# Chapter 9

# Statistical Physics, Home Work Solutions

## 9.1 Problem 9.2

Show that the most probable speed of a gas molecule is

$$v_{mp} = \sqrt{\frac{2k_B T}{m}}$$

Note that the most probable speed corresponds to the point where the Maxwellian speed distribution curve, $n(v)$, has a maximum.

---

### Solution

The Maxwell distribution is given by:

$$n(v) = \frac{4\pi N}{V} \left( \frac{m}{2\pi k_B T} \right)^{3/2} v^2 e^{-mv^2/2k_B T}$$

$n(v)$ has a maximum when $dn(v)/dv = 0$. So,

$$\begin{aligned}
\frac{dn(v)}{dv} &= \frac{4\pi N}{V} \left( \frac{m}{2\pi k_B T} \right)^{3/2} \left[ 2ve^{-mv^2/2k_B T} + v^2 \frac{-2mv}{2k_B T} e^{-mv^2/2k_B T} \right] \\
&= 0 \\
&= \frac{4\pi N}{V} \left( \frac{m}{2\pi k_B T} \right)^{3/2} ve^{-mv^2/2k_B T} \left[ 2 - \frac{mv^2}{k_B T} \right]
\end{aligned}$$

From the last equation, there are three values for $v$ that make $n(v) = 0$, namely:

$$v = 0, \qquad v = \infty, \qquad \text{and} \qquad v = \sqrt{\frac{2k_B T}{m}}$$

Since $n(v)$ is always positive and $n(v) \to 0$ as $v = 0$ or $\infty$, then first two values of $v$ correspond to minima and the third must be a maximum, then

$$v_{mp} = \sqrt{\frac{2k_B T}{m}}$$

## 9.2 Problem 9.8

Use the distribution function given in Exercise 9.1:

$$n(E)dE = \frac{2\pi(N/V)}{(\pi k_B T)^{3/2}} E^{1/2} e^{-E/k_B T} dE$$

to find

(a) the most probable kinetic energy of gas molecules at temperature $T$,

(b) the mean kinetic energy at $T$, and

(c) the root-mean-square kinetic energy at $T$.

---

# Solution

(a) The given distribution applies also to the kinetic energy $K$, so

$$n(K)dK = \frac{2\pi\,(N/V)}{(\pi\,k_B\,T)^{3/2}}\,K^{1/2}\,e^{-K/k_B\,T} dK$$

The most probable value of the kinetic energy $K_{mp}$ occurs when $K^{1/2}e^{-K/k_B T}$ has a minimum.

$$\begin{aligned}
\frac{d[K^{1/2}\,e^{-K/k_B\,T}]}{dK} &= 0 \\
&= \frac{1}{2}\,K^{-1/2}e^{-K/k_B\,T} - \frac{1}{k_B\,T}\,K^{1/2}\,e^{-K/k_B\,T} \\
&= e^{-K/k_B\,T}\left(\frac{1}{2K^{1/2}} - \frac{K^{1/2}}{k_B\,T}\right)
\end{aligned}$$

There are two solution to the above equation:

- $K = \infty$ and this gives $n(\infty) = 0$. This means that this solution is a minimum.
- $\left(\frac{1}{K^{1/2}} - \frac{K^{1/2}}{k_B\,T}\right) = 0$, must be a maximum.

we therfore have:

$$K_{mp} = \frac{1}{2}\,k_B\,T$$

(b) The mean kinetic energy $\overline{K}$ is given by:

$$\begin{aligned}\overline{K} &= \frac{1}{N/V}\int_0^\infty K n(K) dK \\ &= \frac{2\pi}{(\pi\, k_B\, T)^{3/2}}\int_0^\infty K^{3/2}\, e^{-K/k_B\, T} dK\end{aligned}$$

