Chapter 5

Matter Waves. Home Work Solutions

5.1 Problem 5.10 *(In the text book)*

An electron has a de Broglie wavelength equal to the diameter of the hydrogen atom. What is the kinetic energy of the electron? How does this energy compare with the ground-state energy of the hydrogen atom?

Solution

The radius of the hydrogen atom can be taken as the radius of the first orbit or, Bohr radius $a_\circ = 0.5292 \times 10^{-10}\ m$. The de Broglie wave length of the given electron is:

$$\lambda = 2a_\circ = 1.0584 \times 10^{-10}\ m$$

Using $p = h/\lambda$, where p is the momentum of the electron, the kinetic energy K of the electron is:

$$\begin{aligned} K &= \frac{p^2}{2m_e} \\ &= \frac{h^2c^2}{2m_ec^2\lambda^2} \\ &= \frac{(1.240 \times 10^{-6} eV \cdot m)^2}{2 \times 5.11 \times 10^5 (eV) \times (1.0584 \times 10^{-10}\ m)^2} \\ &= 134.3\ eV \end{aligned}$$

The energy of the ground state of the hydrogen atom is 13.6 eV, so the kinetic energy of the given electron is about times the energy of the ground state of the hydrogen atom.

5.2 Problem 5.14 *(In the text book)*

(a) Show that the formula for low-energy electron diffraction (LEED), when electrons are incident perpendicular to a crystal surface, may be written as

$$\sin\phi = \frac{nhc}{d\sqrt{2m_e c^2 K}}$$

where n is the order of the maximum, d is the atomic spacing, m_e is the electron mass, K is the electron's kinetic energy, and ϕ is the angle between the incident and diffracted beams.

(b) Calculate the atomic spacing in a crystal that has consecutive diffraction maxima at $\phi = 24.1°$ and $\phi = 54.9°$ for 100-eV electrons.

Solution

(a) For a maximum in electron diffraction from crystal planes we have:

$$\begin{aligned} n\lambda &= d\sin\phi \\ \sin\phi &= \frac{n\lambda}{d} \end{aligned}$$

where λ is the de Broglie wave length of the electron, and $\lambda = h/p$, where p is the electrons momentum. The kinetic energy of the electron $K = p^2/2m_e$, we th get:

$$\begin{aligned} \sin\phi &= \frac{n}{d}\frac{h}{p} \\ &= \frac{n}{d}\left(\frac{h}{\sqrt{2Km_e}}\right) \\ &= \left(\frac{nhc}{d\sqrt{2Km_e c^2}}\right) \end{aligned}$$

(b) The atomic spacing is;

$$d = \frac{nhc}{\sin\phi\sqrt{2Km_ec^2}}$$

For $\phi = 24.1^\circ$ the atomic spacing d_1 is:

$$\begin{aligned} d_1 &= \frac{n \times 1.240 \times 10^{-6}(eV \cdot m)}{\sin(24.1) \times \sqrt{2 \times 0.511 \times 10^6(eV) \times 100(eV)}} \\ &= 3.00 \times 10^{-10}\ m \\ &= 3.00n\ \text{Å} \end{aligned}$$

For $\phi = 54.9^\circ$ the atomic spacing d_1 is:

$$\begin{aligned} d_2 &= \frac{(n+1) \times 1.240 \times 10^{-6}(eV \cdot m)}{\sin(54.9) \times \sqrt{2 \times 0.511 \times 10^6(eV) \times 100(eV)}} \\ &= 3.00(n+1) \times 10^{-10}\ m \\ &= 3.00(n+1)\ \text{Å} \end{aligned}$$

for the same spacing we should have:

$$\begin{aligned} d_1 &= d_2 \\ 3.00n &= 1.50(n+1) \\ n &= 1 \end{aligned}$$

which means the 24.1° corresponds to $n = 1$ maximum and 54.9° corresponds to $n = 2$ maximum.

5.3 Problem 5.20 *(In the text book)*

We wish to measure simultaneously the wavelength and position of a photon. Assume that the wavelength measurement gives $\lambda = 6000\ \mathring{A}$ with an accuracy of one part in a million, that is, $\Delta\lambda/\lambda = 10^{-6}$. What is the minimum uncertainty in the position of the photon?

