Chapter 4

Physics 205 Solution of Home Work Problems

4.1 Problem 4.14

(a) Show that the formula for Low Energy Electron Diffraction (LEED) when electrons are incident perpendicular to a crystal surface may be written as:

\[ \sin \phi = \frac{n \hbar c}{d \sqrt{2 m_e c^2 K}} \]

where \( n \) is the order of the maximum, \( d \) is the atomic spacing, \( m_e \) is the electron mass, \( K \) is the electron’s kinetic energy, and \( \phi \) is the angle between the incident and diffracted beams.

(b) Calculate the atomic spacing in a crystal that has consecutive diffraction maxima at \( \phi = 24.1^\circ \) and \( \phi = 54.9^\circ \) for 100-eV electrons.

Solution

(a) For a maximum in an interference pattern we have:

\[ n \lambda = d \sin \phi \]

\[ \sin \phi = \frac{n \lambda}{d} \]

\[ = \left( \frac{n}{d} \right) \left( \frac{h}{\lambda} \right) \]

\[ = \left( \frac{n}{d} \right) \left( \frac{h}{\sqrt{2m_e K}} \right) \]

\[ = \left( \frac{n \hbar c}{\sqrt{2m_e c^2 K}} \right) \]
Let us assume that the order of the maximum for the first angle is \( n_1 \) and that for the second angle is \( n_1 + 1 \), we then have:

\[
d_1 = \frac{n_1 hc}{\sin \phi_1 \sqrt{2m_e c^2 K}} = \frac{n_1 \times 12.40 \times 10^{-7} \text{ eV} \cdot \text{m}}{\sqrt{(\sin 24.1^\circ) \times (2 \times 0.511 \times 10^6 \text{ eV} \times 100 \text{ eV})}} \approx n_1 \times 3.00 \times 10^{-10} \text{ m}
\]

\[
d_2 = \frac{(n_1 + 1)hc}{\sin \phi_2 \sqrt{2m_e c^2 K}} = \frac{(n_1 + 1) \times 12.40 \times 10^{-7} \text{ eV} \cdot \text{m}}{\sqrt{(\sin 54.9^\circ) \times (2 \times 0.511 \times 10^6 \text{ eV} \times 100 \text{ eV})}} = (n_1 + 1) \times 1.50 \times 10^{-10} \text{ m}
\]

Since \( d_1 \) must be equal to \( d_2 \) then we get:

\[
d_2 - d_1 = 1.50 \times (n_1 + 1) \times 10^{-10} - 3.00 \times n_1 \times 10^{-10} = 0
\]

\[
0 = (1.5 - 1.5 \times n_1) \times 10^{-10}
\]

\[
n_1 = 1
\]

\[
d = 3.00 \times 10^{-10} \text{ m} = 3.00 \text{ Å}
\]
4.2 Problem 4.23

A proton has a kinetic energy of 1.0 MeV. If its momentum is measured with an uncertainty of 5%, what is the minimum uncertainty in its position?

Solution

The rest mass-energy of a proton is 938.3 MeV. Since the kinetic energy of the proton is 1.0 MeV, then it is only 0.11% of the rest mass-energy. So, we can treat this problem nonrelativistically. The kinetic energy $K$ and momentum are related by:

\[
K = \frac{1}{2}mv^2 \\
= \frac{p^2}{2m} \\
p = \sqrt{2mK} \\
= \sqrt{2 \times 938.3 \text{ MeV}/c^2 \times 1.0 \text{ MeV}} \\
= 43.3 \text{ MeV}/c \\
\Delta p = 0.05p \\
= 2.17 \text{ MeV}/c \\
\Delta p \cdot \Delta x = \frac{h}{2} \\
\Delta x = \frac{h}{2\Delta p} \\
= \frac{\hbar}{2(\Delta p)c} \\
= \frac{1.973 \times 10^{-13} \text{ MeV} \cdot \text{m}}{2 \times 2.17 \text{ MeV}} \\
= 4.55 \times 10^{-14} \text{ m}
\]
4.3 Problem 4.28

An electron of momentum $p$ is at a distance $r$ from a stationary proton. The system has a kinetic energy $K = \frac{p^2}{2m_e}$ and potential energy $U = -\frac{ke^2}{r}$. Its total energy is $E = K + U$. If the electron is bound to the proton to form a hydrogen atom, its average position is at the proton but the uncertainty in its position is approximately equal to the radius $r$ of its orbit. The electron’s average momentum will be zero, but the uncertainty in its momentum will be given by the uncertainty principle. Treat the atom as a one-dimensional system in the following:

(a) Estimate the uncertainty in the electron’s momentum in terms of $r$.

