1. Show that the momentum of a particle having a charge e moving in a circle of radius R in a magnetic field B is given by

$$p = 300BR$$

where p is in MeV/c, B is in teslas, and R is in meters.

Solution

A particle with a charge e moving with a velocity v that is perpendicular to a magnetic field $\mathbf{B}$ will move in a circular path of radius R under a centripetal force of Bev:

$$\frac{mv^2}{R} = Bev$$

and the momentum of the particle is:

$$\begin{aligned} mv &= BeR \\ p &= BeR; \end{aligned} \tag{1}$$

if m is in kg, v in m/s, R in meters, e in Coulomb and B in Tesla then the momentum p is in $kg \cdot m/s$. The momentum units need to be changed to MeV/c, to do that we use:

$$\begin{aligned} \frac{kg \cdot m}{s} &= \frac{(kg \cdot m) \times c}{c} \\ &= \frac{3 \times 10^8 \; kg \cdot m^2}{s^2 \cdot c} \\ &= 3 \times 10^8 \; \frac{J}{c} \\ &= \frac{3 \times 10^8}{1.6 \times 10^{-19}} \; \frac{eV}{c} \\ &= \frac{3}{1.6} \times 10^{27} \times 10^{-6} \; \frac{MeV}{c} \\ &= \frac{3}{1.6} \times 10^{21} \; \frac{MeV}{c} \end{aligned} \tag{2}$$

Substituting in Equation (1) by the value of $e = 1.6 \times 10^{-19}\ C$, the given values of B and R, and using Equation (2) we get:

$$\begin{aligned} p &= 1.6 \times 10^{-19} BR\, \frac{kg \cdot m}{s} \\ &= 1.6 \times 10^{-19} \times \frac{3}{1.6} \times 10^{21}\ BR\ \ \frac{MeV}{c} \\ &= 300\ BR\ \ \frac{MeV}{c} \end{aligned}$$

2. Electrons in projection television are accelerated through a total potential difference of 50 kV.

   (a) Calculate the speed of the electrons using the relativistic form of kinetic energy assuming the electrons start from rest.

   (b) Calculate the speed of the electrons using the classical form of kinetic energy.

   (c) Is the difference in speed significant in the design of this set in your opinion.

Solution

(a) The kinetic energy K of the electron due to a potential difference V is eV. relativistically we get:

$$\begin{aligned}
K &= eV \\
&= mc^2(\gamma - 1) \\
\gamma^2 &= \left(1 + \frac{eV}{mc^2}\right)^2 \\
\frac{v}{c} &= \sqrt{1 - \frac{1}{\left(1 + \frac{eV}{mc^2}\right)^2}} \\
&= \sqrt{1 - \frac{1}{\left(1 + \frac{5\times10^{-2}}{0.511}\right)^2}} \\
&= 0.413 \\
v &= 0.413c
\end{aligned}$$

Classically we get:

$$\begin{aligned}
K &= \frac{1}{2}mv^2 \\
&= eV \\
v &= \sqrt{\frac{2eV}{m}} \\
&= \sqrt{\frac{2 \times 5.0 \times 10^{-2}}{0.511/c^2}} \\
&= 0.442c
\end{aligned}$$

(b) The difference Δv between the two speeds is:

$$\begin{aligned}\Delta v &= \frac{0.442 - 0.413}{4.13} \times 100 \\ &= 7.0\%\end{aligned}$$

The kinetic energy is the same in both cases. The relativistic speed is smaller than the classical speed this means that the electron's mass changes slightly from the rest mass. The bending of the electron beam in a magnetic field depends on the speed and the mass and the effect on the screen depends on the kinetic energy. So this difference does not affect the design of the television set.

3. Gamma rays (high energy photons) of energy 1.02 MeV are scattered from electrons that are initially at rest. If the scattering is symmetric, that is, if $\theta = \phi$, find

   (a) the scattering angle θ.

   (b) the energy of the scattered photons.

