Chapter 1

Relativity I, Solutions of selected Problems

1.1 Problem 1.10

The average lifetime of a $\pi$-meson in its own frame of reference is $2.6 \times 10^{-8}$ s. (This is proper time.) If the meson moves with a speed of 0.95c, what is:

(a) its mean lifetime as measured by an observer on Earth? and,
(b) the average distance it travels before decaying, as measured by an observer on Earth?

Solution

(a)

$$
\tau = \gamma \tau'
$$

where $\tau' = 2.6 \times 10^{-8}$ s is the average lifetime in the pion’s frame, $\tau$ is the average lifetime as measured by an observer at rest, $\gamma = 1/\sqrt{1 - \beta^2}$, and $\beta = v/c$, we then have:

$$
\tau = \frac{\tau'}{\sqrt{1 - \beta^2}}
= \frac{\tau'}{\sqrt{1 - (\frac{v}{c})^2}}
= \frac{2.6 \times 10^{-8}}{\sqrt{1 - (0.95)^2}}
= 8.33 \times 10^{-8} \text{ s}
$$
(b) The distance $d$ travelled by the pions before decaying as measured by an observer at rest is obtained from the rest frame velocity and lifetime, i.e.

$$d = v\tau = 0.95 \times c \times \tau = 0.95 \times 3.0 \times 10^8 \times 8.33 \times 10^{-8} = 24 \text{ m}$$
1.2 Problem 1.14

A rod of length $L_0$ moves with a speed of $v$ along the horizontal direction. The rod makes an angle of $\theta_0$ with respect to the $x'$ axis.

(a) Show that the length of the rod as measured by a stationary observer is given by:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2} \cos^2 \theta_0}$$

(b) Show that the angle that the rod makes with the $x$ axis is given by the expression:

$$\tan \theta = \gamma \tan \theta_0$$

These results show that the rod is both contracted and rotated. (Take the lower end of the rod to be at the origin of the primed coordinate system.)

Solution

![Diagram of rod moving with speed v along x' axis and making an angle $\theta_0$ with it.](image)

Figure 1.14: A rod of length $L_0$ moving with a speed $v$ along $x'$ axis and makes an angle $\theta_0$ with it.

(a) Only the $x$-component of $L_0$ contracts as a result of the motion along the $x$-axis. We have:

$$L_{x'} = L_0 \cos \theta_0$$
and
\[ L_y' = L_o \sin \theta_o = L_y \]
a stationary observer will find \( L_x \) as:
\[
L_x = \frac{L_x'}{\gamma} = \frac{L_o \cos \theta_o}{\gamma}
\]
The total length of the rod \( L \) as measured by a stationary observer is:
\[
L = \sqrt{(L_x)^2 + (L_y)^2} = \sqrt{\left(\frac{L_o \cos \theta_o}{\gamma}\right)^2 + (L_o \sin \theta_o)^2} = L_o \sqrt{\cos^2 \theta_o \left(1 - \frac{v^2}{c^2}\right) + \sin^2 \theta_o} = L_o \sqrt{\cos^2 \theta_o - \cos^2 \theta_o \frac{v^2}{c^2} + \sin^2 \theta_o} = L_o \sqrt{1 - \frac{v^2}{c^2} \cos^2 \theta_o}
\]
(b) The angle of the rod makes with the \( x \) axis as seen by the stationary observer is:
\[
\tan \theta = \frac{L_y}{L_x} = \frac{L_o \sin \theta_o}{L_o \cos \theta_o/\gamma} = \gamma \tan \theta_o
\]
1.3 Problem 1.22

*Speed of light in a moving medium.* The motion of a medium such as water influences the speed of light. This effect was first observed by Fizeau in 1851. Consider a light beam passing through a horizontal column of water moving with a speed \( v \).