Solution

The momentum p of the photon is given by:

$$p = \frac{h}{\lambda}$$

we then get:

$$\begin{aligned}
\frac{\Delta p}{\Delta\lambda} &\approx \frac{dp}{d\lambda} \\
&= -\frac{h}{\lambda^2} \\
\Delta p &= -\frac{h\Delta\lambda}{\lambda^2}
\end{aligned}$$

Since,

$$\begin{aligned}
\Delta p\Delta x &\geq \frac{1}{2}\hbar \\
(\Delta p\Delta x)_{min} &= \frac{1}{2}\hbar \\
|\Delta x|_{min} &= \frac{\hbar}{2\Delta p} \\
&= \frac{\hbar}{2(h\delta\lambda/\lambda^2)} \\
&= \frac{h\lambda}{4\pi(h\Delta\lambda/\lambda)} \\
&= \frac{\lambda}{4\pi(\Delta\lambda/\lambda)} \\
&= \frac{6000\ \mathring{A}}{4' \times 10^{-6}} \\
&= 4.78 \times 10^{8}\ \mathring{A} \\
&= 4.78 \times 10^{-2}\ m
\end{aligned}$$

5.4 Problem 5.30 *(In the text book)*

Robert Hofistadter won the 1961 Nobel prize in physics for his pioneering work in scattering 20-GeV electrons from nuclei.

(a) What is the γ factor for a 20-GeV electron, where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$? What is the momentum of the electron in $kg \cdot m/s$?

(b) What is the wavelength of a 20-GeV electron and how does it compare with the size of a nucleus?

Solution

(a) Using the relativistic definition of the total energy we get:

$$\begin{aligned} E &= \gamma m_e c^2 \\ \gamma &= \frac{E}{m_e c^2} \\ &= \frac{20 \times 10^3 (MeV)}{0.511 (MeV)} \\ &= 3.91 \times 10^4 \end{aligned}$$

Since the energy of the electron is very high, then, $m_e c^2 \ll pc$, and using the relativistic definition of the total energy, in terms of the momentum, we get:

$$\begin{aligned} E^2 &= (pc)^2 + m_e^2 c^4 \\ p &\approx \frac{E}{c} \\ &= \frac{20 \times 10^3 (MeV) \times 1.602 \times 10^{-13} (J/MeV)}{3 \times 10^8 m/s} \\ &= 1.07 \times 10^{-17} \ kg \cdot m/s \end{aligned}$$

(b) The de Broglie wavelength of the 20 GeV electron is:

$$
\begin{aligned}
\lambda &= \frac{h}{p} \\
&= \frac{6.626 \times 10^{-34} (J \cdot s)}{1.07^{-17}\ kg \cdot m/s} \\
&= 6.19 \times 10^{-17}\ m
\end{aligned}
$$

the size of the nucleus is about $10^{-14}\ m$, so the wavelength of the 20 GeV electron is about 1000 times smaller than the nucleus.

5.5 Problem 5.34 *(In the text book)*

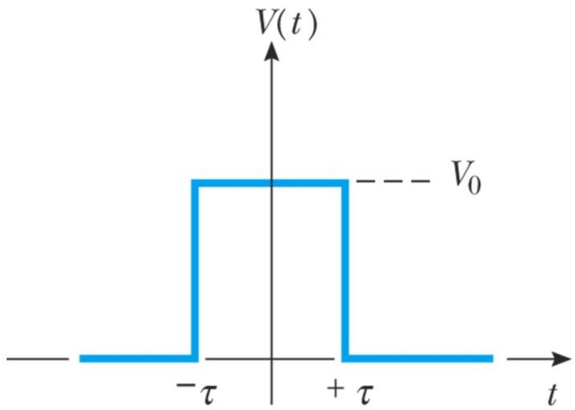

Figure 5.1:

(a) Find and sketch the spectral content of the rectangular pulse of width 2τ- shown in Figure (5.1).

(b) Show that a reciprocity relation $\Delta\omega\Delta t \approx \pi$ holds in this case. Take $\Delta t = \tau$ and define $\Delta\omega$ similarly.

(c) What range of frequencies is required to compose a pulse of width $2\tau = 1\ \mu s$? A pulse of width $2\tau = 1\ ns$?

