Ch. 6. Finding the Probability Distribution of a Function of R.V.

3. The Method of Distribution Function

Univariate Case:

If Y has a p.d.f. $f(y)$ and if U is some function of Y, then we can find Probability Distribution of U, $F_U(u) = P(U \leq u)$ by integrating f(y) over the region for which $U \leq u$.

Bivariate Case:

Let Y_1 and Y_2 be r.v. with jpdf $f(y_1, y_2)$, and let $U = h(y_1, y_2)$.Then for every (y_1, y_2), there corresponds one and only one value of U. If we can find the region of values such that $U \leq u$, then the integral of jpdf over this region gives $F(u) = P(U \leq u)$. The pdf for U is obtained by differentiation.

Exercise 6.2. Let Y be a r.v. with a d.f.

$$f(y) = \frac{3}{2}y^2, \ \ -1 \le y \le 1.$$

We first write

$$F_Y(y) = \int_{-1}^{y} \frac{3}{2}t^2 dt = \frac{1 + y^3}{2}, -1 \le y \le 1.$$

(a) $F_{U_1}(u) = P(3Y \le u) = P(Y \le \frac{u}{3})$

$= F_Y(\frac{u}{3}) = \frac{1}{2}(1 + \frac{u^3}{27}), -3 \le u \le 3.$

We obtain $f_{U_1}(u)$ by differentiating $F_{U_1}(u)$ with respect to u:

$$f_{U_1}(u) = \frac{u^2}{18}, -3 \le u \le 3.$$

(b) $F_{U_2}(u) = P(3 - Y \le u) = P(Y \ge 3 - u)$

$= 1 - F_Y(3 - u) = \frac{1}{2}[1 - (3 - u)^3],$

for $-1 \le 3 - u \le 1$ or $2 \le u \le 4.$

We obtain $f_{U_2}(u)$ by differentiating $F_{U_2}(u)$ with respect to u:

$$f_{U_2}(u) = \frac{3}{2}(3 - u)^2, 2 \le u \le 4.$$

(c) $F_{U_3}(u) = P(Y^2 \le u) =$
$P(-\sqrt{u} \le Y \le \sqrt{u})$

$= F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) = \frac{u^{3/2}}{2},$

for $0 \le y^2 \le 1$ or $0 \le u \le 1$.

We obtain $f_{U_3}(u)$ by differentiating $F_{U_3}(u)$ with respect to u:

$$f_{U_3}(u) = \frac{3}{2} u^{1/2}, 0 \le u \le 1.$$

4. The method of Transformation

Let U be an increasing or decreasing function of the r.v. Y, say $U = h(y)$.

Find the inverse function $Y = h^{-1}(U)$ and $\frac{dy}{du}$. Then

$$f_U(u) = f_Y(y)\left|\frac{dy}{du}\right|.$$

5. The Method of Moment-Generating Functions

1. Let $m_x(t)$ and $m_y(t)$ be the m.g.f. of r.v. X and Y. If $m_x(t) = m_y(t)$ for all values of t, then X and Y have the same probability

distribution.

2. Thus, the method consists of first finding the mgf of U and then comparing it with the mgf of other well known r.v.. If $m_U(t)$ is identical to one of these say, $m_V(t)$, then U and V have the same p.d.

3. Let $Y_1, Y_2, \ldots, Y_n$ be independent r.v. with mgf as $m_{Y_1}(t), m_{Y_2}(t), \ldots, m_{Y_n}(t)$.

If $U = \sum Y_i$, then

$$m_U(t) = m_{Y_1}(t) m_{Y_2}(t) \ldots m_{Y_n}(t).$$

4. Let $Y_1, Y_2, \ldots, Y_n$ be independent normal r.v. with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$. Define $U = \sum a_i Y_i$, where $a_1, a_2, \ldots, a_n$ are constants. Then U is normally distributed r.v. with

$$E(U) = \sum a_i \mu_i$$

$$V(U) = \sum a_i^2 \sigma_i^2.$$

5. Let $Y_1, Y_2, \ldots, Y_n$ be independent normal

r.v. with $E(Y_i) = \mu_i$ and $V(Y_i)=\sigma_i^2$. Define

$$Z_i = \frac{Y - \mu_i}{\sigma_i}.$$

Then $\sum Z_i^2$ has a χ^2 (Chi-square) distribution with n degrees of freedom.