lambda_1 \frac{\sqrt{1+\theta_2^{\ 2}\Delta t}}{\sqrt{1+\theta_1^{\ 2}\Delta t}} \qquad (4.4\text{ b})$$

and

$$\lambda_3 = \lambda_1 \frac{\sqrt{1+\theta_3^{\ 2}\Delta t}}{\sqrt{1+\theta_1^{\ 2}\Delta t}} \quad . \qquad (4.4\text{ c})$$

In order to obtain bounds for the correlation coefficients we set the value of $\lambda_1 = \sqrt{1+\theta_1^{\ 2}\Delta t}$.

Next we obtain bounds for the instantaneous correlation coefficients ρ_{ij} that makes the probabilities non-negative for any value of Δt. Without loss of generality for mathematical tractability we assume that the instantaneous correlation coefficients follow the correlation structure $\rho_{12} = \rho$, $\rho_{13} = \rho + k_1$, $\rho_{23} = \rho + k_2$ where k_1, and k_2 are the constants such that $-1 \le \rho_{ij} \le 1$.

THEOREM 4.2: *For a three-asset MKR model the probabilities will always be non-negative when*

$$\max\left[L_1 \quad , L_2 \quad\right] \le \rho \le \min\left[U_i \quad , \quad \text{for } i = 1,2,....6\right]. \qquad (4.5)$$

In order for a feasible set of jump probabilities we must have

$$L_1 = -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1+\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t+\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(\lambda_{12}+\lambda_{13}+\lambda_{12}\lambda_{13}\right)} \qquad (4.6a)$$

$$L_2 = -\frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1+\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t+\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(\lambda_{12}+\lambda_{13}+\lambda_{12}\lambda_{13}\right)} \qquad (4.6b)$$

$$U_1 = -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1+\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t-\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(\lambda_{12}-\lambda_{13}-\lambda_{12}\lambda_{13}\right)} \qquad (4.6c)$$

$$U_2 = -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1-\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t+\left(k_1\lambda_{13}-k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(-\lambda_{12}+\lambda_{13}-\lambda_{12}\lambda_{13}\right)} \qquad (4.6d)$$

$$U_3 = -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1-\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t+\left(-k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(-\lambda_{12}-\lambda_{13}+\lambda_{12}\lambda_{13}\right)} \qquad (4.6e$$

$$U_4 = \frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1-\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t+\left(-k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(\lambda_{12}+\lambda_{13}-\lambda_{12}\lambda_{13}\right)} \qquad (4.6f)$$

$$U_5 = \frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1-\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t+\left(k_1\lambda_{13}-k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(\lambda_{12}-\lambda_{13}+\lambda_{12}\lambda_{13}\right)} \qquad (4.6g)$$

$$U_6 = \frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1+\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t+\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)\right]}{\left(-\lambda_{12}+\lambda_{13}+\lambda_{12}\lambda_{13}\right)} \qquad (4.6h)$$

where $\lambda_{12} = \lambda_1/\lambda_2$ and $\lambda_{13} = \lambda_1/\lambda_3$.

Using following example we illustrate below that the feasible bounds obtained from Theorem 4.2 gives non negative probabilities.

EXAMPLE 4.1. In this example we let r = 7%, σ_1 = 10%, σ_2 = 5% σ_3 = 40%, Δt = 0.1 (n = 10) , k_1 = 0.10 and k_2 = 0.15 and λ_1 = 1.0209, λ_2 = 1.0904, λ_3 = 1.0000 (value obtained from Theorem 4.1). The feasible bounds for ρ from the Theorem 4.2 are max[-0.68, -0.26] and min[1.46, 0.81, 0.80, 1.19, 1.30, 0.24], given by -0.26≤ $\rho \leq$ 0.24. Let k_1 = 0.10 and k_2 = 0.15 then for ρ_{12} = -0.26, ρ_{13} = -0.16 and ρ_{23} = -0.11. The MKR model gives probabilities as (0.15, 0.21, 0.11, 0.13,0.17,0.16,0.06,0,0) and KR with (λ = 1) probabilities are (0.14, 0.21, 0.12, 0.14,0.19,0.18,0.04,-0.02,0). If the time step Δt is made sufficiently small, then λ_i (i = 1, 2, 3) converges to 1 and the MKR model converges to KR model.