(a) Show that if the beam travels in the same direction as the flow of water, the speed of light measured in the laboratory frame is given by

\[
    u = \frac{c}{n} \left( 1 + \frac{nv}{c} \right) \left( 1 + \frac{v}{nc} \right)
\]

where \( n \) is the index of refraction of the water. (Hint: Use the inverse Lorentz velocity transformation and note that the speed of light with respect to the moving frame is given by \( c/n \).)

(b) Show that for \( v \ll c \), the preceding expression is in good agreement with Fizeau’s experimental result:

\[
    u \approx \frac{c}{n} + v - \frac{v}{n^2}
\]

This proves that the Lorentz velocity transformation and not the Galilean velocity transformation is correct for light.

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Solution

(a) Let us take the (+ve) \( x \)-axis along the direction of the flow of the water, \( u_x \) be the velocity of light as observed in the laboratory, and \( u'_x \) is the speed of light as observed by an observer moving with the water. Using inverse Lorentz velocity transformation we get:

\[
    u_x = \frac{u'_x + v}{1 + u'_x v/c^2}
\]

(1.1)

The ratio of the speed of light in vacuum \( c \) to its speed in a medium \( u'_x \), is the medium’s index of refraction \( n \), that is:

\[
    n = \frac{c}{u'_x}
\]

(1.2)
Using Equation (1.2) in Equation (1.1), we get:

\[ u_x = \frac{(c/n) + v}{1 + v/nc} \]

\[ = \frac{c}{n} \left( \frac{1 + nv/c}{1 + v/nc} \right) \]

(1.3)

(b) when \( v/c \ll 1 \), we can use the binomial expansion in Equation (1.3) to get:

\[ u_x = \frac{c}{n} \left[ 1 + n \left( \frac{v}{c} \right) \right]^{-1} \]

\[ \approx \frac{c}{n} \left[ 1 + n \left( \frac{v}{c} \right) \right] \left[ 1 + \frac{1}{n} \left( \frac{v}{c} \right) \right] \]

\[ \approx \frac{c}{n} \left[ 1 + \frac{nv/c}{nc} - \frac{v}{nc} - \frac{v^2}{c^2} \right] \]

(1.4)

once again since \( v/c \ll 1 \), we ignore last term in Equation (1.4), to get:

\[ u_x \approx \frac{c}{n} \left[ 1 + \frac{nv}{c} - \frac{v}{nc} \right] \]

\[ \approx \frac{c}{n} + v - \frac{v^2}{n^2} \]

(1.5)
1.4 Problem 1.30

An observer in a rocket that is moving toward a mirror at speed $v$ relative to the reference frame labeled by $S$ in Figure (1.31). The mirror is stationary with respect to $S$. A light pulse emitted by the rocket travels toward the mirror and is reflected back to the rocket. The front of the rocket is a distance $d$ from the mirror (as measured by observers in $S$) at the moment the light pulse leaves the rocket. What is the total travel time of the pulse as measured by observers in:

(a) the $S$ frame and
(b) the front of the rocket?

Solution

Figure 1.31:

(a) At time $t = 0$, a light pulse leaves the rocket and travels a distance $d$ to the mirror, which is at rest in frame $S$. The time it takes the pulse to reach the mirror is:

$$ t_1 = \frac{d}{c} \quad (1.6) $$
During the time $t_1$, the rocked traveled a distance $\ell_1$:

$$\ell_1 = t_1 v$$  \hspace{1cm} (1.7)

Using Equation (1.6) in Equation (1.7) we get:

$$\ell_1 = d\frac{v}{c}$$  \hspace{1cm} (1.8)

At the moment the light pulse reaches the mirror the rocket is at distance $\ell_2 = d - \ell_1$, from the mirror using Equation (1.7), $\ell_2$ becomes:

$$\ell_2 = d - \ell_1 = d\left(1 - \frac{v}{c}\right)$$  \hspace{1cm} (1.9)

