1. Consider a particle in the ground state of a box of length a.

   (a) Find the probability density $|\psi|^2$.

   (b) Where is the particle most likely to be found?

   (c) What is the probability of finding the particle in the interval between $x = 0.50a$ and $x = 0.51a$?

   (d) What is it for the interval $[0.75a, 0.76a]$?

   (e) What would be the average result if the position of a particle in the ground state were measured many times?

Solution

(a) The ground state wavefunction ($n = 1$) of a particle in a box of length a is:

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

and the probability density is:

$$|\psi(x)|^2 = \frac{2}{a} \sin^2\left(\frac{\pi x}{a}\right)$$

(b) The most likely position of the particle is that at which $|\psi(x)|^2 =$ maximum. This will happen when $\sin^2(\pi x/a) = 1$ or when $(\pi x/a) = \pi/2$ or when $x = a/2$.

(c) The probability P_1 of finding the particle between $x = 0.50a$ and $x = 0.51a$ is:

$$\begin{aligned}
P_1 &= \int_{0.50a}^{0.51a} |\psi(x)|^2 \, dx \\
&= \int_{0.50a}^{0.51a} \frac{2}{a} \sin^2\left(\frac{\pi x}{a}\right) \, dx \\
&= \frac{2}{a}\left[\frac{x}{2} - \frac{\sin(2\pi x/a)}{4\pi/a}\right]_{x=0.50a}^{x=0.51a} \\
&= \left[\frac{x}{a} - \frac{\sin(2\pi x/a)}{2\pi}\right]_{x=0.50a}^{x=0.51a} \\
&= \left[\frac{0.51a}{a} - \frac{\sin(2\pi 0.51a/a)}{2\pi}\right] - \left[\frac{0.50a}{a} - \frac{\sin(2\pi 0.50a/a)}{2\pi}\right]
\end{aligned}$$

$$
\begin{aligned}
P_1 &= \left[0.51 - \frac{\sin(2\pi 0.51)}{2\pi}\right] - \left[0.50 - \frac{\sin(\pi)}{2\pi}\right] \\
&= 0.01 - \frac{1}{2\pi}\sin(2\pi \times 0.51) \\
&= 0.02 \\
&= 2.0\%
\end{aligned}
$$

(d) The probability P_2 of finding the particle between $x = 0.75a$ and $x = 0.76a$ is:

$$
\begin{aligned}
P_2 &= \int_{0.75a}^{0.76a} |\psi(x)|^2 \, dx \\
&= \int_{0.75a}^{0.76a} \frac{2}{a}\sin^2\left(\frac{\pi x}{a}\right) \, dx \\
&= \frac{2}{a}\left[\frac{x}{2} - \frac{\sin(2\pi x/a)}{4\pi/a}\right]_{x=0.75a}^{x=0.76a} \\
&= \left[\frac{x}{a} - \frac{\sin(2\pi x/a)}{2\pi}\right]_{x=0.750a}^{x=0.76a} \\
&= \left[\frac{0.76a}{a} - \frac{\sin(2\pi 0.76a/a)}{2\pi}\right] - \left[\frac{0.75a}{a} - \frac{\sin(2\pi 0.75a/a)}{2\pi}\right] \\
&= \left[0.76 - \frac{\sin(2\pi 0.76)}{2\pi}\right] - \left[0.75 - \frac{\sin(1.5\pi)}{2\pi}\right] \\
&= \left[0.76 - 0.75 - \frac{\sin(2\pi 0.76)}{2\pi} + \frac{\sin(1.5\pi)}{2\pi}\right] \\
&= [0.01 + 0.159 - 0.159] \\
&= 0.01 \\
&= 1\%
\end{aligned}
$$