COROLLARY 4.3: *When $\theta_1 = \theta_2 = \theta_2 = \theta$ then it implies that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda = \sqrt{1+\theta^2 \Delta t}$ and $\lambda_{12} = \lambda_{13} = 1$. Assuming $\rho_{12} = \rho$, $\rho_{13} = \rho + k_1$, $\rho_{23} = \rho + k_2$ with constants k_1, k_2 then in order to obtain the feasible probabilities we must have*

$$\max\left[-\frac{\left(1+4\theta^2\Delta t+3\lambda\theta\sqrt{\Delta t}\right)+k_1+k_2}{3},\ -\frac{\left(1+4\theta^2\Delta t-3\lambda\theta\sqrt{\Delta t}\right)+k_1+k_2}{3}\right] \leq \rho \leq \min\left[U_i\ ,\quad i=1,2,\ldots\ldots 6\right] \tag{4.7}$$

where

$$U_1 = \left(1+\lambda\theta\sqrt{\Delta t}\right)+k_1+k_2,\quad U_2 = \left(1+\lambda\theta\sqrt{\Delta t}\right)-k_1+k_2,\quad U_3 = \left(1-\lambda\theta\sqrt{\Delta t}\right)+k_1-k_2,$$

$$U_4 = \left(1+\lambda\theta\sqrt{\Delta t}\right)+k_1-k_2,\quad U_5 = \left(1-\lambda\theta\sqrt{\Delta t}\right)-k_1+k_2,\quad U_6 = \left(1-\lambda\theta\sqrt{\Delta t}\right)+k_1+k_2.$$

The feasible bounds from Corollary 4.3 that provides non negative probabilities are illustrated below in example 4.2

EXAMPLE 4.2. Let r = 7%, $\sigma_1 = \sigma_2 = \sigma_3$ = 5%, Δt = 0.1 (n = 10) k_1 = 0.10 and k_2 = 0.15 and λ = 1.0904 (value obtained from equation 4.4a). The feasible bounds for ρ from the Corollary 4.2 are max[-1.14, -0.19] and min[1.22, 1.42, 0.58, 1.52, 0.48, 0.28], given by -0.19≤ $\rho \leq$ 0.28. Let k_1 = 0.10 and k_2 = 0.15 then ρ_{12} = -0.19, ρ_{13} = -0.09 and ρ_{23} = -0.04. The MKR model estimates probabilities as (0.30, 0.15, 0.17, 0.08,0.18,0.07,0.05,0,0) and KR with (λ = 1) probabilities are (0.25, 0.17, 0.20, 0.10,0.21,0.09,0.06,-0.08,0). As before if the time step Δt is made sufficiently small, then λ_i (i = 1, 2, 3) converges to 1 and the MKR model and KR model are identical.

COROLLARY 4.4: *When $\theta_1 = \theta_2 = \theta_2 = \theta$ then it implies that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $\lambda_{12} = \lambda_{13} = 1$. Assuming $\rho_{ij} = \rho$ then in order to obtain the feasible probabilities we must have* $\lambda = \sqrt{1+\theta^2\Delta t}$ *and*

$$\max\left[-\frac{1}{3}\left[\left(1+4\theta^2\Delta t\right)+3\lambda\theta\sqrt{\Delta t}\ \right],\ -\frac{1}{3}\left[\left(1+4\theta^2\Delta t\right)-3\lambda\theta\sqrt{\Delta t}\ \right]\right] \le \rho \le \min\left[1+\lambda\theta\sqrt{\Delta t}\ ,1-\lambda\theta\sqrt{\Delta t}\right] \qquad (4.8)$$

We use the following example to illustrate that feasible bounds from Corollary 4.4 provide the required set of non negative probabilities.