While the light pulse moves toward the rocket, the rocket also moves toward the light pulse. The rocket and the light pulse will then travel along $\ell_2$ until they meet somewhere along $\ell_2$. Let us assume that both the reflected light pulse and the rocket travel a time $t_2$ before the meet. The distance traveled by the light pules after reflection until it reaches the rocket is $x_1 = ct_2$ and the distance traveled by the rocket during the same period is $x_2 = vt_2$, we then have:

$$x_1 + x_2 = \ell_2$$
$$ct_2 + vt_2 = \ell_2$$
$$t_2 = d \left(\frac{1 - v/c}{v + c}\right)$$
$$= \frac{d}{c} \left(\frac{1 - v/c}{1 + v/c}\right)$$  \hspace{1cm} (1.10)

The total time of flight $T$ of the light pulse in frame $S$ is then:

$$T = t_1 + t_2$$
$$= \frac{d}{c} + \frac{d}{c} \left(\frac{1 - v/c}{1 + v/c}\right)$$
$$= \frac{d}{c} \left(1 + \frac{1 - v/c}{1 + v/c}\right)$$
\[ T = \frac{d}{c} \left( \frac{1 + v/c}{1 + v/c} + \frac{1 - v/c}{1 + v/c} \right) \]
\[ = \frac{d}{c} \left( \frac{1 + v/c + 1 - v/c}{1 + v/c} \right) \]
\[ = \frac{d}{c} \left( \frac{2}{1 + v/c} \right) \quad (1.11) \]

(b) For an observe in the rocket that is moving with a speed \( v \) relative to the mirror, the time of travel of the light pulse is \( T' \). The time \( T \) is measured with a clock at rest so it is the proper time, while the clock that measured \( T' \) is moving. So, \( T' \) is given by the time dilation formula:

\[ T' = \frac{T}{\gamma} \]
\[ = \frac{d}{c} \left( \frac{2}{1 + v/c} \right) \sqrt{1 - \frac{v^2}{c^2}} \]
\[ = \frac{2d}{c} \sqrt{\frac{1 - (v/c)^2}{(1 + v/c)^2}} \]
\[ = \frac{2d}{c} \sqrt{\frac{(1 - v/c)(1 + v/c)}{(1 + v/c)^2}} \]
\[ = \frac{2d}{c} \sqrt{\frac{1 - v/c}{1 + v/c}} \quad (1.12) \]
1.5 Problem 1.39

An Earth satellite used in the Global Positioning System moves in a circular orbit with period $11 \, h \, 58 \, min$.

(a) Determine the radius of its orbit.

(b) Determine its speed.

(c) The satellite contains an oscillator producing the principal nonmilitary GPS signal. Its frequency is $1,575.42 \, MHz$ in the reference frame of the satellite. When it is received on the Earth’s surface, what is the fractional change in this frequency due to time dilation, as described by special relativity?

(d) The gravitational blueshift of the frequency according to general relativity is a separate effect. The magnitude of that fractional change is given by

$$\frac{\Delta f}{f} = \frac{\Delta U_g}{mc^2}$$

where $U_g/m$ is the change in gravitational potential energy per unit mass between the two points at which the signal is observed. Calculate this fractional change in frequency.

(e) What is the overall fractional change in frequency? Superposed on both of these relativistic effects is a Doppler shift that is generally much larger. It can be a redshift or a blueshift, depending on the motion of a particular satellite relative to a GPS receiver (Figure (1.40)).

