Ch. 5. Multivariate Probability Distributions

Bivariate & Multivariate P.D.

Many r.v. can be defined over the sample space.

In tossing a pair of dice, the sample space contains (6)(6)=36 sample points, representing the number of ways numbers may appear on the face of the first die (y_1)and the second die (y_2).

The 36 sample points are equiprobable and correspond to the 36 numerical events (y_1, y_2).

The intersection (y_1, y_2) contains only one sample point. Hence the bivariate p.f. is

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2) = \frac{1}{36},$$

for $y_1, y_2 = 1, 2, \ldots, 6.$

Definition

Let Y_1 and Y_2 be discrete r.v.. The joint or bivariate probability distribution for Y_1 and Y_2 is

$$p(y_1, y_2) = P(Y_1 = y_1, Y_2 = y_2)$$

defined for all real numbers y_1 and y_2.

We refer $p(y_1, y_2)$ as the joint probability function ($j.p.f.$).

Properties of J.P.F.

1. $p(y_1, y_2) \geq 0$ for all y_1 and y_2.
2. $\sum_{y_1, y_2} p(y_1, y_2) = 1$.

Joint Distribution Function

Let Y_1 and Y_2 be any r.v.. The joint or bivariate distribution function for Y_1 and Y_2 is

$$F(y_1, y_2) = P(Y_1 \leq y_1, Y_2 \leq y_2).$$

1. For two discrete r.v. Y_1 and Y_2,

$$F(y_1, y_2) = \sum_{t_1=-\infty}^{y_1} \sum_{t_2=-\infty}^{y_2} p(t_1, t_2).$$

2. Let Y_1 and Y_2 be continuous r.v. with j.d.f. $F(y_1,y_2)$. If there exists a nonnegative function $f(y_1,y_2)$ such that

$$F(y_1,y_2) = \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f(t_1,t_2)\, dt_2\, dt_1$$

for any real numbers y_1 and y_2, then Y_1 and Y_2 are said to be jointly continuous r.v..

The function $f(y_1,y_2)$ is called the joint probability density function.

3.$F(-\infty,-\infty) = F(-\infty,y_2) = F(y_1,-\infty) = 0.$

4.$F(\infty,\infty) = 1.$

5. $f(y_1,y_2) \geq 0$, for all y_1,y_2.

6.$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1,y_2)dy_1dy_2 = 1.$

Marginal Probability Distributions

1. Let Y_1 and Y_2 be jointly discrete r.v. with j.p.f. $p(y_1,y_2)$. Then the marginal probability functions of Y_1 and Y_2 are

$$p_1(y_1) = \sum_{y_2} p(y_1, y_2) \text{ and}$$

$$p_2(y_2) = \sum_{y_1} p(y_1, y_2).$$

2. Let Y_1 and Y_2 be jointly continuous r.v. with j.d.f. $f(y_1,y_2)$. Then the marginal density functions of Y_1 and Y_2 are

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1,y_2)dy_2 \text{ and}$$

$$f_2(y_2) = \int_{-\infty}^{\infty} f(y_1,y_2)dy_1.$$

Conditional Probability Distributions

1. Let Y_1 and Y_2 be jointly discrete r.v. with j.p.f. $p(y_1,y_2)$ and the marginal probability functions $p_1(y_1)$ and $p_2(y_2)$.

Then the conditional probability function of Y_1 given Y_2 is

$$p(y_1|y_2) = P(Y_1 = y_1|Y_2 = y_2)$$
$$= \frac{P(Y_1 = y_1, Y_2 = y_2)}{P(Y_2 = y_2)} = \frac{p(y_1, y_2)}{p_2(y_2)}$$

provided that $p_2(y_2) > 0$.