EXAMPLE 4.3. As before we set r = 7%, $\sigma_1 = \sigma_2 = \sigma_3$ = 5%, and now change Δt = 0.01 (n = 100) giving λ = 1.0094 (value obtained from equation 4.4a). The feasible bounds for ρ from the Corollary 4.3 are max[-0.50, -0.22] and min[0.086, 1.14], given by $-0.22 \le \rho \le 0.086$. For ρ = -0.22 the MKR model gives probabilities as (0.10, 0.17, 0.17, 0.13,0.17,0.13,0.13,0,0) and KR with (λ = 1) probabilities are (0.09, 0.17, 0.17, 0.14,0.17,0.14,0.14,-0.01,0).

Also in this case when the time step Δt is made sufficiently small, then λ_i (i = 1, 2, 3) converges to 1 and the MKR model converges to the KR model. In addition to this we observe the following. As Δt is made sufficiently small the bounds for the correlation coefficient become $-1/3 \le \rho \le 1$. Now for both models with $\lambda = 1$, and $\rho = -0.33$, we get the probability values ($p_1 = p_8 = p_9 = 0$) and (p_i =1/6 , for i = 2, ..7)) and with $\rho = 1$, we get ($p_1 = p_8 = 1/2$) and (p_i = 0, for i = 2, ..,7 and p_9 = 0).

5. Computational Effort

To implement the KR and MKR models, one has to decide on the size of the time step in the approximating process Δt = T/n, where T is the duration of the option, and n is the number of iterations. For model comparison purpose we use the total number of nodes generated by the process as a criteria to assess computational effort. Here the number of nodes refers to a possible state generated by the approximating process. For example, a trinomial process generates three nodes (states) after elapse of time Δt; an up node, a horizontal node and a down node. Kamrad and Ritchken (1991) only provide expressions to compute the number of nodes for one and two state models. Here, we first derive the mathematical expressions for computing the total number of node generated for a k-assets model with and without horizontal jumps. Using numerical examples we then compare the computational effort required for implementing a single state, two state and three state KR and MKR models.

We define $N_k(n)$ as the total number of nodes generated in n iterations for a model with k state variables. For the k- state model with 2^k +1 jumps the total number of nodes generated by the process after n iterations we have

$$N_k(n) = \sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)^k = \sum_{j=0}^{n}\left[(j+1) + 2S(j,k) + kS(j,k-1) + \binom{k}{2}S(j,k-2) + \dots\dots\dots\dots + kS(j,1)\right] \quad (5.1a)$$

and with 2^k jumps,

$$N_k(n) = \sum_{i=0}^{n}(i+1)^k = \left[(n+1) + S(n,k) + kS(n,k-1) + \binom{k}{2}S(n,k-2) + \dots\dots\dots\dots + kS(n,1)\right] \quad (5.1b)$$

where,

$$S(j,r) = 1^r + 2^r + \dots\dots\dots\dots + j^r, \qquad r = 0,1,2,3\dots\dots\dots \quad (5.1c)$$

In order to evaluate $S(j,r)$ we use the following result (Theorem A.3, page 562, Miller and Miller 1999),

$$\sum_{r=0}^{k-1}\binom{k}{r}S(j,r)=(j+1)^k-1, \qquad \text{for any positive integers } j \text{ and } k. \tag{5.1d}$$

If negative probabilities occur in the KR model Δt has to be made further small to obtain feasible probabilities. This however, would result in increasing the number of nodes and making it computationally expensive. The numerical examples in the next sub sections best illustrate this point.