Solution

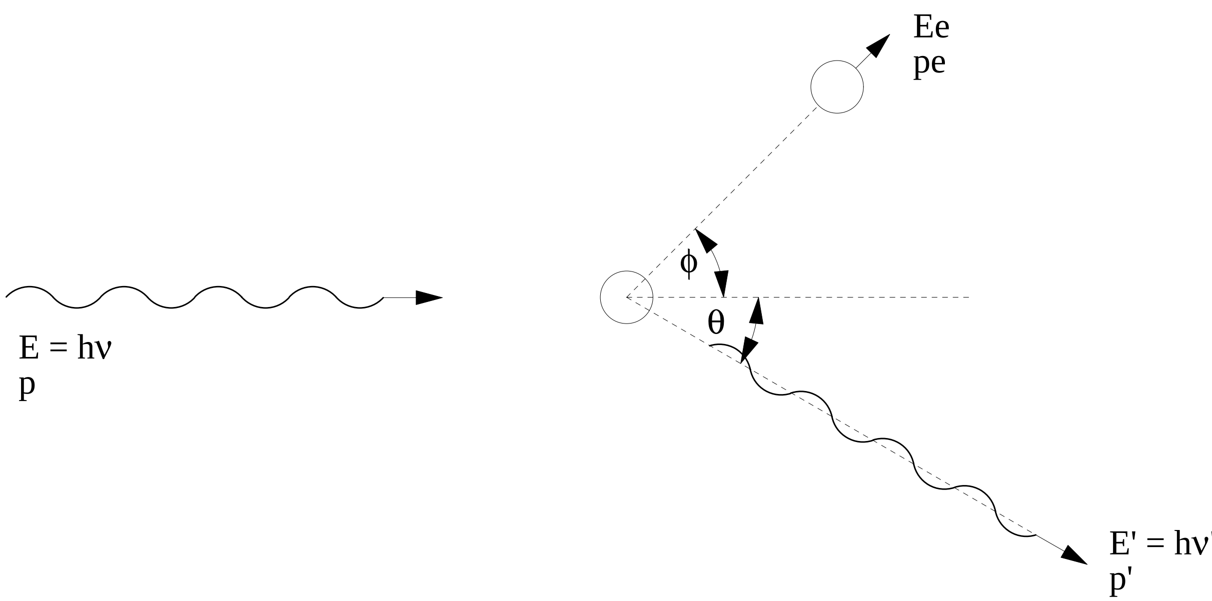

Figure 1: Compton Scattering

(a) In Compton scattering, mass-energy as well as momentum are conserved. Applying momentum conservation we get:

$$\frac{h}{\lambda_\circ} = p_e \cos\phi + \frac{h}{\lambda'} \cos\theta \qquad \text{along the x-axis} \tag{1}$$

$$0 = p_e \sin\phi - \frac{h}{\lambda'} \sin\theta \qquad \text{along the y-axis} \tag{2}$$

for a symmetric scattering $\theta = \phi$ we then get:

$$\begin{aligned} \frac{h}{\lambda_\circ} &= p_e \cos\theta + \frac{h}{\lambda'}\cos\theta \qquad \text{along the x-axis} \quad (3)\\ 0 &= p_e \sin\theta - \frac{h}{\lambda'}\sin\theta \qquad \text{along the y-axis} \quad (4)\\ p_e &= \frac{h}{\lambda'} \quad (5) \end{aligned}$$

substituting Equation 5 into Equation 3 we get:

$$\lambda' = 2\lambda_\circ \cos\theta \quad (6)$$

Using the final Compton scattering formula:

$$\lambda' - \lambda_\circ = \lambda_c(1 - \cos\theta) \quad (7)$$

Combining Equation 6 and Equation 7 we get:

$$\cos\theta = \frac{\lambda_c + \lambda_\circ}{\lambda_c + 2\lambda_\circ} \quad (8)$$

