Chapter 2

Relativity II. Home Work Solutions

2.1 Problem 2.4 (In the text book)

A charged particle moves along a straight line in a uniform electric field $E$ with a speed $v$. If the motion and the electric field are both in the $x$ direction,

(a) show that the magnitude of the acceleration of the charge $q$ is given by

$$a = \frac{dv}{dt} = \frac{qE}{m} \left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

(b) Discuss the significance of the dependence of the acceleration on the speed.

(c) If the particle starts from rest at $x = 0$ at $t = 0$, find the speed of the particle and its position after a time $t$ has elapsed. Comment on the limiting values of $v$ and $x$ as $t \to \infty$

Solution

(a) The electric force on the charged particle is

$$F = \frac{qE}{m}$$

$$= \frac{dp}{dt}$$

$$= \frac{d}{dt} (\gamma mv)$$

$$= \frac{d}{dt} \left(\frac{mv}{\sqrt{1 - v^2/c^2}}\right)$$
\[ \begin{align*}
&= m \left[ \frac{1}{\sqrt{1 - v^2/c^2}} - \frac{1}{2} \frac{-2v^2/c^2}{(1-v^2/c^2)^{3/2}} \right] \frac{dv}{dt} \\
&= m \left[ \frac{1 - v^2/c^2}{(1-v^2/c^2)^{3/2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}} \right] \frac{dv}{dt} \\
&= \frac{m \, dv}{(1-v^2/c^2)^{3/2} \, dt}
\end{align*} \]

since \(dv/dt\) is the acceleration \(a\) we get:

\[ qE = \frac{ma}{(1-v^2/c^2)^{3/2}} \]

\[ a = \frac{qE}{m} \left( 1 - \frac{v^2}{c^2} \right)^{3/2} \]

(b) The larger the speed the smaller the acceleration, and eventually when \(v \rightarrow c\), \(a \rightarrow 0\) and this makes sense since it is not possible to accelerate an object to speeds beyond \(c\). When \(v \ll c\) then \(a \approx qE/m\) which is the classical expression for the acceleration.

(c) Using the expression for the acceleration from section (a), and \(a = dv/dt\), we get:

\[ \frac{dv}{dt} = \frac{qE}{m} \left( 1 - \frac{v^2}{c^2} \right)^{3/2} \]

\[ dv = \frac{qE}{m} \left( 1 - \frac{v^2}{c^2} \right)^{3/2} dt \]

\[ \int_0^v \frac{d(v/c)}{(1-v^2/c^2)^{3/2}} = \int_0^t \frac{qE}{mc} dt \]

\[ \left[ \frac{(v/c)}{\sqrt{1-v^2/c^2}} \right]_0^v = \left[ \frac{qEt}{mc} \right]_0^t \]

\[ \frac{v/c}{\sqrt{1-v^2/c^2}} = \frac{qEt}{mc} \]

\[ \frac{v^2/c^2}{1-v^2/c^2} = \left( \frac{qEt}{mc} \right)^2 \]

\[ \frac{v^2}{c^2} = \left( 1 - \frac{v^2}{c^2} \right) \left( \frac{qEt}{mc} \right)^2 \]
\[
\frac{v^2}{c^2} \left[ 1 + \left( \frac{qEt}{mc} \right)^2 \right] = \left( \frac{qEt}{mc} \right)^2
\]

\[
\frac{v^2}{c^2} = \frac{(qEt/mc)^2}{1 + (qEt/mc)^2}
\]

\[
= \frac{1}{1 + (mc/qEt)^2}
\]

From the last equation we can see that when \( t \to 0 \), \( v \to 0 \) and when \( t \to \infty \), \( v \to c \).

using \( v = dx/dt \) we get:

\[
\frac{dx}{dt} = v
\]

\[
= c \sqrt{\frac{(qEt/mc)^2}{1 + (qEt/mc)^2}}
\]

\[
= \frac{qEt}{\sqrt{(mc)^2 + (qEt)^2}}
\]

\[
dx = \frac{qEt \ dt}{\sqrt{(mc)^2 + (qEt)^2}}
\]

\[
\int_0^x \ dx = \frac{c}{qE} \int_0^t \frac{(qEt) \ d(qEt)}{\sqrt{(mc)^2 + (qEt)^2}}
\]

\[
[x]_0^x = \frac{c}{qE} \left[ \frac{\sqrt{(mc)^2 + (qEt)^2}}{l} \right]_0^t
\]

\[
x = \frac{c}{qE} \left[ \frac{\sqrt{(mc)^2 + (qEt)^2}}{-mc} \right]
\]

The last equation shows that as \( t \to 0 \), \( x \to 0 \) and as \( t \to \infty \), \( x \to \infty \).
2.2 Problem 2.14 (In the text book)

*How long will the Sun shine, Nellie?* The Sun radiates about $4.0 \times 10^{26} \text{ J}$ of energy into space each second.