(e) The average position $< x >$ of the particle within the box is:

$$
\begin{aligned}
< x > &= \int_0^a x|\psi(x)|^2 \, dx = \frac{2}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\right) \, dx \\
&= \frac{2}{a}\left[\frac{x^2}{4} - \frac{x\sin\left(\frac{2\pi x}{a}\right)}{4\left(\frac{\pi}{a}\right)} - \frac{\cos\left(\frac{2\pi x}{a}\right)}{8\left(\frac{\pi}{a}\right)^2}\right]_0^a \\
&= \frac{2}{a}\left[\frac{a^2}{4} - \frac{a\sin\left(\frac{2\pi a}{a}\right)}{4\left(\frac{\pi}{a}\right)} - \frac{\cos\left(\frac{2\pi a}{a}\right)}{8\left(\frac{\pi}{a}\right)^2} - \frac{0^2}{4} + \frac{0\sin\left(\frac{2\pi 0}{a}\right)}{4\left(\frac{\pi}{a}\right)} + \frac{\cos\left(\frac{2\pi 0}{a}\right)}{8\left(\frac{\pi}{a}\right)^2}\right] \\
&= \frac{2}{a}\left[\frac{a^2}{4} - 0 - \frac{1}{8\left(\frac{\pi}{a}\right)^2} - 0 + \frac{1}{8\left(\frac{\pi}{a}\right)^2}\right] = \frac{1}{2}a
\end{aligned}
$$

2. Positronium is a hydrogen-like atom consisting of a positron (a positively charged electron) and an electron revolving around each other. Using the Bohr model, find the allowed radii (relative to the center of mass of the two particles) and the allowed energies of the system. Use the reduced mass of the system.

Solution

The reduced mass μ of the electron-positron system is:

$$\mu = \frac{m_e m_{pos}}{m_e + m_{pos}}$$

Since $m_e = m_{pos} = 511\ keV$, then μ becomes, $\mu = \frac{1}{2}m_e$. According to Bohr's theory the allowed atomic radii are given by:

$$r_n = \frac{n^2\hbar^2}{\mu Z k e^2} \qquad n = 1,\ 2,\ 3,\ \cdots$$

where Ze is the positive charge of the atom, and for positronium $Z = 1$ and $\mu = \frac{1}{2}m_2$, so the atomic radii of the positronium becomes:

$$\begin{aligned} r_{n-pos} &= \frac{2n^2\hbar^2}{m_e k e^2} \\ &= 2n^2 a_\circ \\ &= 2r_{n-hyd} \end{aligned}$$

where $a_\circ$ is the Bohr radius $a_\circ = \hbar^2/m_e k e^2$.

The allowed energies of the electron in a positronium are given by:

$$\begin{aligned} E_{n-pos} &= -\frac{ke^2}{2r_{n-pos}} \\ &= -\frac{1}{2}\frac{ke^2}{2r_{n-hyd}} \\ &= -\frac{1}{2}E_{hyd} \\ &= -\frac{1}{2}13.6n^2 = -\frac{6.80}{n^2}\ eV \end{aligned}$$

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3. An atom in an excited state 1.8 eV above the ground state remains in that excited state 2.0 μs before moving to the ground state. Find

   (a) the frequency of the emitted photon,

   (b) its wave length, and

   (c) its approximate uncertainty in its energy.

Solution

(a) The frequency of the emitted photon is:

$$f = \frac{E}{h} = \frac{1.8 \times 1.602 \times 10^{-19}}{6.626 \times 10^{-34}} = 4.352 \times 10^{14}\ Hz$$

(b) The wave length of the photon is:

$$\lambda = \frac{c}{f} = \frac{3 \times 10^{8}}{4.352 \times 10^{14}} = 6.893 \times 10^{-7}\ m = 689.3\ nm$$

(c) The time the electron spends in the excited state is the time available to measure the energy of the excited state, which is then the uncertainty Δt in time. Using Heisenberg's uncertainty principle, the uncertainty in the energy of the excited state ΔE is:

$$\Delta E \geq \frac{\hbar}{2\Delta t} = \frac{6.582 \times 10^{-16}(eV \cdot s)}{2 \times 2.0 \times 10^{-6} s} = 1.646 \times 10^{-10}\ eV$$

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4. In a Compton scattering event, the scattered photon has an energy of 120 keV and the recoiling electron has an energy of 40 keV. Find

   (a) The wave length of the incident photon,

   (b) The angle θ at which the photon is scattered, and

   (c) The recoil angle ϕ of the electron.