We also have:

$$\begin{aligned} E &= \frac{hc}{\lambda_\circ}\\ &= 1.02\ MeV\\ \lambda_\circ &= \frac{hc}{E}\\ &= \frac{1240\ (eV \cdot nm)}{1.02 \times 10^6\ (eV)}\\ &= 1.22 \times 10^{-3}\ nm \quad (9) \end{aligned}$$

The compton wavelength is, $\lambda_c = \frac{h}{m_e c} = 2.43 \times 10^{-3}\ nm$. Using this and Equation (9) in Equation (8), we get:

$$\cos\theta = 0.7495$$

This gives,

$$\theta = 41.5^\circ$$

(b) Using Equation (7) we get,

$$\begin{aligned}\lambda' &= \lambda_\circ + \lambda_c(1 - \cos\theta) \\ &= 1.22 \times 10^{-3} + 2.43 \times 10^{-3}\,[1 - \cos(41.5^\circ)] \\ &= 1.83 \times 10^{-3}\ nm\end{aligned}$$

The energy of the scattered photon is:

$$\begin{aligned}E' &= \frac{hc}{\lambda'} \\ &= \frac{1240\ (eV \cdot nm)}{1.83 \times 10^{-3}\ (nm)} \\ &= \frac{1240\ (eV \cdot nm)}{1.83 \times 10^{-3}\ (nm)} \\ &= 0.679\ MeV\end{aligned}$$

4. Consider a spaceship of proper length 100 m that moves along the positive x-axis at $0.9c$ with respect to the ground. If S is a coordinate frame fixed to the ground and S' is a coordinate frame fixed to the ship, then the origins are set so that at $t = t' = 0$ the front of the ship is at $x = x' = 0$. Where is the back of the ship at $t = 0$, according to an observer on the ground, and at $t = 0$ what time T' will a clock fixed to the back of the ship read, according to a ground-based observer? Do the problem

   (a) classically and

   (b) according to special relativity

Solution

(a) Classically the end of the spaceship is at $x = x' = -100\ m$. The ship based clock reads the same time as the earth based clock, $T' = 0$.

(b) In the S frame there is one event, the location of the front of the ship at $x = 0$ and $t = 0$. In the S' frame there are two events, the first is the location of the front of the ship at $x' = 0$ and t', and the second event is the location of the back of the ship $x' = x'_b = -100\ m$ and $t' = 0$. The missing event is the location of the back of the ship x_b at $t = 0$ in the S frame. Using Lorentz transformation we get:

$$x'_b = \gamma(x_b - vt) = \gamma x_b$$

and

$$x_b = \frac{x'_b}{\gamma} = x'_b\sqrt{1 - \frac{v^2}{c^2}} = -100 \times \sqrt{1 - (0.9)^2} = -43.6\ m$$

Now the clock in the S frame is located at $x = x_b$ and $t = 0$ and Lorentz transformation gives:

$$T' = \gamma(t - \frac{v}{c^2}x) = \gamma(0 - \frac{v}{c^2}x) = \frac{-(v/c^2)x}{\sqrt{1 - v^2/c^2}} = \frac{-0.9 \times 43.6}{c\sqrt{1 - (0.9)^2}} = -3 \times 10^{-7}\ s$$

5. Consider a rod of proper length d. What is its length, as seen by an observer moving with velocity v relative to the rod, when the direction of the velocity makes an angle θ with the direction of the rod? Check that your result has the proper limits when θ is zero and 90°.

