Another look at growth equations *

Oscar García

1 Introduction

Finding equations that "unify" a number of growth equations is a popular pastime. By introducing additional parameters it is easy to devise equations that include two or more well-known models as special cases. For example, with $0 \leq \theta \leq 1$, $y = f(t)$ and $y = g(t)$ could be combined as

$$y = f(t)^{\theta} g(t)^{1-\theta} ,$$

among many other possibilities.

Although this sort of generalization is no great achievement, including a number of particular models in one over-all equation can simplify the study of their properties. The presence and nature of asymptotes and inflection points, for example, can then be studied once and for all in the general equation, without having to consider separately a large number of special cases. I analyze here an expression that contains most of the univariate growth models used in forestry, with two "essential" parameters (i.e. excluding possible linear transformations in y and t.) I shall focus on sigmoidal curves, with one inflection point and a finite upper asymptote. Although I must disagree with the assertion of Shvets and Zeide (1996) that nonasymptotic curves are not acceptable for growth modelling; for example, there is strong evidence that growth in volume per hectare in even-aged stands is not asymptotic, and the gross accumulated volume may not show any inflection points.

Note that the hunt for growth equations is essentially the same as that for probability distributions with an explicit formula for the cumulative (Burr, 1942). In fact, the general model here is closely related to Burr's distributions with range $(0, \infty)$.

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2 The Box-Cox transformation

Most growth equations contain powers, logarithms, and/or exponentials. These functions can conveniently be treated together using the Box-Cox transformation (Box & Cox, 1964):

$$u = \begin{cases} \frac{x^c - 1}{c} & \text{if } c \neq 0 \\ \ln x & \text{if } c = 0 \end{cases}$$

for $x \geq 0$ and $-\infty < c < \infty$. It will be slightly more convenient to use the negative of this, and to write it as

$$B(x, c) = \lim_{p \to c} \frac{1 - x^p}{p} \ . \tag{1}$$

As an example, consider the Richards equation $y = a(1 - \frac{k}{c} e^{-bt})^c$. We can write it as

$$B(\frac{y}{a}, \frac{1}{c}) = k e^{-bt} \ ,$$

which with $c \to \infty$ (i. e. $\frac{1}{c} = 0$) includes also the Gompertz $y = a \exp(-k e^{-bt})$.

The inverse of the transformation is clearly

$$B^{-1}(x, c) = \lim_{p \to c} (1 - px)^{1/p} \ , \tag{2}$$

defined for $cx \leq 1$, or $x \leq 1/c$ if $c > 0$ and $x \geq 1/c$ if $c < 0$. Note that $B^{-1}(x, 0) = e^{-x}$.

It will be useful to note here also the derivative:

$$\frac{\mathrm{d}B(x, c)}{\mathrm{d}x} = -x^{c-1} \ . \tag{3}$$

3 The general model

For simplicity, in what follows I shall ignore positive scale factors in y and t in a growth equation $y = f(t)$. That is, that equation will be taken as representative of $y = \alpha f(\beta t)$ for any positive α and β. If it exists, the asymptote will be 1 (corresponding to α in the scaled version), so that y will lie in the interval $0 \leq y \leq 1$.

Apart from linear transformations in y and t, most forestry univariate growth models express a power or logarithmic function of y in terms of power or exponential functions of t. Using the Box-Cox transformation, consider

$$B(B(y, a), b) = t \ , \tag{4}$$

for $0 \leq y \leq 1$ and $t \geq 0$, or

$$y = B^{-1}(B^{-1}(t,b),a) \ , \tag{5}$$

or, more explicitly,

$$y = \lim_{p \to a, q \to b} [1 - p(1-qt)^{1/q}]^{1/p} \ , \tag{6}$$

where a and b are parameters, and there may be some linear transformation implicit in t. As a shorthand, write $G(a,b)$ for this relationship between y and t. Write also $G_-(a,b)$ for the same relationship, but with $1-y$ in the place of y, that is, with the y-axis reversed.

It is easily verified that these two relationships include all but one of the models listed by Zeide (1993). Of those, the three most general are the Richards,

$$G(\frac{1}{c}, 0): \qquad y = (1-e^{-t})^c$$

or

$$G(-\frac{1}{c}, 0): \qquad y = (1+e^{-t})^{-c} \ ,$$

Levakovic I[1],

$$G(-\frac{1}{c}, -d): \qquad y = (\frac{t^d}{1+t^d})^c \ ,$$

and Weibull,

$$G_-(0, \frac{1}{c}): \qquad y = 1 - e^{-t^c} \ ,$$

where the parameters c and d are positive. The others are special instances of these (with Zeide's terminology and notation): Hossfeld IV and Yoshida I $G(-1,-c)$, Gompertz $G(0,0)$, Logistic $G(-1,0)$, Monomolecular $G(1,0)$, Bertalanffy $G(1/3,0)$, Levakovic III $G(-1/c,-2)$, and Korf $G(0,-c)$. As will be seen, some apparently new extensions in the range of the parameters are possible.