Solution

(a) Conservation of energy gives:

$$E_\circ = E' + K_e = 120 + 40 = 160\ keV$$

where $E_\circ$ is the energy of the incident photon, E' is the energy of the scattered photon, and K_e is the energy of the recoiling electron.

The wavelength $\lambda_\circ$ of the incident photon is given by:

$$\lambda_\circ = \frac{hc}{E_\circ} = \frac{1.240 \times 10^3}{160 \times 10^3} = 7.750 \times 10^{-3}\ nm$$

(b) The angle θ at which the photon is scattered can be calculated from the Compton scattering formula:

$$\lambda' - \lambda_\circ = \lambda_c(1 - \cos\theta)$$

where λ' is the wavelength of the scattered photon and $\lambda_c = 2.43 \times 10^{-3}\ nm$ is the Compton wavelength. The wavelength of the scattered photon can be obtained from:

$$\lambda' = \frac{hc}{E'} = \frac{1.24\ (keV \cdot nm)}{120(keV)} = 1.03 \times 10^{-2}\ nm$$

Using the Compton formula we get:

$$\begin{aligned} \lambda' - \lambda_\circ &= \lambda_c(1 - \cos\theta) \\ \cos\theta &= 1 - \frac{\lambda' - \lambda_\circ}{\lambda_c} \end{aligned}$$

Substituting the numerical values we get:

$$\begin{aligned}\cos\theta &= 1 - \frac{1.03\times 10^{-2} - 7.75\times 10^{-3}}{2.43\times 10^{-3}} \\ &= -4.94\times 10^{-2} \\ \theta &= 92.8^\circ\end{aligned}$$

(c) Conservation of momentum gives:

$$\begin{aligned}p &= p'\cos\theta + p_e\cos\phi \\ 0 &= p'\sin\theta - p_e\sin\phi \\ p_e\cos\phi &= p - p'\cos\theta \\ p_e\sin\phi &= p'\cos\theta\end{aligned}$$

where $p = h/\lambda_\circ$ and $p' = h/\lambda'$ are the momenta of the incident and scattered photon respectively, p_e is the momentum of the recoiling electron, and ϕ is the angle at which the electron recoils. Dividing the last two equations we get:

$$\begin{aligned}\tan\phi &= \frac{p'\sin\theta}{p - p'\cos\theta} \\ &= \frac{\sin\theta}{(p/p') - \cos\theta} \\ &= \frac{\sin\theta}{(\lambda'/\lambda_\circ) - \cos\theta} \\ &= \frac{\sin 92.8}{(1.03\times 10^{-2}/7.75\times 10^{3}) - \cos 92.8} \\ &= 0.725 \\ \phi &= 35.9^\circ\end{aligned}$$

5. Which of the following functions are eigenfunctions of the momentum operator $[p]$? For those that are eigenfunctions, what are the eigenvalues?

   (a) $A\sin(kx)$

   (b) $A\sin(kx) - A\cos(kx)$

   (c) $A\cos(kx) + iA\sin(kx)$

   (d) $Ae^{ik(x-a)}$

Solution

Applying the momentum operator $[p_x] = -i\hbar(d/dx)$ on each wavefunction we get:

(a) $$[p_x]\left\{A\sin(kx)\right\} = -i\hbar\frac{d}{dx}A\sin(kx) = -i\hbar k\left\{A\cos(kx)\right\}$$

(b) $$[p_x]\left\{A\sin(kx) - A\cos(kx)\right\} = -i\hbar k\left\{A\cos(kx) + A\sin(kx)\right\}$$

(c) $$[p_x]\left\{A\cos(kx) + iA\sin(kx)\right\} = -i\hbar k\left\{-A\sin(kx) + iA\cos(kx)\right\} = \hbar k\left\{A\cos(kx) + iA\sin(kx)\right\}$$

(d) $$[p_x]\left\{e^{ik(x-a)}\right\} = -i\hbar(ik)\left\{e^{ik(x-a)}\right\} = \hbar k\left\{e^{ik(x-a)}\right\}$$

only the functions in (c) and (d) produces a constant multiplied by the function after operated on by the momentum operator. So only these two function are eigenfunction of the momentum operator $[p_x]$. The eigenvalue in both cases is $\hbar k$.