(b) Estimate the electron’s kinetic, potential, and total energies in terms of $r$.

(c) The actual value of $r$ is the one that minimizes the total energy, resulting in a stable atom. Find the value of $r$ and the resulting total energy. Compare your answer with the predictions of the Bohr theory.

Solution

(a) Since $\Delta x \approx r$ and $p \approx \Delta p$, then the uncertainty principle gives

$$\Delta p \cdot \Delta x = \frac{\hbar}{2}$$

$$\Delta p \approx \frac{\hbar}{2r},$$

(b) We then get for the kinetic energy $K$, the potential energy $U$ and the total energy $E$:

$$K = \frac{p^2}{2m} = \frac{(\Delta p)^2}{2m} = \frac{\hbar^2}{8mr^2}$$

$$U = -\frac{ke^2}{r}$$

$$E = \frac{\hbar^2}{8mr^2} - \frac{ke^2}{r}$$
(c) We minimize $E$ by setting $dE/dr = 0$:

$$0 = -\frac{\hbar^2}{4mr^3} + \frac{ke^2}{r^2}$$

$$\frac{ke^2}{r^2} = \frac{\hbar^2}{4mr^3}$$

$$r = \frac{\hbar^2}{4mke^2}$$

$$= \frac{1}{4a_o}$$

$$= \frac{1}{4} \times 0.5292 \times 10^{-10} \text{ m}$$

$$= 1.323 \times 10^{-11} \text{ m}$$

where $a_o = \hbar^2/(mke^2)$ is the Bohr radius. Substituting for $r$ in the total energy equation we get:

$$E = \frac{\hbar^2}{8m} \times \frac{16m^2k^2e^4}{\hbar^4} - ke^2 \times \frac{4mke^2}{\hbar^2}$$

$$= \frac{2mk^2e^4}{\hbar^2} - \frac{4mk^2e^4}{\hbar^2}$$

$$= -\frac{2mk^2e^4}{\hbar^2}$$

$$= -2ke^2 \times \frac{mke^2}{\hbar^2}$$

$$= -4 \times \frac{ke^2}{2a_o}$$

$$= -4 \times 13.6 \text{ eV}$$

The radius of this atom is one quarter of the Bohr radius and the energy is 4 times the energy of the $n = 1$ state of the Bohr hydrogen atom.
4.4 Problem 4.29

An excited nucleus with a lifetime of 0.100 ns emits a $\gamma$-ray of energy 2.00 MeV. Can the energy width (uncertainty in energy $\Delta E$) of this 2.00-MeV $\gamma$ emission line be directly measured if the best gamma detectors can measure energies to $\pm 5$ eV?

Solution

The energy width can be found from the uncertainty principle:

$$\Delta E \cdot \Delta t = \frac{\hbar}{2}$$

$$\Delta E = \frac{\hbar}{2\Delta t}$$

$$= \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{2 \times 0.100 \times 10^{-9} \text{ s}}$$

$$= 3.29 \times 10^{-6} \text{ eV}$$

Since the width is much smaller than the best energy resolution of the gamma detectors, then this line can not be directly detected.
A $\pi^0$ meson is an unstable particle produced in high-energy particle collisions. It has a mass-energy equivalent of about 135 MeV, and it exists for an average lifetime of only $8.7 \times 10^{-17}$ s before decaying into two gamma rays. Using the uncertainty principle, estimate the fractional uncertainty $\Delta m/m$ in its mass determination.

**Solution**

From the uncertainty principle we get:

\[
\begin{align*}
\Delta E \cdot \Delta t &= \frac{\hbar}{2} \\
\Delta mc^2 \Delta t &= \frac{\hbar}{2} \\
\frac{\Delta m}{m} &= \frac{\hbar}{2 \Delta t mc^2} \\
&= \frac{\hbar}{2 \Delta t E_{\text{rest}}} \\
&= \frac{6.582 \times 10^{-22} \text{ MeV} \cdot s}{2 \times 8.7 \times 10^{-17} \text{ s} \times 135 \text{ MeV}} \\
&= 2.80 \times 10^{-8}
\end{align*}
\]