5.1 One State Model

We first consider the one state model. When k =1 the total number of nodes for a trinomial model can be computed by

$$N_1(n)=\sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)^1=\sum_{j=0}^{n}\left[\frac{(j+1)(j+2)}{2}\right]=\frac{1}{6}\left(n^3+6n^2+11n+6\right) \tag{5.1.1a}$$

and for the binomial model is given by

$$N_1(n)=\sum_{i=0}^{n}(i+1)^1=\frac{1}{2}\left(n^2+3n+2\right). \tag{5.1.1b}$$

EXAMPLE 5.1 Here we consider our previous Example 2.1. To make the KR jump probabilities feasible, we need to change Δt = 0.15 when λ = 1. This yields the following feasible set of probabilities (0.95, 0, 0.05). When λ = 1.2247 we need to change Δt = 0.12 in order to obtain the positive probabilities (0.66, 0.33, 0.01). It is therefore evident that the only way to obtain a feasible set of probabilities without increasing λ arbitrarily is to increase the number of nodes. Notice that in Example 2.1 the MKR model provides a feasible set of probabilities for any Δt. In the limiting case when for any Δt = 0.0001 (n = 10^4) the probabilities from both model are (0.5, 0, 0.5) since λ in the MKR converges to 1.

5.2 Two State Model

Next we consider a two state model. When k =2 then the total number of nodes generated in a five jump model is

$$N_2(n)=\sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)^2=\sum_{j=0}^{n}\left[\frac{2j^3+9j^2+13j+6}{6}\right]=\frac{1}{12}\left(n^4+8n^3+23n^2+28n+12\right) \tag{5.1.2a}$$

and in a four jump model is

$$N_2(n)=\sum_{i=0}^{n}(i+1)^2=\frac{1}{6}\left(2n^3+9n^2+13n+6\right). \tag{5.1.2b}$$

The following example provides an insight.

EXAMPLE 5.2. We consider r = 7%, σ_1 = 10%, σ_2 = 5%, and λ = 1.0515 (value obtained from equation 3.3b). For a set of increasing Δt values we evaluate the KR probabilities as indicated in Table 4.

Insert Table 4 here

Notice that the number of iterations required to implement the KR model, is n = 1667. From equations 5.1.2a and 5.1.2b, the resulting total number of nodes generated by the process after n iterations for a 5-jump model and a four-jump model will be very large and impose heavy computational burden.

5.3 Three State Model

As a final example we consider a three state model. When k =3 then the total number of nodes in a 9 jump model is given by

$$N_3(n) = \sum_{j=0}^{n}\sum_{i=0}^{j}(i+1)^3 = \sum_{j=0}^{n}\left[\frac{j^4+6j^3+13j^2+12j+4}{4}\right]$$

$$= \frac{1}{60}\left(3n^5+30n^4+115n^3+210n^2+182n+60\right) \qquad (5.1.3a)$$

and in an eight jump model is given by

$$N_3(n) = \sum_{j=0}^{n}(i+1)^3 = \left[\frac{(n+1)(n+2)}{2}\right]^2 \qquad (5.1.3b)$$

The following example illustrates the computational effort that is required.

EXAMPLE 5.3 For the last example we take r = 7%, σ_1 = 10%, σ_2 = 5%, σ_3 = 60%, and employ the KR model with λ = 1. Table 5 provides the KR probabilities for a set of increasing Δt values.

Insert Table 5 here

The number of iteration necessary to obtain the feasible probabilities in the KR model is n = 17. From equations 5.1.3a and 5.1.3b, the resulting total number of nodes for 9-jump model and an 8-jump model are 123,224 and 29241. This would require an enormous computational effort.

Hence our theoretical results in section 2 through section 4 can be exploited to improve computational effort.

6. Conclusion

We included an omitted second order term in computing the transition probabilities in the KR model to value contingent claims whose value depends on multiple sources of uncertainty. We demonstrated the accuracy of the MKR model over the KR model using option prices on the maximum of two assets. Our analysis suggest that the KR model should be used when the time step is very small and when there are fewer state variables implying it is only then the model becomes computationally inexpensive. We proved that bounds for the stretch parameter and the correlation coefficient could be exploited to obtain a feasible set of probabilities without imposing heavy computational burden. In addition, we developed general expressions to determine the number of nodes generated in the approximation process when there are *k* state variables.