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6. Suppose a particle of mass m is free within the region $0 < x < L$, but can not go beyond that region because of high potential walls. Suppose, moreover, that the particle is in the ground state of this one dimensional box, so that its wave function is given by:

$$u_1(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}$$

Calculate $< x >, < p >$, and $< p^2 >$.

Solution

$< x >$ is the same as in Problem 1 where $< x > = L/2$, and $< p >$ is :

$$\begin{aligned}
< p > &= \int_0^L u_1^*(x)[p]u_1 \, dx \\
&= \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \left(-i\hbar \frac{d}{dx}\right) \sin\left(\frac{\pi x}{L}\right) \, dx \\
&= \frac{2}{L} \left(-i\hbar \frac{\pi}{L}\right) \int_0^L \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) \, dx \\
&= \frac{2}{L} \left(-i\hbar \frac{\pi}{L}\right) \int_0^L \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) \, dx \\
&= -i\hbar \frac{\pi}{L^2} \frac{L}{2\pi} \left[-\cos\left(\frac{\pi x}{L}\right)\right]_0^L \\
&= -i\hbar \frac{\pi}{L^2} \frac{L}{2\pi} [1 - 1] = 0
\end{aligned}$$

Similarly:

$$\begin{aligned}
< p^2 > &= \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \left(-i\hbar \frac{d}{dx}\right)^2 \sin\left(\frac{\pi x}{L}\right) \, dx \\
&= \frac{2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \left(-\hbar^2 \frac{d^2}{dx^2}\right) \sin\left(\frac{\pi x}{L}\right) \, dx \\
&= \frac{2\hbar^2 \pi^2}{L^3} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) \, dx \\
&= \frac{2\hbar^2 \pi^2}{L^3} \left[\frac{x}{2} - \frac{\sin\left(\frac{2\pi x}{L}\right)}{4\left(\frac{2\pi^2}{L}\right)}\right]_0^L = \frac{2\hbar^2 \pi^2}{L^3} \frac{L}{2} = \frac{\pi^2 \hbar^2}{L^2}
\end{aligned}$$

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7. Consider a free particle inside a box with length L_1, L_2, and L_3 along the x, y, and z axes respectively. The particle is considered to be inside the box.
   (a) Find the wave function and energies. Then find the ground state and first excited state energies and wave functions.
   (b) Use your results to predict the same quantities for a cube with sides L.

Solution

Inside the box $U(x) = 0$, so Schrödinger equation becomes:

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x, y, z) = E\psi(x, y, z)$$

The wave function$\psi(x, y, z)$ must continuous, single valued, and finite everywhere. It should also be zero at the walls of the box since it is confined inside. The wave function can be separable, i.e.

$$\psi(x, y, z) = \psi_1(x)\psi_2(y)\psi_3(z)$$

and Schrödinger equation can be split into three equations:

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} &= E_1\psi_1(x) \\ -\frac{\hbar^2}{2m}\frac{d^2\psi_2(y)}{dy^2} &= E_2\psi_2(y) \\ -\frac{\hbar^2}{2m}\frac{d^2\psi_3(z)}{dz^2} &= E_3\psi_3(z) \end{aligned}$$

Possible solutions for these equations are:

$$\begin{aligned} \psi_1(x) &= \sin(k_1 x) \\ \psi_2(y) &= \sin(k_2 y) \\ \psi_3(z) &= \sin(k_3 z) \end{aligned}$$

and then,

$$\psi(x, y, z) = A\sin(k_1 x)\sin(k_2 y)\sin(k_3 z)$$

where A is a normalization constant and $k_1^2 = 2mE_1/\hbar^2$, $k_2^2 = 2mE_2/\hbar^2$, $k_3^2 = 2mE_3/\hbar^2$, and $E = E_1 + E_2 + E_3$. Applying the boundary condition at $\psi_1(L_1) = 0$, $\psi_2(L_2) = 0$, and $\psi_3(L_3) = 0$ we get:

$$k_1 = \frac{n_1\pi}{L_1} \qquad k_2 = \frac{n_2\pi}{L_2} \qquad k_3 = \frac{n_2\pi}{L_3}$$

where $n_1,\ n_2$, and $n_3 = 1,\ 2,\ 3,\ \cdots$.