2. Let Y_1 and Y_2 be jointly continuous r.v. with j.d.f. $f(y_1,y_2)$. Then the conditional distribution function of Y_1 given $Y_2 = y_2$ is

$$F(y_1|y_2) = P(Y_1 \leq y_1|Y_2 = y_2).$$

3. Let Y_1 and Y_2 be jointly continuous r.v. with j.d.f. $f(y_1,y_2)$ and marginal densities $f_1(y_1)$ and $f_2(y_2)$. Then the conditional density of Y_1 given $Y_2 = y_2$ is

$$f(y_1|y_2) = \frac{f(y_1,y_2)}{f_2(y_2)}\ ,\ f_2(y_2) > 0$$

and the conditional density of Y_2 given $Y_1 = y_1$ is

$$f(y_2|y_1) = \frac{f(y_1,y_2)}{f_1(y_1)}\ ,\ f_1(y_1) > 0.$$

4. Independent R.V.

1. Let Y_1 and Y_2 be continuous r.v. with j.d.f. $F(y_1, y_2)$. Also let Y_1 have d.f. $F_1(y_1)$ and let Y_2 have d.f. $F_2(y_2)$. Then Y_1 and Y_2 are said to be independent iff

$$F(y_1, y_2) = F_1(y_1)F_2(y_2),$$

for every pair of real numbers (y_1, y_2).

2. Let Y_1 and Y_2 be discrete r.v. with j.p.f. $p(y_1, y_2)$ and m.d.f. $p_1(y_1)$ and $p_2(y_2)$, respectively. Then Y_1 and Y_2 are said to be independent iff

$$p(y_1, y_2) = p_1(y_1)p_2(y_2),$$

for every pair of real numbers (y_1, y_2).

3. Let Y_1 and Y_2 be continuous r.v. with j.d.f. $f(y_1, y_2)$ and m.d.f. $f_1(y_1)$ and $f_2(y_2)$, respectively. Then Y_1 and Y_2 are said to be independent iff

$$f(y_1, y_2) = f_1(y_1)f_2(y_2),$$

for every pair of real numbers (y_1, y_2).

5. Expected Value of a Function of R.V.

1. Let $g(Y_1, Y_2, \dots)$ be a function of the discrete r.v. $Y_1, Y_2, \dots$ with j.p.f. $p(y_1, y_2, \dots)$. Then

$$E[g(Y_1, Y_2, \dots)]$$
$$= \sum_{y_k} \dots \sum_{y_1} g(y_1, y_2, \dots) p(y_1, y_2, \dots).$$

2. Let $g(Y_1, Y_2, \dots)$ be a function of the continuous r.v. $Y_1, Y_2, \dots$ with j.d.f. $f(y_1, y_2, \dots)$. Then

$$E[g(Y_1, Y_2, \dots)]$$
$$= \int_{y_k} \dots \int_{y_1} g(y_1, y_2, \dots) f(y_1, y_2, \dots) dy_1 .. dy_k.$$

3. If c is a constant, then $E(c) = c$.

4. Let $g(Y_1, Y_2)$ be a function of the r.v. Y_1, Y_2 and c be a constant. Then

$$E[cg(Y_1, Y_2)] = cE[g(Y_1, Y_2)].$$

5. Let Y_1 and Y_2 be continuous r.v. with

j.d.f. $f(y_1, y_2)$ and let $g_1(Y_1, Y_2)$, $g_1(Y_1, Y_2), \ldots$ be functions of the r.v. Y_1, Y_2. Then

$$E[g_1(Y_1, Y_2) + g_2(Y_1, Y_2) + \ldots]$$
$$= E[g_1(Y_1, Y_2)] + E[g_2(Y_1, Y_2)] + \ldots$$

6. Let Y_1 and Y_2 be independent r.v. with a j.d.f. $f(y_1, y_2)$. Let $g(Y_1)$, and $h(Y_2)$ be functions of only Y_1 and Y_2.

$$E[g(Y_1)h(Y_2)] = E[g(Y_1)]E[h(Y_2)].$$

7. Covariance of Two R.V.

We think the dependence of two r.v. Y_1 and Y_2 as implying that one variable say Y_1 either increases or decreases as other variable say Y_2 changes.