Solution

(a) Using Fourier integrals:

$$
\begin{aligned}
g(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(t)\, e^{-i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(t)\, (\cos\omega t - i\sin\omega t)\, dt
\end{aligned}
$$

$V(t)\sin\omega t$ is an odd function in t so its integral vanishes, we then get:

$$
g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} V(t)\ \cos\omega t\ dt
$$

from Figure (5.1), $V(t) = V_\circ$ when $-\tau < t < \tau$ and $V(t) = 0$ everywhere else. Also, $V(t)$ is symmetric around $t = 0$:

$$
\begin{aligned}
g(\omega) &= \frac{2}{\sqrt{2\pi}} \int_0^\tau V_\circ \, \cos\omega t \, dt \\
&= \sqrt{\frac{2}{\pi}} V_\circ \left[\frac{\sin\omega t}{\omega}\right]_0^\tau \\
&= \sqrt{\frac{2}{\pi}} V_\circ \, \frac{\sin\omega\tau}{\omega}
\end{aligned}
$$

A plot of $g(\omega)$ is shown in Figure (5.2).

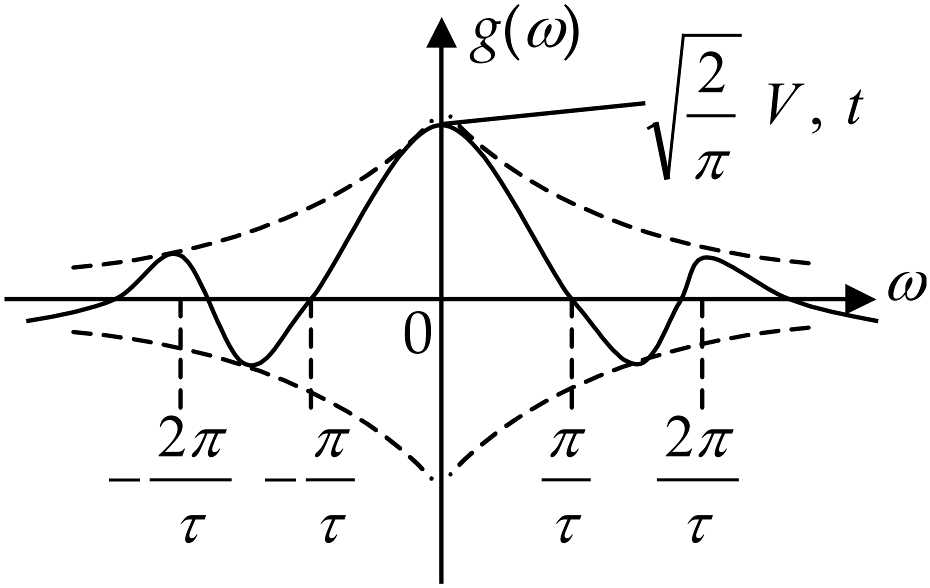

Figure 5.2:

(b) Since the major contribution to the pulse comes from $-\pi/\tau < t < +\pi/\tau$, we take $\Delta\omega$ as π/τ and since $\Delta t = \tau$, we then get:

$$\Delta\omega\Delta t = \frac{\pi}{\tau}\tau = \pi$$

(c) The frequency range required to compose a pulse of width $2\tau = 1\ \mu s$, $2\Delta f$ is:

$$
\begin{aligned}
\omega &= 2\pi f \\
\Delta\omega &= 2\pi\Delta f \\
\Delta\omega\Delta t &= \pi \\
2\Delta f &= \frac{\Delta\omega}{\pi} \\
&= \frac{\pi}{\pi\Delta t} \\
&= \frac{1}{\Delta t} \\
&= \frac{1}{0.5 \times 10^{-6}} \\
&= 2 \times 10^{6}\ Hz
\end{aligned}
$$

and for a 1 ns pulse:

$$
\begin{aligned}
2\Delta f &= \frac{1}{\Delta t} \\
&= \frac{1}{0.5 \times 10^{-9}} \\
&= 2 \times 10^{9}\ Hz
\end{aligned}
$$

Notice that to make the pulse sorter, i.e. localized in a smaller time range you need a much larger frequency range.