(a) The wave function becomes:

$$\psi(x,y,z) = A\sin\left(\frac{n_1\pi x}{L_1}\right)\sin\left(\frac{n_2\pi y}{L_2}\right)\sin\left(\frac{n_3\pi z}{L_3}\right)$$

and the energy of the system is then:

$$E = E_1 + E_2 + E_3 = \frac{\hbar^2 k_1^2}{2m} + \frac{\hbar^2 k_2^2}{2m} + \frac{\hbar^2 k_3^2}{2m} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2}\right)$$

The ground state has $n_1 = n_2 = n_3 = 1$, the ground state wavefunction $\psi_{gs}(x,y,z)$ is then:

$$\psi_{gs}(x,y,z) = A\sin\left(\frac{\pi x}{L_1}\right)\sin\left(\frac{\pi y}{L_2}\right)\sin\left(\frac{\pi z}{L_3}\right)$$

and the ground state energy E_{gs} is:

$$E_{gs} = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{L_1^2} + \frac{1}{L_2^2} + \frac{1}{L_3^2}\right)$$

The first excited state has one of $n's$ equals 2 and the other two equals 1, so the first excited state will have three energies namely:

$$\begin{aligned}
E_{11} &= \frac{\pi^2\hbar^2}{2m}\left(\frac{2}{L_1^2} + \frac{1}{L_2^2} + \frac{1}{L_3^2}\right)\\
E_{12} &= \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{L_1^2} + \frac{2}{L_2^2} + \frac{1}{L_3^2}\right)\\
E_{13} &= \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{L_1^2} + \frac{1}{L_2^2} + \frac{3}{L_3^2}\right)
\end{aligned}$$

and the corresponding wave functions are:

$$\begin{aligned}\psi 11 &= A\sin\left(\frac{2\pi x}{L_1}\right)\sin\left(\frac{\pi y}{L_2}\right)\sin\left(\frac{\pi z}{L_3}\right)\\ \psi 12 &= A\sin\left(\frac{\pi x}{L_1}\right)\sin\left(\frac{2\pi y}{L_2}\right)\sin\left(\frac{\pi z}{L_3}\right)\\ \psi 13 &= A\sin\left(\frac{\pi x}{L_1}\right)\sin\left(\frac{\pi y}{L_2}\right)\sin\left(\frac{2\pi z}{L_3}\right)\end{aligned}$$

Notice that each wave function has its own energy, so there is no degeneracy.

(b) For a cube $L_1 = L_2 = L_3 = L$, so the wave function then becomes:

$$\psi(x,y,z) = A\sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n_2\pi y}{L}\right)\sin\left(\frac{n_3\pi z}{L}\right)$$

and the energy for a cube is:

$$E = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{L^2}+\frac{n_2^2}{L^2}+\frac{n_3^2}{L^2}\right) = \frac{\pi^2\hbar^2}{2mL^2}\left(n_1^2+n_2^2+n_3^2\right)$$

So the ground state wavefunction, becomes:

$$\psi_{gs}(x,y,z) = A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)$$

and the ground state energy:

$$E_{gs} = \frac{\pi^2\hbar^2}{2m}\left(\frac{1}{L^2}+\frac{1}{L^2}+\frac{1}{L^2}\right) = \frac{3\pi^2\hbar^2}{2mL^2}$$

The first excited state has three different combinations of $n's$, namely $(n_1 = 2,\ n_2 = 1,\ n_3 = 1)$, $(n_1 = 1,\ n_2 = 2,\ n_3 = 1)$, and $(n_1 = 1,\ n_2 = 1,\ n_3 = 2)$ and the corresponding wavefunctions are:

$$\begin{aligned}\psi 11 &= A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)\\ \psi 12 &= A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)\sin\left(\frac{\pi z}{L}\right)\\ \psi 13 &= A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{2\pi z}{L}\right)\end{aligned}$$

However, three different combination of n^2-values have the same sum, so there is only one value of E_1,

$$E_1 = \frac{\pi^2\hbar^2}{2mL^2}(2^2+1^2+1^2) = \frac{\pi^2\hbar^2}{2mL^2}(1^2+2^2+1^2) = \frac{\pi^2\hbar^2}{2mL^2}(1^2+1^2+2^2) = \frac{3\pi^2\hbar^2}{mL^2}$$

In a cube there are three distinct first excited state wavefunctions and all have the same energy. This means that the first excited state is three-fold degenerate.