Solution

Only the length of the rod in the direction of the motion is affected. Let us take the direction of the motion to be the x- direction. The proper length of the rod can then be split into two components in the rest frame, one along the x-direction (the direction of the motion) ℓ_x and the other along the y-direction ℓ_y:

$$\begin{aligned}
\ell_x &= d\cos\theta \\
\ell_y &= d\sin\theta
\end{aligned}$$

For an observer moving along the x-axis withe a speed v will see the x-component of the rod's different from the the one seen in the rest frame, so:

$$\begin{aligned}
\ell'_x &= \frac{\ell_x}{\gamma} \\
&= \ell_x\sqrt{1-\frac{v^2}{c^2}} \\
&= d\cos\theta\sqrt{1-\frac{v^2}{c^2}} \\
\ell'_y &= \ell_y \\
&= d\sin\theta
\end{aligned}$$

The total length of the rod d' as seen by the moving observer is then,

$$
\begin{aligned}
d' &= \sqrt{\ell_x'^2 + \ell_y'^2} \\
&= \sqrt{d^2 \cos^2\theta \left(1 - \frac{v^2}{c^2}\right) + d^2 \sin^2\theta} \\
&= d\sqrt{\sin^2\theta + \cos^2\theta \left(1 - \frac{v^2}{c^2}\right)} \\
&= d\sqrt{\sin^2\theta + \cos^2\theta - \left(\frac{v^2}{c^2}\right)\cos^2\theta} \\
&= d\sqrt{1 - \left(\frac{v^2}{c^2}\right)\cos^2\theta}
\end{aligned}
$$

6. Consider the metals lithium, beryllium, and mercury, which have work functions of 2.3 eV, 3.9 eV, and 4.5 eV respectively. If light of wavelength 300 nm is incident on each of these metals, determine

   (a) which metals exhibit the photoelectric effect, and

   (b) the maximum kinetic energy for the photoelectron in each case.

Solution

The energy of a photon with a wavelength of $\lambda = 300\ nm$ is:

$$
\begin{aligned}
E &= h\nu \\
&= \frac{hc}{\lambda} \\
&= \frac{1240\ (eV \cdot nm)}{300\ (nm)} \\
&= 4.13\ eV
\end{aligned}
$$

(a) Lithium and beryllium has work functions that are smaller than the incident photon energy of 4.13 eV. So it is possible for these two metals to exhibit photoelectric effect. Mercury on the other hand has a work function of 4.5 eV, so the incident photon can not provide enough energy to liberate an electron from a mercury atom. So, mercury will not exhibit photoelectric effect using a photon with wavelength $\lambda = 300\ nm$.

(b) The maximum kinetic energy is given by:

$$
\begin{aligned}
K_{max} &= \frac{hc}{\lambda} - \phi \\
&= E - \phi \\
K_{max}(Li) &= 4.13 - 2.3 \\
&= 1.83\ eV \\
K_{max}(Be) &= 4.13 - 3.9 \\
&= 0.23\ eV
\end{aligned}
$$

7. In a system of two particles with energies and momenta (E_1, p_1) and (E_2, p_2) respectively, the quantity

$$s^2 = (E_1 + E_2)^2 - c^2(p_1 + p_2)^2$$

is invariant; that is, it has the same numerical value in all inertial frames.

(a) Consider a center-of-mass collision of a proton and an antiproton ($mc^2 = 938.3$ MeV). What is the minimum momentum required to produce a particle with mass $Mc^2 = 91.2 \times 10^3$ MeV?

(b) In a fixed-target accelerator, an antiproton projectile collides with a proton target at rest. What is the minimum energy that the antiproton must have to create the new particle of part (a)?

Solution

The collision of a proton and antiproton to produce one particle can written as:

$$p + \overline{p} \longrightarrow X$$

where X is the particles that is produced with $M_X c^2 = 91.2 \times 10^3$ MeV. Doing this reaction with a minimum incoming momentum means that particle X is produced at rest.