The probability estimates from the MKR model are more accurate than the probability estimates based on the KR model which ignore the second order terms of the time step. We have however shown that the two models provide identical probability estimates when the time step is very small. The previous research including Kamrad and Ritchken (1991) do not provide an objective way of selecting the stretch parameter. The examples that we selected show that negative probabilities may occur limiting the usefulness of the KR model. Also, option values may be different for different stretch parameters. Adapting the framework that we suggested as we have shown reduces the computational effort. More importantly it provides a model where positive probabilities are guaranteed by constraining the correlation coefficient. Knowing the feasible range of the correlation coefficient indicates when the multinomial approximation model can be applied.

Appendix

Two state Model

Proof of Theorem 3.1

From equations 3.1c and 3.1(d) we have

$$\frac{\lambda_1^2}{1+\theta_1^{\,2}\Delta t}=\frac{\lambda_2^2}{1+\theta_2^{\,2}\Delta t}$$

$$\frac{\lambda_1}{\lambda_2}=\frac{\sqrt{1+\theta_1^{\,2}\Delta t}}{\sqrt{1+\theta_2^{\,2}\Delta t}}=\lambda^*. \tag{A1}$$

From equations 3.2 e and 3.3a we get $\lambda_1 \geq \sqrt{1+\theta_1^{\,2}\Delta t}$ for $p_5 \geq 0$ and $\lambda_2=\lambda_1\frac{\sqrt{1+\theta_2^{\,2}\Delta t}}{\sqrt{1+\theta_1^{\,2}\Delta t}}$.

Proof of Theorem 3.2

From equation 3.2 a., for $p_1 \geq 0$, we must have

$$\frac{\theta_1\sqrt{\Delta t}}{\lambda_1}+\frac{\theta_2\sqrt{\Delta t}}{\lambda_2}+\frac{1+\theta_1^2\Delta t}{\lambda_1^{\,2}}+\frac{\rho+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}\geq 0. \tag{A2}$$

Multiplying A2 by $\lambda_1\lambda_2$ and using equation A.1 we obtain

$$\lambda_1\left(\lambda^*\theta_1+\theta_2\right)\sqrt{\Delta t}+\lambda^*\left(1+\theta_1^2\Delta t\right)+\rho+\theta_1\theta_2\Delta t\geq 0$$

that in-turn provides

$$\rho \geq \left[-\lambda_1\left(\lambda^*\theta_1+\theta_2\right)\sqrt{\Delta t}-\lambda^*\left(1+\theta_1^2\Delta t\right)-\theta_1\theta_2\Delta t\right]=L_1 \tag{A3}$$

Similarly we obtain the following bounds that make $p_i \geq 0$, for i = 2,3, and 4.

$$\rho \geq \left[\lambda_1\left(\lambda^*\theta_1-\theta_2\right)\sqrt{\Delta t}+\lambda^*\left(1+\theta_1^2\Delta t\right)-\theta_1\theta_2\Delta t\right]=U_1 \tag{A4}$$

$$\rho \geq \left[\lambda_1\left(\lambda^*\theta_1+\theta_2\right)\sqrt{\Delta t}-\lambda^*\left(1+\theta_1^2\Delta t\right)-\theta_1\theta_2\Delta t\right]=U_2 \tag{A5}$$

$$\rho \geq \left[\lambda_1\left(-\lambda^*\theta_1 + \theta_2\right)\sqrt{\Delta t} + \lambda^*\left(1 + \theta_1^2\Delta t\right) - \theta_1\theta_2\Delta t\right] = L_{.2} \tag{A6}$$

Therefore, in the two asset model the probabilities will always be non negative when

$$\max\left[L_1 \quad , L_2 \quad\right] \leq \rho \leq \min\left[U_1 \quad , \quad U_2\right].$$

Three State Model

Proof of Theorem 4.1

By equating the pair-wise covariance terms we obtain

$$\frac{\lambda_1^2}{1 + \theta_1{}^2\Delta t} = \frac{\lambda_2^2}{1 + \theta_2{}^2\Delta t} = \frac{\lambda_3^2}{1 + \theta_3{}^2\Delta t}.$$