8. Calculate $< 1/r >$ for an electron in the ground state of hydrogen, and use your results to calculate the average kinetic energy of the electron.

Solution

We can calculate $< 1/r >$ from:

$$\left\langle \frac{1}{r} \right\rangle = \int_0^\infty R_{10} \frac{1}{r} R_{10} r^2 \, dr$$

where rR_{10} is the ground state ($n = 1,\ \ell = 0$) radial wavefunction of the hydrogen atom, and R_{10} is given by:

$$R_{10}(r) = \left(\frac{Z}{a_\circ}\right)^{3/2} 2e^{-Zr/a_\circ}$$

$< 1/r >$ then becomes:

$$\left\langle \frac{1}{r} \right\rangle = \int_0^\infty \left(\frac{Z}{a_\circ}\right)^3 4e^{-2Zr/a_\circ} r \, dr = 4\left(\frac{Z}{a_\circ}\right)^3 \left(\frac{a_\circ}{2Z}\right)^2 = \frac{Z}{a_\circ} = \frac{1}{a_\circ}$$

The average kinetic energy $< K >$ is given by:

$$< K > = < E - U(r) > = < E > - < U(r) > = E - < U(r) >$$

we used the fact that the total energy $< E >$ is sharp and then $< E > = E$. Let us now find $< U(r) >$.

$$\begin{aligned} U(r) &= -\frac{kZe^2}{r} \\ < U(r) > &= -\left\langle \frac{kZe^2}{r} \right\rangle \\ &= -kZe^2 \left\langle \frac{1}{r} \right\rangle = -\frac{kZ^2e^2}{a_\circ} \end{aligned}$$

Using the above equation and

$$E = -\frac{kZ^2e^2}{2a_\circ}$$

The average kinetic energy then becomes:

$$\begin{aligned} < K > &= -\frac{kZ^2e^2}{2a_\circ} + \frac{kZ^2e^2}{a_\circ} \\ &= \frac{kZ^2e^2}{2a_\circ} \end{aligned}$$

9. A π° meson is an unstable particle produced in high energy particle collisions. It has a mass-energy equivalent 135 MeV, and it exists for an average life-time of only 8.7×10^{-17} s before decaying into two *gamma* rays. Using the uncertainty principle estimate the fractional uncertainty in its mass determination.

Solution

Since:

$$\begin{aligned} E &= m_\pi c^2 \\ \Delta E &= \Delta(m_\pi)c^2 \end{aligned}$$

Applying the uncertainty principle we get:

$$\begin{aligned} \Delta E \Delta t &\geq \tfrac{1}{2}\hbar \\ \Delta(m_\pi)c^2\Delta t &= \frac{1}{2}\hbar \\ \Delta m - \pi &= \frac{\hbar}{2\Delta t c^2} \\ \frac{\Delta m_\pi}{m_\pi} &= \frac{\hbar}{2\Delta t m_\pi c^2} \\ &= \frac{6.582 \times 10^{-16}(eV \cdot s)}{2 \times 8.7 \times 10^{-17}(s) \times 135 \times 10^6 (eV)} \\ &= 2.80 \times 10^{-8} \end{aligned}$$

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10. A hydrogen atom is in the $6g$ state.
    (a) What is the principal quantum number?
    (b) What is the energy of the atom?
    (c) What are the values for the orbital quantum number and the magnitude of the electron's orbital angular momentum?
    (d) What are the possible values for the magnetic quantum number? For each value, find the corresponding z component of the electron's orbital angular momentum and the angle that the orbital angular momentum vector makes with the z axis.