(a) Doing this reaction in the center of mass system means that the center of mass of the proton and antiproton is kept at rest. This means that the two particles have equal and opposite momenta, i.e. $p_1 = -p_2$. Since s^2 is an invariant quantity, then its magnitude will not change before and and after the reaction. Let $m_p = 938.3$ MeV/c^2 be the rest mass of the each proton, p is the magnitude of the momentum of each particle, and $E_p^2 = m_p^2 c^4 + p^2 c^2$ is the total relativistic energy of each particle. Let us now calculate s^2 before and after the reaction:

$$s_{after}^2 = M_X^2 c^4$$

and

$$
\begin{aligned}
s^2_{before} &= (E_1 + E_2)^2 - c^2(p_1 + p_2)^2 \\
&= (E_1 + E_2)^2 \\
&= 4E_p^2 \\
&= 4(m_p^2 c^4 + p^2 c^2)
\end{aligned}
$$

Since $s^2_{before} = s^2_{after}$, we get:

$$
\begin{aligned}
M_X^2 c^4 &= 4m_p^2 c^4 + p^2 c^2 \\
pc &= \frac{1}{2}\sqrt{M_X^2 c^4 - 4m_p^2 c^4} \\
&= \frac{1}{2}\sqrt{(91.2 \times 10^3)^2 - 4(938.3)^2} \\
&= 45.6 \times 10^3 \ MeV \\
&= 45.6 \ GeV \\
p &= 45.6 \ GeV/c
\end{aligned}
$$

(b) Now we consider the case where the proton (the target) is at rest. s^2_{after} is still same as that of the center of mass case. However, s^2_{before} is considerably different. Since now, $E_1 = m_p c^2$, $p_1 = 0$, $p_2 \neq 0$, so $E_2 = \sqrt{m_p^2 c^4 + p_2^2 c^2}$, then s^2_{before}:

$$
\begin{aligned}
s^2_{before} &= (E_1 + E_2)^2 - c^2(p_1 + p_2)^2 \\
&= E_1^2 + E_2^2 + 2E_1 E_2 - p_2^2 c^2 \\
&= m_p^2 c^4 + m_p^2 c^4 + p_2^2 c^2 + 2m_p c^2 \sqrt{m_p^2 c^4 + p_2^2 c^2} - p_2^2 c^2 \\
&= 2m_p^2 c^4 + 2m_p c^2 \sqrt{m_p^2 c^4 + p_2^2 c^2}
\end{aligned}
$$

Since, $s^2_{before} = s^2_{after}$, we get:

$$\begin{aligned}
M_X^2c^4 &= 2m_p^2c^4 + 2m_pc^2\sqrt{m_p^2c^4 + p_2^2c^2} \\
\sqrt{m_p^2c^4 + p_2^2c^2} &= \frac{M_X^2c^4 - 2m_p^2c^4}{2m_pc^2} \\
m_p^2c^4 + p_2^2c^2 &= \left(\frac{M_X^2c^4 - 2m_p^2c^4}{2m_pc^2}\right)^2 \\
p_2^2c^2 &= \left(\frac{M_X^2c^4 - 2m_p^2c^4}{2m_pc^2}\right)^2 - m_p^2c^4 \qquad (10)
\end{aligned}$$

Equation (10) gives the minimum momentum the antiproton should have to create M_X, we should now find the corresponding minimum kinetic energy, for this we use:

$$\begin{aligned}
E &= T + m_pc^2 \\
E^2 &= m_p^2c^4 + p^2c^2 \\
(T + m_pc^2)^2 &= m_p^2c^4 + p^2c^2 \qquad (11)
\end{aligned}$$

Using Equation (10) in Equation 11 we get:

$$\begin{aligned}
(T + m_pc^2)^2 &= m_p^2c^4 + \left(\frac{M_X^2c^4 - 2m_p^2c^4}{2m_pc^2}\right)^2 - m_p^2c^4 \\
&= \left(\frac{M_X^2c^4 - 2m_p^2c^4}{2m_pc^2}\right)^2 \\
T &= \frac{M_X^2c^4 - 2m_p^2c^4}{2m_pc^2} - m_pc^2 \\
&= \frac{(91.2 \times 10^3)^2 - 2(938.3)^2}{2 \times 938.3} - 938.3 \\
&= 4.43 \times 10^6\ MeV \\
&= 4.43\ TeV
\end{aligned}$$

8. Calculate the frequency of the photon emitted by a hydrogen atom making a transition from the $n = 4$ to the $n = 3$ state. Compare your result with the frequency of revolution for the electron in these two Bohr orbits.