Figure 1.40:

Solution
(a) For the satellite to stay in orbit, the gravitational force on it must be the centripetal force that acts on it to keep bending the path of the satellite, that is:

$$\frac{GM_Em}{r^2} = \frac{mv^2}{r} \quad (1.13)$$

where $G = 6.67 \times 10^{-11} \, N \cdot m^2/kg^2$ is the gravitational constant, $M_E = 5.98 \times 10^{24} \, kg$ is the mass of earth, $m$ is the mass of the satellite, $r$ is the radius of the satellite’s orbit, and $v$ is the satellite’s linear velocity. The velocity $v$ can be related to the satellite’s period $T$ and radius $r$, since the satellite travels the circumference of the orbit in one period:

$$v = \frac{2\pi r}{T} \quad (1.14)$$

Substituting for $v$ from Equation (1.14) into Equation (1.13) we get:

$$\frac{GM_Em}{r^2} = \frac{mv}{r} \left(\frac{2\pi r}{T}\right)^2$$

$$GM_ET^2 = 4\pi^2r^3$$

$$r = \left(\frac{GM_ET^2}{4\pi^2}\right)^{1/3} \quad (1.15)$$

substituting the numerical values of $G$, $M_E$, $r$ and $T = 1\, hours\ 58\, minutes = 11.967\, hour = 11.97 \times 60 \times 60\, seconds = 4.31 \times 10^4\, s$ in Equation (1.15) we get:

$$r = \left(\frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times (4.31 \times 10^4)^2}{4\pi^2}\right)^{1/3}$$

$$= 2.66 \times 10^7 \, m \quad (1.16)$$

(b) Using Equation (1.14) we can calculate the velocity $v$ of the satellite as:

$$v = \frac{2\pi r}{T}$$

$$= \frac{2\pi \times 2.66 \times 10^7}{4.31 \times 10^4}$$

$$= 3.88 \times 10^3 \, m/s \quad (1.17)$$
(c) The oscillator frequency of \( f = 1,575.42 \text{MHz} \) was measured with the oscillator at rest. In orbit the oscillator is moving with a speed of \( v = 3.88 \times 10^3 \text{m/s} \) as a result the oscillator will slow down and its period \( T = 1/f \) becomes longer due to relativistic time dilation. The period of the moving oscillator as measured on earth is \( T' \) where:

\[
T' = \frac{T}{\gamma} = \frac{1}{f'}
\]

The fractional change in the frequency is:

\[
\frac{\Delta f}{f} = \frac{f - f'}{f} = \frac{(1/T) - (1/T')}{1/T}
\]

Using Equation (1.18) in Equation (1.19), we get:

\[
\frac{\Delta f}{f} = \frac{(1/T) - (\gamma/T)}{1/T} = 1 - \gamma = 1 - \frac{1}{\sqrt{1 - (v/c)^2}} = 1 - \frac{1}{\sqrt{1 - (3.88 \times 10^3/3 \times 10^8)^2}} = -8.00 \times 10^{-11}
\]

(d) The gravitational potential energy \( U_g \) of an object of mass \( m_s \) at a position \( r \) from the center of Earth is:

\[
U_g = -\frac{GM_Em_s}{r}
\]

where \( m \) in our case is the mass of the satellite and \( r \) is the radius of its orbit. The change in the potential energy of the satellite as it has been taken from the surface of earth to orbit is the difference between it’s potential energy at earth’s surface \( r = r_e \) and its potential energy in orbit, \( r = r_s \):

\[
\Delta U_g = U_g(r = r_s) - U_g(r = r_e) = -\frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times m_s}{2.66 \times 10^7} + \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times m_s}{6.37 \times 10^6} = 4.76 \times 10^7 \times m_s \text{J/m}
\]
The fractional change in the frequency due to the change in gravitation is:

\[
\frac{\Delta f}{f} = \frac{\Delta U_g}{m_s c^2} = \frac{4.76 \times 10^7 \times m_s}{m_s \times (3 \times 10^8)^2} = 5.29 \times 10^{-10}
\]  

(1.23)

(e) the total fractional change \((\Delta f/f)_{tot}\) due to time dilation and gravitation change is:

\[
(\frac{\Delta f}{f})_{tot} = -8.00 \times 10^{-11} + 5.29 \times 10^{-10} = 4.49 \times 10^{-10}
\]  

(1.24)