Using "Maple" to evaluate the integral we get

$$\begin{aligned}\overline{K} &= \frac{2\pi}{(\pi\, k_B\, T)^{3/2}} \lim_{K\to\infty} - K^{3/2}\, e^{(-\frac{K}{k_B\, T})}\, k_B\, T \\ &+3\, k_B\, T\left(-\frac{1}{2}\, k_B\, T\, \sqrt{K}\, e^{(-\frac{K}{k_B\, T})} + \frac{1}{4}\,\frac{k_B\, T\, \sqrt{\pi}\, \mathrm{erf}(\sqrt{\frac{1}{k_B\, T}}\, \sqrt{K})}{\sqrt{\frac{1}{k_B\, T}}}\right)\end{aligned}$$

Since $e^{-\infty} = 0$ and $\mathrm{erf}(\infty) = 1$ we get:

$$\begin{aligned}\overline{K} &= \frac{2\pi}{(\pi k_B\, T)^{3/2}}\,\frac{3(k_B\, T)^2\, \sqrt{\pi}}{4\sqrt{\frac{1}{k_B\, T}}} \\ &= \frac{6}{4}\,\frac{\pi}{(\pi k_B\, T)^{3/2}}\, \sqrt{\pi}\, (k_B\, T)^{5/2} \\ &= \frac{3}{2}\, k_B\, T\end{aligned}$$

(c) The root-mean-square energy $E_{rms}$ is given by:

$$\begin{aligned}K_{rms}^2 &= \frac{1}{N/V}\int_0^\infty K^2 n(K)\, dK \\ &= \frac{2\pi}{(\pi k_B\, T)^{2/3}}\int_0^\infty K^{5/2}\, e^{-K/k_B\, T}\, dK\end{aligned}$$

Using "Maple" to evaluate the integral, we get:

$$
\begin{aligned}
K_{rms}^2 &= \frac{2\pi}{(\pi k_B\,T)^{3/2}} \lim_{K\to\infty} - K^{5/2}\, e^{(-\frac{K}{k_B\,T})}\, k_B\,T \\
&\quad + 5\,(k_B\,T)^2 \left( -\frac{1}{2}\, K^{3/2}\, e^{(-\frac{K}{k_B\,T})} + \frac{3}{2}\, k_B\,T \left( -\frac{1}{2}\sqrt{K}\, e^{(-\frac{K}{k_B\,T})} + \frac{1}{4}\, \frac{\sqrt{\pi}\,\mathrm{erf}(\sqrt{\frac{1}{k_B\,T}}\,\sqrt{K})}{\sqrt{\frac{1}{k_B\,T}}} \right)\right) \\
&= \frac{2\pi}{(\pi k_B\,T)^{3/2}}\, \frac{15}{8}\, \sqrt{\pi}\, (k_B\,T)^{7/2} \\
&= \frac{15}{4}\, (k_B\,T)^2 \\
K_{rms} &= \sqrt{\frac{15}{4}}\, (k_B\,T)
\end{aligned}
$$

## 9.3 Problem 9.12

(a) Find the average energy per photon for photons in thermal equilibrium with a cavity at temperature $T$.

(b) Calculate the average photon energy in electron volts at $T = 6000\,K$. *Hint:* Two useful integrals are

$$\int_0^\infty \frac{z^2 dz}{e^z - 1} \approx 2.41 \qquad \text{and} \qquad \int_0^\infty \frac{z^3 dz}{e^z - 1} \approx \frac{\pi^4}{15}$$

---

# Solution

(a) The average energy per photon $\overline{E}$ can be found from Bose-Einestein distribution $F_{BE}$ and the density of states $g(E)$ as:

$$\begin{aligned}
\overline{E} &= \frac{\int_0^\infty E\, F_{BE}\, g(E)\, dE}{N/V} \\
&= \left(\int_0^\infty E\, F_{BE}\, g(E)\, dE\right) \Big/ \left(\int_0^\infty F_{BE}\, g(E)\, dE\right) \\
&= \left(\int_0^\infty \frac{8\,\pi\,E^3}{(hc)^3}\, \frac{1}{e^{E/k_B\,T} - 1}\, dE\right) \Big/ \left(\int_0^\infty \frac{8\,\pi\,E^2}{(hc)^3}\, \frac{1}{e^{E/k_B\,T} - 1}\, dE\right) \\
&= \left(\frac{8\,\pi\,(k_B\,T)^4}{(hc)^3} \int_0^\infty \left(\frac{E}{k_B\,T}\right)^3 \frac{1}{e^{E/k_B\,T} - 1}\, \frac{dE}{k_B\,T}\right) \Big/ \\
&\quad \left(\frac{8\,\pi\,(k_B\,T)^3}{(hc)^3} \int_0^\infty \left(\frac{E}{k_B\,T}\right)^2 \frac{1}{e^{E/k_B\,T} - 1}\, \frac{dE}{k_B\,T}\right)
\end{aligned}$$

Substituting $z$ for $E/(k_B\,T)$ we get:

$$\begin{aligned}
\overline{E} &= \left(k_B\,T \int_0^\infty \frac{z^3}{e^z - 1}\, dz\right) \Big/ \left(\int_0^\infty \frac{z^2}{e^z - 1}\, dz\right) \\
&= k_B\,T\, \frac{\pi^4}{15 \times 2.41}
\end{aligned}$$

(b) The average energy per photon at 6000 $K$ is:

$$\begin{aligned}\overline{E} &= \frac{k_B\,T\,\pi^4}{15\times 2.41}\\ &= \frac{8.62\times 10^{-5}\,(eV/K)\times 6000\,(K)\,\pi^4}{36.15}\\ &= 1.39\,eV\end{aligned}$$

## 9.4 Problem 9.17

Show that the average kinetic energy of a conduction electron in a metal at 0 $K$ is given by $\overline{E} = 3E_F/5$. By way of contrast, note that all of the molecules in an ideal gas at 0 $K$ have zero energy! *Hint:* Use the standard definition of an average given by

$$\overline{E} = \frac{V}{N}\int_0^\infty E\, g(E)\, f_{FD}(E) dE$$

---

### Solution

At $T = 0\,K$, $f_{FD} = 0$ for $E > E_F$ and $f_{FD} = 1$ for $E < E_f$, and $g(E) = D\, E^{1/2}$. Since:

$$\frac{N}{V} = \int_0^\infty \; g(E)\, f_{FD} dE$$

we then have:

$$\begin{aligned}
\overline{E} &= \left( D \int_0^{E_F} \; E^{3/2}\, dE \right) \Bigg/ \left( D \int_0^{E_F} \; E^{1/2}\, dE \right) \\
&= \left( \frac{2}{5}\, E_F^{5/2} \right) \Bigg/ \left( \frac{2}{3}\, E_F^{3/2} \right) \\
&= \frac{3}{5}\, E_F
\end{aligned}$$

## 9.5 Problem 9.23

Calculate the energy of a conduction electron in silver at 800 $K$ if the probability of finding the electron in that state is 0.95. Assume that the Fermi energy for silver is 5.48 $eV$ at this temperature.

---

### Solution

$E_F = 5.48\,eV$ for silver at 800 $K$, we then have:

$$\begin{aligned}
f_{FD} &= 0.95 \\
&= \frac{1}{e^{(E-E_f)/k_B\,T}+1} \\
e^{(E-E_f)/k_B\,T} &= \frac{1}{0.95} - 1 \\
\frac{E-E_F}{k_B\,T} &= \ln\left(\frac{1}{0.95} - 1\right) \\
&= \ln(0.05263) \\
&= -2.944 \\
E - E_F &= -2.944 \times\ k_B\,T \\
&= -2.944 \times 8.617 \times 10^{-5}(eV/K)\ \times 800\,(K) \\
&= -0.203\,eV \\
E &= -0.203 + E_F \\
&= -0.203 + 5.48 \\
&= 5.28\,eV
\end{aligned}$$