Mathematical Modelling in Forestry

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Abstract

Industrial forest plantations can contribute to the conservation of the remaining world forests. Mathematical models are fundamental for the rational management of forests, and especially for the intensive management of such plantations. Some examples and modelling issues will be examined, focusing on two types of problems. One is the prediction of tree growth and development under various silvicultural alternatives. Conceptual, data gathering, and statistical aspects will be discussed. The other concerns cutting and planting decisions at the enterprise, regional, or national level. Fluctuating production over time from mixtures of units having various ages, productivities and silvicultural treatments gives rise to complex decision problems. Simulation and Linear Programming based approaches are commonly used.

Forestry information and modelling systems

A forestry organization dealing with intensively managed wood production plantations should have a system similar to that depicted in Fig. 1. Arrows indicate outputs used by other components. The focus here is in decision support for forest management planning. Obviously, there is much more necessary information that has not been included in this view, such as the technical details of establishment techniques, nutrition, tree improvement, etc.

The functions of this information and modelling system are prediction, control, and monitoring. That is, figuring out what will happen, what is happening, and what has happened. A stand database keeps track of the current state of the forest. This data is collected through periodic forest inventories, and it may be kept up-to-date with the help of growth models. In most instances the evaluation of stand management alternatives on a per-hectare basis is inadequate, and planning must take care of forest-wide considerations (or region-wide, or nation-wide, depending of the agency doing the planning). This is the purpose of the forest planning or forest estate models [1, 2]. They use the stand data, and require also growth predictions, and financial and other resource information. Conversion and logging models may help to estimate yields and costs as a function of projected stand conditions. Growth models [1, 3] are developed and validated with data from remeasured, or permanent, sample plots (PSPs).

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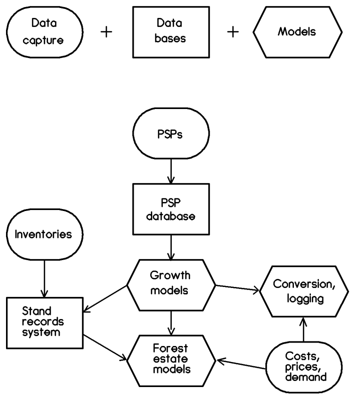

Figure 1: Forestry DSS

A tractable nonlinear system

The theory and practice of linear systems is well developed. Unfortunately, for many purposes a linear model is not realistic. With a single state variable x, more flexibility can be achieved by substituting in the transition equation a power transformation x^c in place of x, where c is a constant. That is, using

$$\frac{\mathrm{d}x^c}{\mathrm{d}t} = ax^c + b$$

instead of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + b \; .$$

This happens to be equivalent to the von Bertalanffy - Richards equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \eta x^m - \kappa x \; .$$

The growth curves derived from it are one of the most popular nonlinear models used in Forestry, Fisheries, etc.

For an n-dimensional state vector $\mathbf{x}$ we can use, in place of the state variables x_i, power transformations of the form

$$x_1^{c_{i1}} x_2^{c_{i2}} \cdots x_n^{c_{in}} \; .$$

Assembling the transformation exponents c_{ij} into a matrix C, and defining

$$\exp[\mathrm{C} \ln \mathbf{x}] \equiv \mathbf{x}^{\mathrm{C}} \; ,$$

the extended model can be written

$$\frac{\mathrm{d}\mathbf{x}^{\mathrm{C}}}{\mathrm{d}t} = \mathrm{A}\mathbf{x}^{\mathrm{C}} + \mathbf{b} \; ,$$

the "multivariate Richards model" [3].

It works fairly well. Full maximum likelihood estimation is possible under a reasonable stochastic differential equation model [3, 5]. One problem is that to ensure monotonic trends many of the coefficients have to be made zero in an *ad hoc* way (we do not like trees to shrink, or numbers of trees to increase). Finding the most general conditions that would ensure monotonicity (between the origin and the asymptotes / stationary point) is an open question. It would also be nice to have alternative flexible, closed-form integrable models.

Some open questions in forest planning

Consider the following model:

Characterize a homogenous forest at time t by its age distribution, that is, the number of hectares x_{it} in each age class $i = 1, \ldots, n$. At each $t = 0, 1, 2, \ldots$, we can cut $0 \leq u_{it} \leq x_{it}$ hectares from each age class, yielding volumes of y_i cubic meters per hectare. Therefore, the volume cut in period t is $v_t = \sum y_i u_{it}$. The remaining areas move into the next age class in the next time period (if necessary we may take the last age class as open, or n as infinite, for example). The area cut goes into the first age class in the next period.

If necessary, we can assume that the yield function y_i is positive, increasing, and perhaps concave over the range where cuts are made.