(a) How much mass is released as radiation each second?

(b) If the mass of the Sun is $2.0 \times 10^{30} \text{ kg}$, how long can the Sun survive if the energy release continues at the present rate?

**Solution**

(a) Since $E = mc^2$, the radiation released from the was produced by changing part of the sun’s mass into energy, the equivalent mass released every second is then:

$$m = \frac{E}{c^2} = \frac{4.0 \times 10^{26}}{9 \times 10^{16}} = 4.4 \times 10^9 \text{ kg}$$

(b) The life time of the sun if it keeps radiating at the current rate is:

$$t = \frac{2.0 \times 10^{30}}{4.4 \times 10^9} = 4.5 \times 10^2 \text{ s} = \frac{4.5 \times 10^{20}}{60 \times 60 \times 24 \times 365} \text{ years} = 1.4 \times 10^{13} \text{ years}$$
2.3 Problem 2.18 (In the text book)

Consider the decay $^{24}_{55}$Cr $\rightarrow$ $^{25}_{55}$Mn + $e^-$, where $e^-$ is an electron. The $^{55}$Cr nucleus has a mass of 54.9279 u, and the $^{55}$Mn nucleus has a mass of 54.9244 u.

(a) Calculate the mass difference in MeV.

(b) What is the maximum kinetic energy of the emitted electron?

Solution

(a) The mass difference $\Delta m$ in atomic mass units u is:

$$\Delta m = m_{^{55}Cr} - m_{^{55}Mn}$$
$$= 54.9279 - 54.9244$$
$$= 3.5 \times 10^{-3} \text{ u}$$

Given that $1u = 931.5 \text{ MeV/c}^2$, then the mass difference in MeV is:

$$\Delta m = 3.5 \times 10^{-3} \times 931.5$$
$$= 3.26 \text{ MeV/c}^2$$

and the equivalent energy $\Delta E$ is:

$$\Delta E = 3.26 \text{ MeV}$$

(b) The electron needs to be created, i.e. its rest mass energy has to come from somewhere. $\Delta E$ the total amount of energy available to create the electron and as kinetic energy for the electron and $^{55}$Mn. The maximum kinetic energy $KE_{e-\text{max}}$ the electron can get is when the nucleus of $^{55}$Mn is created at rest, i.e.

$$KE_{e-\text{max}} = \Delta E - m_e c^2$$
$$= 3.26 - 0.511$$
$$= 2.75 \text{ MeV}$$
2.4 Problem 2.22 *(In the text book)*

The $K^0$ meson is an uncharged member of the particle “zoo” that decays into two charged pions according to $K^0 \rightarrow \pi^+ + \pi^-$. The pions have opposite charges, as indicated, and the same mass, $m = 140 \text{ MeV}/c^2$. Suppose that a $K^0$ at rest decays into two pions in a bubble chamber in which a magnetic field of 2.0 T is present (see Figure (2.1)). If the radius of curvature of the pions is 34.4 cm, find

(a) the momenta and speeds of the pions and

(b) the mass of the $K^0$ meson.

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**Solution**

(a) The momentum of each pion as it moved in the magnetic field is:

$$p_\pi = q_\pi BR_\pi$$

where $q_\pi$ is the charge of the pion $q_\pi = 1.6 \times 10^{-19} \text{ C}$, $B$ is the magnetic field and $R_\pi$ is the radius of the pion’s path. Using the numbers we get:

$$p_\pi = 1.6 \times 10^{-19} \times 2.0 \times 34.4 \times 10^{-2} = 1.1 \times 10^{-19} \text{ kg} \cdot \text{m/s}$$
we need to change the units from $\text{kg} \cdot \text{m/s}$ to the more relevant units to nuclear reactions $\text{MeV}/c$. This is done as follows:

\[
\frac{\text{kg} \cdot \text{m}}{\text{s}} = \frac{\text{kg} \cdot \text{c} \cdot v}{c} = 3 \times 10^8 \frac{\text{kg} \cdot \text{m}^2}{\text{c} \cdot \text{s}^2} = 3 \times 10^8 \frac{\text{J}}{c} = 3 \times 10^8 \frac{\text{eV}}{1.6 \times 10^{-19} \frac{\text{c}}{\text{MeV}}} = 3 \times 10^8 \times 10^{-6} \text{MeV} = 1.875 \times 10^{21} \frac{\text{MeV}}{c}
\]