Solution

The energy of the photon emitted in a transition in a hydrogen atom is given by:

$$\begin{aligned}\Delta E &= h\nu \\ &= 13.6\left[\frac{1}{n_f^2}-\frac{1}{n_i^2}\right] \\ \nu &= \frac{13.6}{h}\left[\frac{1}{n_f^2}-\frac{1}{n_i^2}\right] \\ &= \frac{13.6\ (eV)}{4.14\times10^{-15}\ (eV\cdot s)}\left[\frac{1}{9}-\frac{1}{16}\right] \\ &= 1.60\times10^{14}\ Hz\end{aligned}$$

The period T_{rev}, and frequency f_{rev} of revolution of an electron rotating in a circular path with a velocity v and radius r is given by:

$$\begin{aligned}T_{rev} &= \frac{2\pi r}{v} \\ f_{rev} &= \frac{1}{T_{rev}} \\ &= \frac{v}{2\pi r}\end{aligned}$$

The velocity of of the electron can be obtained from the centripetal force that is provided by the coulomb force:

$$\begin{aligned}\frac{m_e v^2}{r} &= \frac{ke^2}{r^2} \\ v &= \sqrt{\frac{ke^2}{r m_e}}\end{aligned}$$

For $n = 3$, $r_3 = (3)^2 a_\circ$ where $a_\circ = 5.29\times10^{-11}\ m$ and,

$$\begin{aligned} f_{rev} &= \sqrt{\frac{ke^2}{(2\pi)^2 m_e (9a_\circ)^3}} \\ &= \sqrt{\frac{(8.99\times 10^9)(1.602\times 10^{-19})^2}{(2\pi)^2\times 9.11\times 10^{-31}\times (9\times 5.29\times 10^{-11})^3}} \\ &= 2.44\times 10^{14}\ Hz \end{aligned}$$

repeating for $n = 4$ we get:

$$f_{rev} = 1.03\times 10^{14}\ Hz$$

The photon frequency resulting from a transition from $n = 4$ state to $n = 3$ state is about half way between the revolution frequencies of the two states.

9. Hydrogen, which consists of a single electron bound to a nucleus of charge $Z = 1$, comes in several varieties. One of the varieties is ordinary hydrogen, in which the nucleus is a proton. Two other varieties exist: deuterium, for which the nucleus has $Z = 1$ and a nuclear mass $M \approx 2m_p$, and tritium, for which $Z = 1$ and $M \approx 3m_p$. Calculate the wavelength for the transition $n = 2$ to $n = 1$ in

   (a) deuterium and

   (b) tritium.

   (c) Compare your results with the value for the wavelength for the same transition in hydrogen. A measurement of a shifted spectrum in this transition led to the discovery of deuterium by the American chemist Harold Urey.

Solution

If the motion of the nucleus is considered in Bohr's model then the wavelength emitted in a transition from a state n_i to a state n_f, can be written as:

$$\begin{aligned}
\frac{1}{\lambda} &= \left(\frac{\mu}{m_e}\right) R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \\
&= \mu \left(\frac{R}{m_e}\right) \left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \\
&= 9.031284 \times 10^{36} \mu \\
\lambda &= \frac{1.107262 \times 10^{-37}}{\mu}
\end{aligned}$$

(a) For hydrogen atom the reduced mass is:

$$\begin{aligned}
\mu_{1H} &= \frac{m_e M_{1H}}{m_e + M_{1H}} \\
&= \frac{9.109390 \times 10^{-31} \times 1.673533 \times 10^{-27}}{9.109390 \times 10^{-31} + 1.673533 \times 10^{-27}} \\
&= 9.104434 \times 10^{-31}\ kg
\end{aligned}$$

and the wavelength is:

$$\begin{aligned}
\lambda_{1H} &= \frac{1.107262 \times 10^{-37}}{\mu_{1H}} \\
&= \frac{1.107262 \times 10^{-37}}{9.104434 \times 10^{-31}} \\
&= 1.216179 \times 10^{-7}\ m \\
&= 121.6179\ nm
\end{aligned}$$