The exception is Sloboda's model $y = a\exp[-b\exp(-ct^d)]$. There are other models that are not covered in this scheme, such as $y = (\tanh t)^c$ and $y = \exp(-at^b e^{-t})$. This last one, used by Stage (1963) and currently being investigated by Per Holten-Andersen (pers. comm.), is quite flexible but not so convenient for some uses because it cannot be solved analytically for t. It would be interesting to see if a similar range of curve shapes could be represented by (5).

[1]This is equivalent to one of Burr's probability distributions (Burr, 1942).

4 Differential forms

Growth equations can be thought as the result of integrating differential equations expressing the growth rate. There is considerable confusion about this in the literature, however, apparently due to a failure in realizing that "the" differential equation for a given growth equation is not unique. Given $y = f(t)$, a differential equation in terms of t can be obtained as $\mathrm{d}y/\mathrm{d}t = f'(t)$. But we can also substitute $t = f^{-1}(y)$ to obtain $\mathrm{d}y/\mathrm{d}t = f'[f^{-1}(y)]$, a differential equation depending only of y (the same equation can be obtained by differentiating $t = f^{-1}(y)$ with respect to y.) In fact, there is an infinity of differential equations containing both y and t that will produce the same result on integration[2].

Here the most convenient form for studying the model properties is probably the one without t. From (4),

$$\frac{\mathrm{d}B(B(y,a),b)}{\mathrm{d}t} = 1 \ . \tag{7}$$

Differentiating we can also write

$$B'(B(y,a),b)\frac{\mathrm{d}B(y,a)}{\mathrm{d}t} = 1$$

$$\frac{\mathrm{d}B(y,a)}{\mathrm{d}t} = -B(y,a)^{1-b} \ , \tag{8}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^{1-a}B(y,a)^{1-b} \ , \tag{9}$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \begin{cases} y^{1-a}(\frac{1-y^a}{a})^{1-b} & \text{if } a \neq 0 \\ y(-\ln y)^{1-b} & \text{if } a = 0 \end{cases}$$

Instead of (4) – (6), equations (7), (8) or (9) could be taken as the general model definition. Then the growth curve equations may be obtained by integration with appropriate initial conditions. With (t_0, y_0) we obtain from (7)

$$B(B(y,a),b) = B(B(y_0,a),b) + t - t_0 \ . \tag{10}$$

In particular, for $y_0 = 0$,

$$B(B(0,a),b) = \begin{cases} B(1/a,b) & \text{if } a > 0 \\ 1/b & \text{if } a \leq 0 \text{ and } b < 0 \\ -\infty & \text{if } a \leq 0 \text{ and } b \geq 0 \end{cases} \tag{11}$$

[2]More striking is the equivalence between the growth equations obtained from Schnute's second-order differential equation and those from Richards' first-order differential equation (Yuancai, Marques and Macedo, to appear in *Forest Ecology and Management*.)

For a curve through the origin this value must be added to t in (6). It follows that for $a \leq 0$ and $b \geq 0$ the curve cannot go through the origin (as is well-known for the Gompertz, in which $a = b = 0$.) Similarly, it is seen that $G_{-}(a, b)$ cannot go through the origin if $b \leq 0$.

5 Sigmoids

In order to have an asymptote at $y = 1$ the derivative in (9) must be zero at that point. For having an inflection point the derivative must be zero also at $y = 0$. Some analysis of the various possibilities shows that this is the case if and only if $a < 1$, $b < 1$, and $ab < 1$. As noted before, if we want curves through the origin we must also exclude the range $a \leq 0$, $b \geq 0$.

Within the sigmoidal parameter range, the y-position of the inflection point corresponds to the maximum of (9). Differentiating and equating to zero,

$$y_{\text{infl}} = \left(\frac{1-a}{1-ab}\right)^{\frac{1}{a}} . \qquad (12)$$

The inflection point for $G_{-}(a, b)$ is located at 1 minus that, and, from symmetry, the same conditions on the parameters ensure a sigmoidal shape.