Solution

For a $6g$ state we have:

(a) $n = 6$

(b) The energy of a state n in a hydrogen atom is:

$$\begin{aligned} E_n &= \frac{-13.6\ eV}{n^2} \\ E_6 &= \frac{-13.6\ eV}{36} \\ &= -0.378\ eV \end{aligned}$$

(c) For a g-state $\ell = 4$. The orbital angular momentum is given by:

$$\begin{aligned} L &= \sqrt{\ell(\ell+1)}\hbar \\ &= \sqrt{4 \times 5}\hbar \\ &= \sqrt{20}\hbar \\ &= 4.47\ \hbar \end{aligned}$$

(d) The values of magnetic quantum number m_ℓ are:

$$\begin{aligned} m_\ell &= 0, \pm1, \pm2, \pm3, \cdots \pm \ell \\ &= -4, -3, -2, -1, 0, 1, 2, 3, 4 \end{aligned}$$

The z component of the electron's orbital angular momentum, L_z, and the angle, θ, that the orbital angular momentum vector makes with the z axis are given by:

$$\begin{aligned} L_z &= m_\ell \hbar \\ \cos\theta &= \frac{L_z}{L} \\ &= \frac{m_\ell}{\sqrt{\ell(\ell+1)}} \\ &= \frac{m_\ell}{4.47} \end{aligned}$$

The values of m_ℓ and the corresponding values of L_z, θ in degrees, and θ in radians are shown in the following table.

m_ℓ	-4	-3	-2	-1	0	1	2	3	4
L_z	-4$\hbar$	-3$\hbar$	-2$\hbar$	-1$\hbar$	0	1$\hbar$	2$\hbar$	3$\hbar$	4$\hbar$
θ	153.4°	132.1°	116.6°	102.9°	90°	77.1°	63.4°	47.9°	26.6°
θ (rad)	2.677	2.306	2.035	1.796	1.571	1.346	1.107	0.836	0.464

11. An electron has a wavefunction

$$\psi(x) = Ce^{-|x|/x_\circ}$$

where $x_\circ$ is a constant and $C = 1/\sqrt{x_\circ}$ for normalization. For this case, obtain expressions for $< x >$ and Δx in terms of $x_\circ$. Also calculate the probability that the electron will be found within a standard deviation of its average position, that is, in the range $< x > - \Delta x$ to $< x > + \Delta x$, and show that this is independent of $x_\circ$.

Solution

The expectation value $< x >$ is given by:

$$< x >= \int_{-\infty}^{\infty} x|\psi(x)|^2\, dx = C^2 \int_{-\infty}^{\infty} xe^{-2|x|/x_\circ}\, dx$$

The function $\psi(x) = e^{-i|x|/x_\circ}$ is symmetric about $x = 0$, i.e. it is an even function. However, the function $x|\psi(x)|^2$ is an odd function i.e. antisymmetric about $x = 0$. thus the contribution from $x > 0$ exactly cancels the contribution from $x < 0$, and as a result $< x >= 0$.

On the other hand the function $x^2|\psi(x)|^2$ is an even function, i.e. symmetric about $x = 2$ and the contributions from the negative values of x are identical to those from $x > 0$, so , x^2 is:

$$< x^2 >= \int_{-\infty}^{\infty} x^2|\psi(x)|^2\, dx = 2C^2 \int_{0}^{\infty} x^2e^{-2|x|/x_\circ}\, dx$$

Using the standard integral:

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n!}{a^{n+1}}$$

putting $n = 2$ and $a = x_\circ/2$, we get:

$$< x^2 >= 2C^2 2\left\{\frac{x_\circ}{2}\right\}^3 = \frac{4}{x_\circ}\left\{\frac{x_\circ}{2}\right\}^3 = \frac{1}{2}x_\circ$$

Δx can then be calculated from:

$$\Delta x = \sqrt{< x^2 > - < x >^2} = \sqrt{< x^2 > -0} = \frac{x_\circ}{\sqrt{2}}$$

The probability P the electron will be found in the interval $< x > -\Delta x = -\Delta x$ and $< x > +\Delta x = \Delta x$ is given by:

$$P = \int_{-\Delta x}^{+\Delta x} |\psi(x)|^2 \, dx = 2C^2 \int_0^{\Delta x} e^{-2x/x_\circ} = 2C^2 \left(\frac{x_\circ}{2}\right) \left[e^{-2x/x_\circ}\right]_0^{\Delta x} = 1 - e^{-\sqrt{2}} = 0.757$$

Of course the P is independent of $x_\circ$.