Two common measure of dependence:

Covariance

1. The covariance of r.v. Y_1 and Y_2 is

$$Cov(Y_1, Y_2) = E[(Y_1 - \mu_1)(Y_2 - \mu_2)]$$

where $\mu_1 = E(Y_1)$ and $\mu_2 = E(Y_2)$.

2. The larger the absolute value of $Cov(Y_1, Y_2)$, the greater the linear dependence between Y_1 and Y_2.

3. Positive values indicate that Y_1 increases as Y_2 increases.

4. Negative values indicate that Y_1 decreases as Y_2 increases.

5. A zero value indicates no linear dependence between Y_1 and Y_2.

Coefficient of Correlation

1. Covariance value depends upon the scale of measurement which makes hard to determine whether it is large.

This problem is eliminated by standardizing its value and using the simple coefficient of correlation

$$\rho = \frac{Cov(Y_1, Y_2)}{\sigma_1 \sigma_2},$$

where σ_1 and $\sigma_2$are the standard deviations of Y_1 and Y_2.

2. $-1 \leq \rho \leq 1$.

3. $\rho = 1$ or -1 indicates perfect correlation, with all points falling on a straight line.

4. $\rho = 0$ indicates no correlation and zero covariance.

5. The sign of the covariance is the same as the sign of the coefficient of correlation.

6. A positive ρ indicates that Y_1 increases as Y_2 increases.

7. Negative ρ indicates that Y_1 decreases as Y_2 increases.

8. Let Y_1 and Y_2 be r.v. with a j.d.f. $f(y_1, y_2)$. Then

$$\begin{aligned} Cov(Y_1, Y_2) &= E[(Y_1 - \mu_1)(Y_2 - \mu_2)] \\ &= E(Y_1 Y_2) - E(Y_1)E(Y_2) \\ &= E(Y_1 Y_2) - \mu_1 \mu_2. \end{aligned}$$

9. If Y_1 and Y_2 are independent r.v. then $Cov(Y_1, Y_2) = 0$.

8. Expected value 7 variance of Linear

Functions of R.V.

Let $Y_1, Y_2, \ldots, Y_n$ and $X_1, X_2, \ldots, X_m$ be r.v. with $E(Y_i) = \mu_i$ and $E(X_i) = \xi_i$.

Define

$$U_1 = \sum a_i Y_i \text{ and } U_2 = \sum b_j X_j,$$

where a_i and b_j are constants. Then

$$1. \qquad E(U_1) = \sum a_i \mu_i.$$

2. $V(U_1) = \sum a_i^2 V(Y_i) + 2 \sum \sum_{i<j} a_i a_j Cov(Y_i,$

3. $Cov(U_1, U_2) = \sum_i \sum_j a_i b_j Cov(Y_i, X_j)$

11. Conditional Expectations

1. Let Y_1 and Y_2 be two r.v., then the conditional expectation of $g(Y_1)$ given $Y_2 = y_2$ is

$$E[g(y_1)|Y_2 = y_2)] = \sum_{y_1} g(y_1) p(y_1|y_2),$$

if Y_1 and Y_2 are discrete r.v. and

$$E[g(y_1)|Y_2 = y_2)] = \int_{-\infty}^{\infty} g(y_1)f(y_1|y_2)dy_2,$$

if Y_1 and Y_2 are continuous r.v..

2. Let Y_1 and Y_2 be two r.v.. Then

$$E(Y_1) = E[E(Y_1|Y_2)],$$

where on the RHS, the inside expectation is with respect to the conditional distribution of Y_1 given $Y_2 = y_2$ and the outside expectation is with respect to the distribution of Y_2.

3. Let Y_1 and Y_2 be two r.v.. Then

$$V(Y_1) = E[V(Y_1|Y_2)] + V[E(Y_1|Y_2)].$$