(b) Similarly for deuterium ^{2}H, using $M_{2H} = 3.344495 \times 10^{-27}\ kg$, then reduced mass is:

$$\begin{aligned}
\mu_{2H} &= \frac{m_e M_{2H}}{m_e + M_{2H}} \\
&= \frac{9.109390 \times 10^{-31} \times 3.344495 \times 10^{-27}}{9.109390 \times 10^{-31} + 3.344495 \times 10^{-27}} \\
&= 9.106912 \times 10^{-31}\ kg
\end{aligned}$$

and the wave length is

$$\begin{aligned}
\lambda_{2H} &= \frac{1.107262 \times 10^{-37}}{\mu_{2H}} \\
&= \frac{1.107262 \times 10^{-37}}{9.106912 \times 10^{-31}} \\
&= 1.215848 \times 10^{-7}\ m \\
&= 121.5848\ nm
\end{aligned}$$

(c) Similarly for tritium ^{3}H, using $M_{3H} = 5.008234 \times 10^{-27}\ kg$, then reduced mass is:

$$\begin{aligned}
\mu_{3H} &= \frac{m_e M_{3H}}{m_e + M_{3H}} \\
&= \frac{9.109390 \times 10^{-31} \times 5.008234 \times 10^{-27}}{9.109390 \times 10^{-31} + 5.008234 \times 10^{-27}} \\
&= 9.107733 \times 10^{-31}\ kg
\end{aligned}$$

and the wave length is

$$\begin{aligned}
\lambda_{3H} &= \frac{1.107262 \times 10^{-37}}{\mu_{3H}} \\
&= \frac{1.107262 \times 10^{-37}}{9.107733 \times 10^{-31}} \\
&= 1.215739 \times 10^{-7}\ m \\
&= 121.5739\ nm
\end{aligned}$$

10. For what kinetic energy will a particle's de Broglie wavelength equal its Compton wavelength?

Solution

The Compton wavelength λ_c of a particle with a mass m is given by:

$$\lambda_c = \frac{h}{mc}$$

and its de Broglie wavelength λ_{db} is given by:

$$\lambda_{db} = \frac{h}{p}$$

where h is Planck's constant and p is the particles momentum. For $\lambda_c = \lambda_{db}$ we get:

$$\frac{h}{mc} = \frac{h}{p}$$

or

$$p = mc$$

Now, we find the relation between the momentum p and kinetic energy T.

$$\begin{aligned}
E &= T + mc^2 \\
E^2 &= m^2c^4 + p^2c^4 \\
(T + mc^2)^2 &= m^2c^4 + p^2c^2 \\
T^2 + m^2c^4 + 2Tmc^2 &= m^2c^4 + p^2c^2 \\
T^2 + 2Tmc^2 - p^2c^2 &= 0
\end{aligned}$$

since for $\lambda_c = \lambda_{db}$ we must have $p = mc$, the last equation then becomes:

$$\begin{aligned}
T^2 + 2Tmc^2 - m^2c^4 &= 0 \\
T &= -mc^2 \pm \frac{1}{2}\sqrt{4m^2c^4 + 4m^2c^4} \\
&= mc^2(-1 \pm \sqrt{2})
\end{aligned}$$

Since the kinetic energy is a positive quantity, we neglect the overall negative solution, to get

$$T = (\sqrt{2} - 1)mc^2 = 0.414mc^2$$

So for the compton wavelength of a particle to be equal to its de Broglie wavelength, it kinetic enertgy must be 41.4% of its rest mass energy.