Also, cuts may be restricted to be done "oldest first", i.e., cut all of a class before advancing to the next lower one. This is optimal under fairly general conditions.

An analogous continuous model — with continuous age distribution, continuous yield function, and continuous time — could be used and might be easier to handle in some instances.

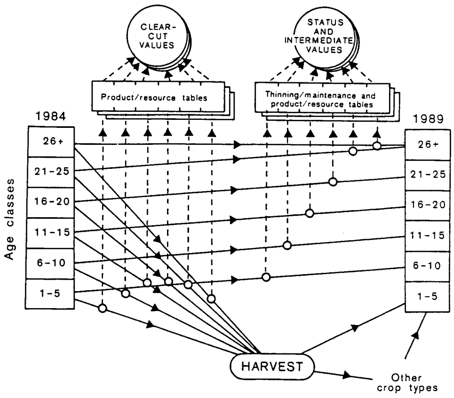

Figure 2: Forest planning problem structure

Notes: Of course, this model is an oversimplification, but finding nice solutions for it would be a big step toward handling more realistic problems. In practice there are different "crop types", with different yields, etc, and usually also multiple products (see Fig. 2), financial objectives, and additional constraints. Class/period width varies between 1 and 10 years, depending on growth rates, problem size, and computing limitations.

Sustainability

To be specific and avoid unnecesary complications, take 1-year classes/periods, and $\sum_i^n x_{it} = 1$.

Assume that the annual cut $v_t = v$, taken oldest first, is constant. Clearly, the initial distribution and v (and the fixed yield function) determine the evolution of the forest.

It is easily seen that a uniform distribution with equal areas over the first m classes and v equal to the volume in the last class ($v = y_m/m$) is stationary (an equilibrium point). This is the classical "normal forest", and y_i/i is called a *mean annual increment*, MAI. Maximum volume production is obtained with a normal forest where m (the *rotation age*) corresponds to the *maximum MAI*, that is, the maximum of y_i/i over i.

Much of forest management theory over the last few centuries has revolved around these concepts, and the goal of aproaching normal forests is still taken very seriously in many countries. For many reasons this is not very relevant in modern plantation management, but it is a useful theoretical starting point.

What happens If we keep cutting v from an arbitrary initial distribution? There are two eventual outcomes: (a) the forest is eventually exhausted, or (b) it tends to a stable normal forest (except possibly for the last age class in the discrete model). Obviously, for the equilibrium point (b) to exist v must be not larger than the maximum MAI. Roughly, if v is smaller than the Maximum MAI, there are two equilibrium normal forests corresponding to two solutions of $y_i/i = v$ (at least in the continuous version of the model, which is less messy for this analysis). It would seem that the one with the lower rotation is always unstable, and the other stable.

All this is easy to verify by simulation. Given an initial distribution, cutting a v equal or slightly less than the maximum MAI may lead to exhaustion, or to an equilibrium. However, finding a condition for a distribution to be able to sustain a given cut is, as far as I know, an open question.

Some simulations seem to suggest interesting dynamics (chaos?), especially close to critical points. Unfortunately, the dimensionality of the system difficults visualization, although at least the three age classes case could be managed more easily. All this is largely unexplored.

A variant of the problem is, as before, to find the maximum sustainable cut if this is less than the maximum MAI. Otherwise, finding the maximum amount of unplanted land that can be added to the forest (say to age class 1) before that cut becomes unsustainable. These are the basis of Allison's *Equivalent Normal Forest* (ENF) measures of "maturity" [1]. Again, it is not known how to compute these values without recurring to simulation.

Non-decreasing yield, etc

Having a constant cut is rather too restrictive and artificial. Production fluctuations are costly, particularly when there are supply commitments, but it is mainly the down-movements that have to be avoided. It is common, then to impose a non-decreasing production constraint, that is, $v_t \leq v_{t+1}$. The objective may be to maximize a discounted value proportional to $\sum \alpha^t v_t$, subject to these and possibly other constraints. This is done by a "brute force" approach through Linear Programming. It seem that it should be possible, however, to exploit the considerable special structure to do somewhat better.

Consider the following "bare bones" problem. We have an initial distribution $x_{i0} \equiv b_i$, and denote by z_{it} the area from initial age class i cut in period t. Just ignore what happens after the first rotation. We can then formulate the following LP problem:

$$\max \sum_t \alpha^t v_t$$

subject to:

$$\begin{aligned}
\sum_t z_{it} &= a_i \\
\sum_i y_{t-i} z_{it} &= v_t \\
v_t &\leq v_{t+1} \\
z_{it} &\geq 0
\end{aligned}$$

We can use that y_j is positive, increasing, and concave. It is also known that the *oldest first* principle is optimal here [2]. It seems increadible that even this simple-looking problem could not be solved by other than a full-blown LP approach!

References

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