Chapter 26

Capacitance and Dielectrics. Solutions of Selected Problems

26.1 Problem 26.11 *(In the text book)*

A 50.0-m length of coaxial cable has an inner conductor that has a diameter of 2.58 mm and carries a charge of 8.10 μC. The surrounding conductor has an inner diameter of 7.27 mm and a charge of $-8.10\ \mu C$.

(a) What is the capacitance of this cable?

(b) What is the potential difference between the two conductors? Assume the region between the conductors is air.

Solution

(a) The capacitance of the cable is:

$$C = \frac{\ell}{2k_e \ln(b/a)}$$

where $\ell = 50.0\ m$ is the length of the cable, $a = 2.58\ mm$ is the diameter of the inner conductor and $b = 7.27\ mm$ is the diameter of the outer conductor. The capacitance of the cable then becomes:

$$C = \frac{50.0}{2 \times 8.99 \times 10^9 \times \ln(7.27/2.58)} = 2.68 \times 10^{-9}\ F = 2.68\ nF$$

(b) The potential difference can be obtain in two ways. One way is to use the potential difference between two charged concentric shells and the other from the capacitance of the cable. Using the first method gives:

$$\Delta V = 2k_e\lambda \ln\left(\frac{b}{a}\right)$$

where $\lambda = q/\ell$ is the charge per unit length, so:

$$\Delta V = 2 \times 8.99 \times 10^9 \times \frac{8.10 \times 10^{-6}}{50.0} \times \ln\left(\frac{7.27}{2.58}\right) = 3.02 \times 10^3\ V = 3.02\ kV$$

Using the second method we get:

$$\Delta V = \frac{Q}{C} = \frac{8.10 \times 10^{-6}}{2.68 \times 10^{-9}} = 3.02 \times 10^3\ V = 3.02\ kV$$

26.2 Problem 26.30 *(In the text book)*

Some physical systems possessing capacitance continuously distributed over space can be modeled as an infinite array of discrete circuit elements. Examples are a microwave waveguide and the axon of a nerve cell. To practice analysis of an infinite array, determine the equivalent capacitance C between terminals X and Y of the infinite set of capacitors represented in Figure (26.30). Each capacitor has capacitance $C_\circ$. Suggestion: Imagine that the ladder is cut at the line AB, and note that the equivalent capacitance of the infinite section to the right of AB is also C.

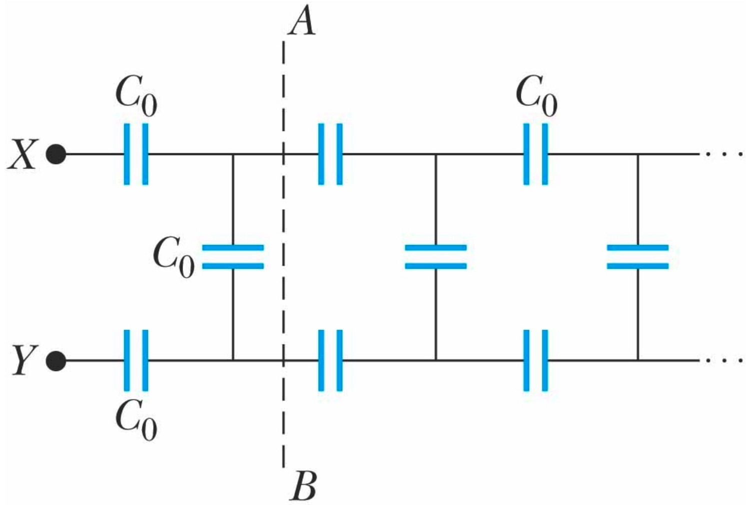

Figure 26.30:

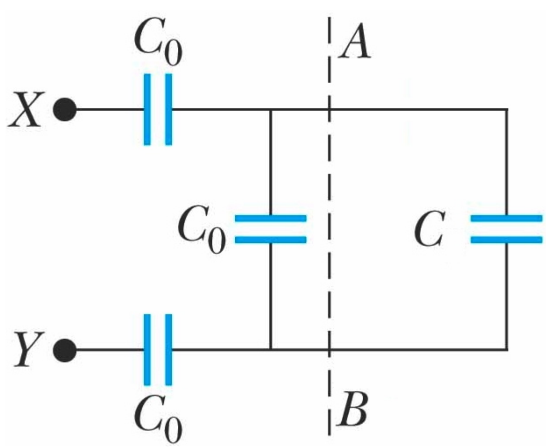

Figure 26.31:

Solution

The capacitance of the infinite series is C so cutting off the three capacitors to the left of line AB leaves the reminder of the ladder as infinite with a capacitance of C. So the ladder can then be reduced to one capacitor with capacitance C attached to the three capacitors to the left of line AB as shown in Figure (26.31). In addition, since the capacitance of the whole ladder between X and Y is C, then the equivalent capacitance of the circuit shown in Figure (26.31) is also C. Moreover the two vertical capacitors $C_\circ$ and C in Figure (26.31) are connected in parallel so their equivalent capacitance $C_1 = C_\circ + C$ is connected in series withe remaining two capacitors, we then get:

$$\begin{aligned}
\frac{1}{C} &= \frac{1}{C_\circ} + \frac{1}{C_1} + \frac{1}{C_\circ} \\
&= \frac{1}{C_\circ} + \frac{1}{C + C_\circ} + \frac{1}{C_\circ} \\
&= \frac{C + C_\circ + C_\circ + C + C_\circ}{C(C + C_\circ)} \\
C_\circ C + C_\circ^2 &= 2C^2 + 3C_\circ C \\
2C^2 + 2C_\circ C - C_\circ^2 &= 0
\end{aligned}$$

we then get:

$$C = \frac{-2C_\circ \pm \sqrt{4C_\circ^2 + 4(2C_\circ^2)}}{4}$$

The negative leads to a negative capacitance, so only the positive root is an acceptable solution:

$$C = \frac{-2C_\circ + \sqrt{4C_\circ^2 + 4(2C_\circ^2)}}{4} = \frac{1}{2}C_\circ\left(\sqrt{3} - 1\right)$$

26.3 Problem 26.33 *(In the text book)*

Two capacitors, $C_1 = 25.0\ \mu F$ and $C_2 = 5.00\ \mu F$, are connected in parallel and charged with a 100-V power supply.

(a) Draw a circuit diagram and calculate the total energy stored in the two capacitors.

(b) What If? What potential difference would be required across the same two capacitors connected in series in order that the combination stores the same amount of energy as in (a)? Draw a circuit diagram of this circuit.

Solution

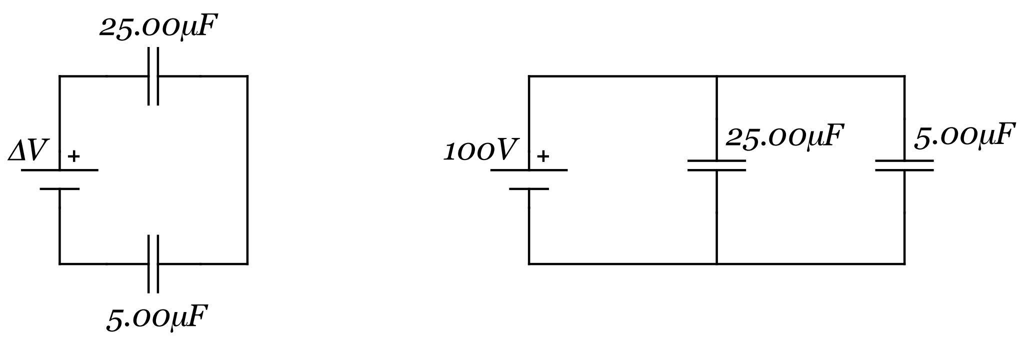

Figure 26.33:

The total energy stored in the capacitors is:

$$U = \frac{1}{2}C(\Delta V)^2$$

where C is the equivalent capacitor. The circuit diagrams are shown in Figure (26.33), the series connection to the left and the parallel connection to the right.