The pion momentum is, then:

\[
p_\pi = 1.1 \times 10^{-19} \times 1.875 \times 10^{21} = 206 \text{ MeV}/c
\]

The $K^\circ$ meson was at rest before the decay, so conservation of momentum requires that the total momentum after decay be zero. In other words the pions should have equal and opposite momenta. Since the two pions have equal masses then they should have equal and opposite velocities. To find the velocity $u_\pi$ of the pions we use $p_\pi = \gamma m_\pi u_\pi$ and $E_\pi = \gamma m_\pi c^2$, where $p$ and $E$ are the momentum and total energy of each pion,

\[
\frac{p_\pi}{E_\pi} = \frac{\gamma m_\pi u_\pi}{\gamma m_\pi c^2} = \frac{u_\pi}{c^2}
\]

\[
\frac{u_\pi}{c} = \frac{p_\pi c}{E_\pi} = \frac{p_\pi c}{\sqrt{p_\pi^2 c^2 + (m_\pi c^2)^2}}
\]

\[
= \frac{206}{\sqrt{(206)^2 + (140)^2}} = 0.827
\]
(b) Conservation mass energy requires:

\[ E_K = 2E_\pi \]

where \( E_K \) is the total relativistic energy of the the \( K^\circ \) meson and \( E_\pi \) is the total relativistic energy of one of the pions. Since \( K^\circ \) decayed while at rest we get:

\[
m_{K^\circ}c^2 = 2\sqrt{p^2c^2 + (m_\pi c^2)^2} = 2\sqrt{(206)^2 + (140)^2} = 498 \text{ MeV} \]

\[
m_{K^\circ} = 498 \text{ MeV}/c^2
\]
2.5 Problem 2.29 (*In the text book*)

An object disintegrates into two fragments. One of the fragments has mass 1.00 MeV/c\(^2\) and momentum 1.75 MeV/c in the positive \(x\) direction. The other fragment has mass 1.50 MeV/c\(^2\) and momentum 2.005 MeV/c in the positive \(y\) direction. Find

(a) the mass and
(b) the speed of the original object.

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**Solution**

The total energy of the first fragment is:

\[
E_1 = \sqrt{p_{1x}^2 c^2 + (m_1 c^2)^2} = \sqrt{(1.75)^2 + (1.00)^2} = 2.02 \text{ MeV}
\]

and for the second fragment we have:

\[
E_2 = \sqrt{p_{2y}^2 c^2 + (m_2 c^2)^2} = \sqrt{(2.005)^2 + (1.50)^2} = 2.50 \text{ MeV}
\]

(a) Mass-energy conservation requires that the total energy of the object before disintegration \(E_o = E_1 + E_2 = 4.52 \text{ MeV}\). The momentum after disintegration \(p\) is given by:

\[
p^2 = p_{1x}^2 + p_{2y}^2 = (1.75)^2 + (2.00)^2 = 7.06 \text{ (MeV/c)}^2
\]

The original object must also have the is momentum, so:
\[ E_o^2 = p^2 c^2 + (m_o c^2)^2 \]
\[ m_o c^2 = \sqrt{E_o^2 - p^2 c^2} \]
\[ = \sqrt{(4.52)^2 - 7.06} \]
\[ = 3.66 \text{ MeV} \]
\[ m_o = 3.66 \text{ MeV/c}^2 \]

(b) for the original object we have:

\[ E_o = \gamma m_o c^2 \]
\[ \frac{1}{\gamma} = \frac{m_o c^2}{E_o} \]
\[ \sqrt{1 - \frac{u^2}{c^2}} = \frac{m_o c^2}{E_o} \]
\[ \frac{u^2}{c^2} = 1 - \left( \frac{m_o c^2}{E_o} \right)^2 \]
\[ \frac{u}{c} = \sqrt{1 - \left( \frac{m_o c^2}{E_o} \right)^2} \]
\[ = 0.587 \]