Then

$$\frac{\lambda_1}{\lambda_2} = \frac{\sqrt{1 + \theta_1{}^2\Delta t}}{\sqrt{1 + \theta_2{}^2\Delta t}} = \lambda_{12} \tag{A7}$$

$$\frac{\lambda_1}{\lambda_3} = \frac{\sqrt{1 + \theta_1{}^2\Delta t}}{\sqrt{1 + \theta_3{}^2\Delta t}} = \lambda_{13}. \tag{A8}$$

From equation (4.3i) for $p_9 \geq 0$ we get

$$\lambda_1 \geq \sqrt{1 + \theta_1{}^2\Delta t} \tag{A9}$$

$$\lambda_2 = \lambda_1 \frac{\sqrt{1 + \theta_2{}^2\Delta t}}{\sqrt{1 + \theta_1{}^2\Delta t}} \tag{A10}$$

$$\lambda_3 = \lambda_1 \frac{\sqrt{1 + \theta_3{}^2\Delta t}}{\sqrt{1 + \theta_1{}^2\Delta t}}. \tag{A11}$$

Next using $\lambda_1 = \sqrt{1+\theta_1^{\,2}\Delta t}$ and λ_2, λ_3 from equations A10 and A11 we obtain bounds for the instantaneous correlation coefficients ρ_{ij} that makes the probabilities non-negative for any value of Δt.

Proof of Theorem 4.2

From equations 4.3 a., for $p_1 \geq 0$, we must have

$$\sqrt{\Delta t}\left(\frac{\theta_1}{\lambda_1}+\frac{\theta_2}{\lambda_2}+\frac{\theta_3}{\lambda_3}\right)+\frac{1+\theta_1^2\Delta t}{\lambda_1^{\,2}}+\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\geq 0$$

$$\text{or}\quad \frac{1+\theta_1^2\Delta t}{\lambda_1^{\,2}}+\frac{\theta_1}{\lambda_1}\sqrt{\Delta t}\geq -\left[\left(\frac{\theta_2\sqrt{\Delta t}}{\lambda_2}+\frac{\theta_3\sqrt{\Delta t}}{\lambda_3}\right)+\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1\lambda_2}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1\lambda_3}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_2\lambda_3}\right]$$

$$\text{or}\quad \frac{1}{\lambda_1}\left(\theta_1\sqrt{\Delta t}+\frac{1+\theta_1^2\Delta t}{\lambda_1}\right)\geq -\left[\frac{1}{\lambda_2}\left(\theta_2\sqrt{\Delta t}+\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_3}\right)+\frac{1}{\lambda_3}\left(\theta_3\sqrt{\Delta t}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1}\right)\right]$$

$$\text{or}\quad \theta_1\sqrt{\Delta t}+\frac{1+\theta_1^2\Delta t}{\lambda_1}\geq -\left[\frac{\lambda_1}{\lambda_2}\left(\theta_2\sqrt{\Delta t}+\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_3}\right)+\frac{\lambda_1}{\lambda_3}\left(\theta_3\sqrt{\Delta t}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1}\right)\right].$$

From equation A.7 and A.8 we obtain

$$\theta_1\sqrt{\Delta t}+\frac{1+\theta_1^2\Delta t}{\lambda_1}\geq\left[\lambda_{12}\left(\theta_2\sqrt{\Delta t}+\frac{\rho_{12}+\theta_1\theta_2\Delta t}{\lambda_1}+\frac{\rho_{23}+\theta_2\theta_3\Delta t}{\lambda_3}\right)+\lambda_{13}\left(\theta_3\sqrt{\Delta t}+\frac{\rho_{13}+\theta_1\theta_3\Delta t}{\lambda_1}\right)\right].$$

Assume that the instantaneous correlation coefficients to follow correlation structure $\rho_{12} = \rho$, $\rho_{13} = \rho + k_1$, $\rho_{23} = \rho + k_2$ with constants k_1, k_2 we obtain the required result as

$$\rho \geq -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1+\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]+\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)}{\left(\lambda_{12}+\lambda_{13}+\lambda_{12}\lambda_{13}\right)} = L_1$$

(A12)

Similarly we obtain the following bounds that make $p_i \geq 0$, for i = 2,3, and 8.