(a) The equivalent capacitance in the parallel connection C_p is:

$$C_p = C_1 + C_2 = 25.0 + 5.00 = 30.0\ \mu F$$

The energy stored is then:

$$U = \frac{1}{2} \times 30.0 \times 10^{-6} \times (100)^2 = 0.150\ J$$

(b) The equivalent capacitance C_s in the series connection is:

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} \qquad \text{or} \qquad C_s = \frac{C_1 C_2}{C_1 + C_2} = \frac{5.00 \times 10^{-6} \times 25.0 \times 10^{-6}}{5.00 \times 10^{-6} + 25.0 \times 10^{-6}} = 4.17\ \mu F$$

and,

$$U = \frac{1}{2} C_s (\Delta V)^2 \qquad \text{or} \qquad \Delta V = \sqrt{\frac{2U}{C_s}} = \sqrt{\frac{2 \times 0.150}{4.17 \times 10^{-6}}} = 268\ V$$

26.4 Problem 26.51 *(In the text book)*

A small object with electric dipole moment $\boldsymbol{p}$ is placed in a nonuniform electric field $\boldsymbol{E} = E(x)\hat{\boldsymbol{i}}$. That is, the field is in the x direction and its magnitude depends on the coordinate x. Let θ represent the angle between the dipole moment and the x direction.

(a) Prove that the dipole feels a net force

$$F = p\left(\frac{dE}{dx}\right)\cos\theta$$

in the direction toward which the field increases.

(b) Consider a spherical balloon centered at the origin, with radius 15.0 cm and carrying charge 2.00 μC. Evaluate dE/dx at the point (16 cm, 0, 0). Assume a water droplet at this point has an induced dipole moment of $6.30\hat{\boldsymbol{i}}\ nC\cdot m$. Find the force on it.

Solution

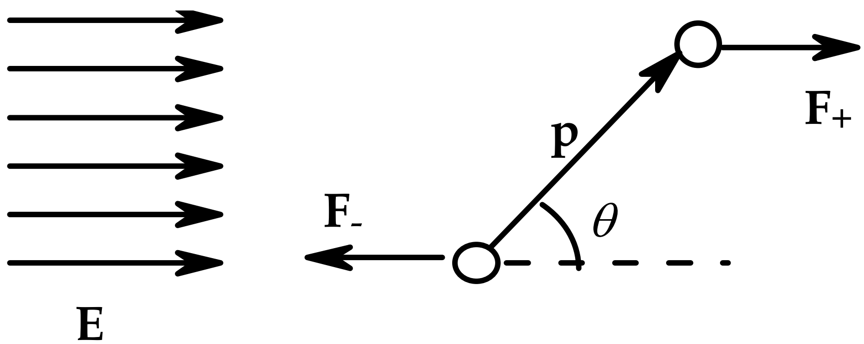

Figure 26.51:

(a) Let x represent the coordinate of the negative charge. Then $x = 2 + a\cos\theta$ is the coordinate of the positive charge(see Figure (26.51)). The force on the negative charge is:

$$\boldsymbol{F}_{-} = -qE(x)\hat{\boldsymbol{i}}$$

While the force on the positive charge is:

$$\boldsymbol{F}_{+} = +qE(x + 2a\cos\theta)\hat{\boldsymbol{i}}$$

Since we don not know the exact way E depends on x we then use the definition of derivatives to evaluate $E(x + 2a\cos\theta)$, since $2a\cos\theta$ is small, i.e. for $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad \text{or} \qquad f(x+\Delta x) = f(x) + \Delta x\frac{dy}{dx}$$

or

$$E(x + 2a\cos\theta)\hat{\boldsymbol{i}} = E(x)\hat{\boldsymbol{i}} + 2a\cos\theta\frac{dE}{dx}\hat{\boldsymbol{i}}$$