12. Compute the probability that a $2s$ electron of hydrogen will be found inside the Bohr radius for this state, $4a_\circ$. Compare this with the probability of finding a $2p$ electron in the same region.

Solution

The radial wavefunction of the $2s$ state ($n = 2,\ \ell = 0$) of the hydrogen atom is given by:

$$rR_{20} = r\left(\frac{Z}{2a_\circ}\right)^{3/2}\left(2 - \frac{Zr}{a_\circ}\right)e^{-Zr/2a_\circ}$$

and the radial wavefunction of the $2p$ state ($n = 2,\ \ell = 1$) of the hydrogen atom is given by:

$$rR_{21} = r\left(\frac{Z}{2a_\circ}\right)^{3/2}\frac{Zr}{\sqrt{3}a_\circ}e^{-Zr/2a_\circ}$$

The probability P_{2s} of finding the electron inside the Bohr radius of the $2s$ state is given by, taking $Z = 1$:

$$\begin{aligned}
P_{2s} &= \int_0^{4a_\circ} |rR_{20}|^2\, dr \\
&= \int_0^{4a_\circ} \frac{1}{8a_\circ}\left(\frac{r}{a_\circ}\right)^2\left(2 - \frac{r}{a_\circ}\right)^2 e^{-r/a_\circ}\, dr \\
&= \frac{1}{8a_\circ}\int_0^{4a_\circ}\left(\frac{r}{a_\circ}\right)^2\left(2 - \frac{r}{a_\circ}\right)^2 e^{-r/a_\circ}\, dr
\end{aligned}$$

Changing variables to $z = r/a_\circ$ and $dz = dr/a_\circ$ the last equation becomes:

$$\begin{aligned}
P_{2s} &= \frac{1}{8}\int_0^4 z^2(2-z)^2 e^{-z}\, dz \\
&= \frac{1}{8}\int_0^4 (4z^2 - 4z^3 + z^4)e^{-z}\, dz
\end{aligned}$$

The above integration can be carried out by successively using the following standard form:

$$\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx$$

We then get for P_{2s}:

$$\begin{aligned}
P_{2s} &= \frac{1}{8}\left[\left\{-(4z^2-4z^3+z^4)-(8z-12z^2+4z^3)-(8-24z+12z^2)-(24+24z)-(24)\right\}e^{-z}\right]_0^4 \\
&= \frac{1}{8}\left[-(64+96+104+72+24)e^{-4}+8\right] \\
&= 0.176
\end{aligned}$$

For the $2p$ state electron, the probability of finding it inside $r = 4a_\circ$, using $Z = 1$, is:

$$\begin{aligned}
P_{2p} &= \int_0^{4a_\circ} |rR_{21}|^2\, dr \\
&= \int_0^{4a_\circ}\left[\left(\frac{1}{2a_\circ}\right)^{3/2}\frac{r^2}{\sqrt{3}a_\circ}e^{-r/2a_\circ}\right]^2 dr \\
&= \int_0^{4a_\circ}\frac{1}{24a_\circ}\left(\frac{r^4}{a_\circ^4}\right)e^{-r/a_\circ}\, dr
\end{aligned}$$

Once again we change variables to $z = r/a_\circ$ to get:

$$P_{2p} = \frac{1}{24}\int_0^4 z^4 e^{-z}\, dz$$

Using the general integral given above, P_{2p} becomes:

$$\begin{aligned}
P_{2p} &= \frac{1}{24}\left[\left\{-z^4-4z^3-12z^2-24z-24\right\}e^{-z}\right]_0^4 \\
&= \frac{1}{24}(-824e^{-4}+24) \\
&= 0.371
\end{aligned}$$