$$\rho \geq -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1+\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]-\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)}{\left(\lambda_{12}-\lambda_{13}-\lambda_{12}\lambda_{13}\right)}=U_1$$

(A13)

$$\rho \geq -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1-\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]+\left(k_1\lambda_{13}-k_2\lambda_{12}\lambda_{13}\right)}{\left(-\lambda_{12}+\lambda_{13}-\lambda_{12}\lambda_{13}\right)}=U_2$$

(A14)

$$\rho \geq -\frac{\left[\left(1+\theta_1^2\Delta t\right)+\lambda_1\left(\theta_1-\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]+\left(-k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)}{\left(-\lambda_{12}-\lambda_{13}+\lambda_{12}\lambda_{13}\right)}=U_3$$

(A15)

$$\rho \geq \frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1-\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]+\left(-k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)}{\left(\lambda_{12}+\lambda_{13}-\lambda_{12}\lambda_{13}\right)}=U_4$$

(A16)

$$\rho \geq \frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1-\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(-\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]+\left(k_1\lambda_{13}-k_2\lambda_{12}\lambda_{13}\right)}{\left(\lambda_{12}-\lambda_{13}+\lambda_{12}\lambda_{13}\right)}=U_5$$

(A17)

$$\rho \geq \frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1+\theta_2\lambda_{12}-\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}-\theta_1\theta_3\lambda_{13}-\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]+\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)}{\left(-\lambda_{12}+\lambda_{13}+\lambda_{12}\lambda_{13}\right)}=U_6$$

(A18)

$$\rho \geq -\frac{\left[\left(1+\theta_1^2\Delta t\right)-\lambda_1\left(\theta_1+\theta_2\lambda_{12}+\theta_3\lambda_{13}\right)\sqrt{\Delta t}+\left(\theta_1\theta_2\lambda_{12}+\theta_1\theta_3\lambda_{13}+\theta_2\theta_3\lambda_{12}\lambda_{13}\right)\Delta t\right]+\left(k_1\lambda_{13}+k_2\lambda_{12}\lambda_{13}\right)}{\left(\lambda_{12}+\lambda_{13}+\lambda_{12}\lambda_{13}\right)}=L_2$$

(A19)

For a three-asset model the probabilities will always be non negative when

$$\max\left[L_1 \quad , L_2 \quad\right] \leq \rho \leq \min\left[U_i \quad , \quad \text{for } i=1,2,....6\right].$$

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Table 1

Errors Resulting by Ignoring *O*(Δt)

Probability	Error
p_1	$\theta^2 \Delta t / 2\lambda^2$
p_2	$-\theta^2 \Delta t / \lambda^2$
p_3	$\theta^2 \Delta t / 2\lambda^2$

Table 2

Percent Absolute Relative Errors

σ = 8%, Δt = 0.25	σ = 8%, Δt = 0.1	σ = 5%, Δt = 0.5
2.90	2.89	6.25
0	0	0
6.45	6.67	25

Table 3

Option Prices with the KR and the MKR Models

T	True Option Price	KR Model	MKR Model
1 Year	9.94	9.68 (-2.8%)	9.70 (-2.6%)
10 Years	40.54	37.39 (-7.7%)	38.03 (-6.2%)

Table 4

Process Iterations and Probabilities for a Two State Model

Δt	n	**KR Probabilities**
0.25	4	0.252, 0.355, -0.229, 0.527, 0.096
0.0007	1420	0.024, 0.436, -0.001, 0.445, 0.096
0.0006	1667	0.23, 0.437, 0, 0.445, 0.096

Table 5

Process Iterations and Probabilities for a Three State Model

Δt	n	**KR Probabilities**
0.25	4	0.3, 0.2, 0.06, 0.1, 0.17, 0.17, -0.07, 0.07, 0
0.07	14	0.24, 0.14, 0.08, 0.12, 0.16, 0.14, -0.01, 0.12, 0
0.06	17	0.24, 0.13, 0.09, 0.12, 0.16, 0.14, 0, 0.13, 0