The force on the positive charge becomes:

$$F_+ = qE(x)\hat{\boldsymbol{i}} + 2aq\cos\theta\frac{dE}{dx}\hat{\boldsymbol{i}} = qE(x)\hat{\boldsymbol{i}} + p\frac{dE}{dx}\cos\theta\hat{\boldsymbol{i}}$$

where $p = 2aq$ is the dipole moment and $2a$ is the separation between the dipole charges. The net force on the dipole is then:

$$\boldsymbol{F} = \boldsymbol{F}_- + \boldsymbol{F}_+ = -qE(x)\hat{\boldsymbol{i}} + qE(x)\hat{\boldsymbol{i}} + p\frac{dE}{dx}\cos\theta\hat{\boldsymbol{i}} = p\frac{dE}{dx}\cos\theta\hat{\boldsymbol{i}}$$

(b) The balloon creates an electric field E_b along the x-axis given by:

$$E_b(x) = \frac{k_e q}{x^2}qE(x)\hat{\boldsymbol{i}} \qquad \text{and} \qquad \frac{dE_b}{dx} = -\frac{2k_e q}{x^3}$$

at $x = 16\ cm$, get:

$$\frac{dE_b(x=16)}{dx} = -\frac{2\times 8.99\times 10^9\times 2.00\times 10^{-6}}{(0.160)^3} = -8.78\times 10^6\ N/C\cdot m$$

The force on water droplet dipole due to E_b is:

$$\boldsymbol{F} = p\frac{dE_b}{dx}\cos\theta\ \hat{\boldsymbol{i}} = 6.30\times 10^{-9}\times -8.78\times 10^6\cos 0\ \hat{\boldsymbol{i}} = -55.3\times 10^{-3}\ \hat{\boldsymbol{i}}\ N$$

26.5 Problem 26.58 *(In the text book)*

A 2.00-nF parallel-plate capacitor is charged to an initial potential difference $\Delta V_i = 100\ V$ and then isolated. The dielectric material between the plates is mica, with a dielectric constant of 5.00.

(a) How much work is required to withdraw the mica sheet?

(b) What is the potential difference of the capacitor after the mica is withdrawn?

Solution

(a) Let U_i and C_i be the potential energy and the capacity of the capacitor with mica between plates and U_f and C_f be the potential energy and capacitance after the mica is withdrawn. We then have:

$$U_i = \frac{1}{2}\left(\frac{Q^2}{C_i}\right) \qquad \text{and} \qquad U_f = \frac{1}{2}\left(\frac{Q^2}{C_f}\right)$$

If κ is the dielectric constant of mica, then:

$$C_i = \kappa C_f \qquad \text{and} \qquad U_f = \frac{1}{2}\kappa\left(\frac{Q^2}{C_i}\right)$$

The work needed to be done by an external force to remove the mica should be equal to the change in potential energy, i.e.

$$W = U_f - U_i = \frac{1}{2}\kappa\left(\frac{Q^2}{C_i}\right) - \frac{1}{2}\left(\frac{Q^2}{C_i}\right) = \frac{1}{2}\left(\frac{Q^2}{C_i}\right)(\kappa - 1)$$

Since the capacitor is isolated, the charge on it will not change by withdrawing the mica. The capacitance and the potential difference, however, do change. Since $Q = C_i(\Delta V_i)$, we get:

$$W = \frac{1}{2}C_i(\Delta V_i)^2(\kappa - 1) = \frac{1}{2} \times 2.00 \times 10^{-9} \times (100)^2 \times (5.00 - 1) = 4 \times 10^{-5}\ J$$

The positive result indicates that the final energy of the capacitor is greater than the initial energy. The extra energy comes from the work done on the system by the external force that pulled out the dielectric.

(b) The final potential difference is:

$$\Delta V_f = \frac{Q}{C_f} = \kappa\frac{Q}{C_i} \qquad \text{since} \qquad Q = C_i\Delta V_i \qquad \text{then} \qquad \Delta V_f = \kappa\Delta V_i = 5.00 \times 100 = 500\ V$$