This range of parameter values includes the Levakovic I, which is defined with $a < 0$ and $b < 0$ (Zeide, 1993). It, and the formula (12) for the inflection point, agree with those known for the Richards ($b = 0$), and for the Weibull ($a = 0$). The possibilities $0 < a < 1$, $0 < b < 1$, and $G_{-}(a, b)$ with $a \neq 0$, do not seem to have been previously explored.

Contour curves for the height of the inflection point can be easily drawn by solving (12) for b:

$$b = [1 - (1-a)/y_{\text{infl}}^{a}]/a .$$

These are shown in Figure 1.

In addition to the inflection height, another easily interpretable property for characterizing the various curves would be their "steepness". For curves through the origin it might be defined as the slope at $y = 1/2$, for example, after scaling for $y = 1/2$ at $t = 1$. It can be conveniently obtained from

$$t = B(B(y, a), b) - B(B(0, a), b)$$

as the reciprocal of $(\frac{\mathrm{d}t}{\mathrm{d}y})/t = \frac{\mathrm{d}\ln t}{\mathrm{d}y}$ evaluated at $y = 1/2$. We find

$$\text{steepness} = (1/2)^{1-a} B(1/2, a)^{1-b} [B(B(1/2, a), b) - B(B(0, a), b)]$$

(see Figure 2.)

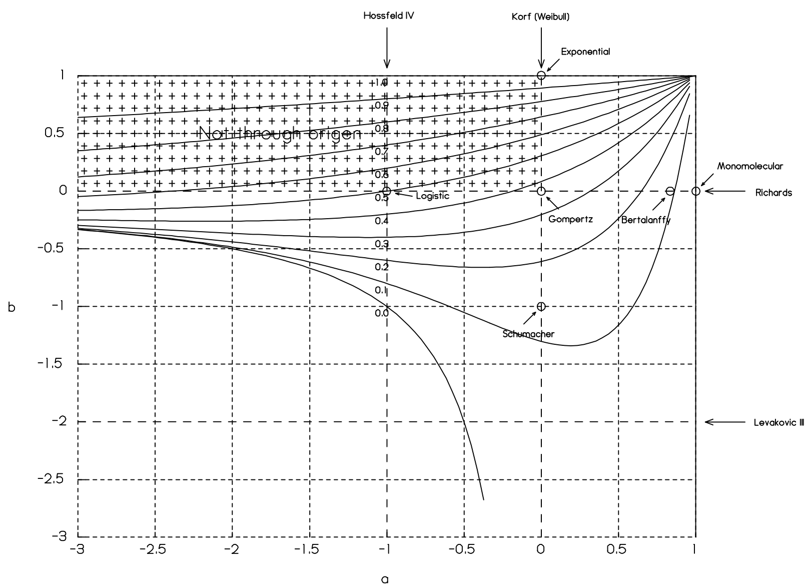

Figure 1: Height of the inflection point of $G(a, b)$. The corresponding inflection height for $G_{-}(a, b)$ is 1 minus this.

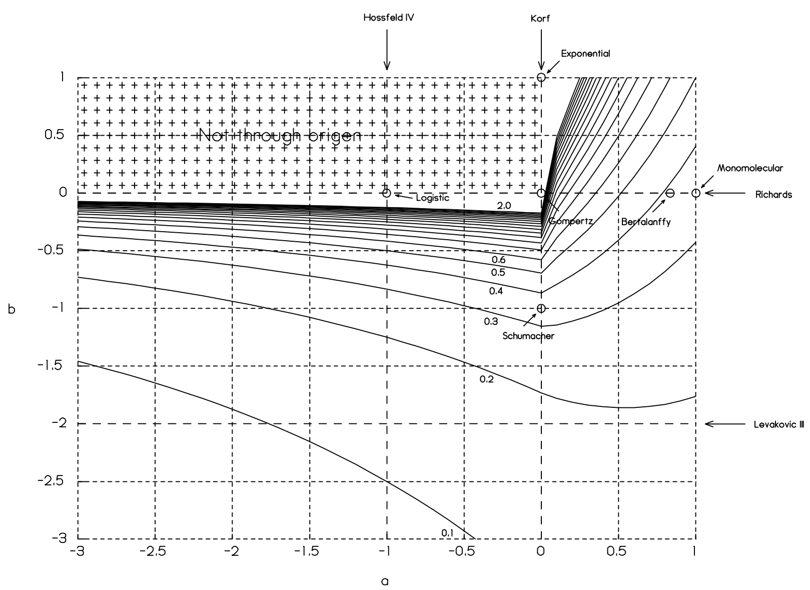

Figure 2: "Steepness" of curves from the family $G(a, b)$.

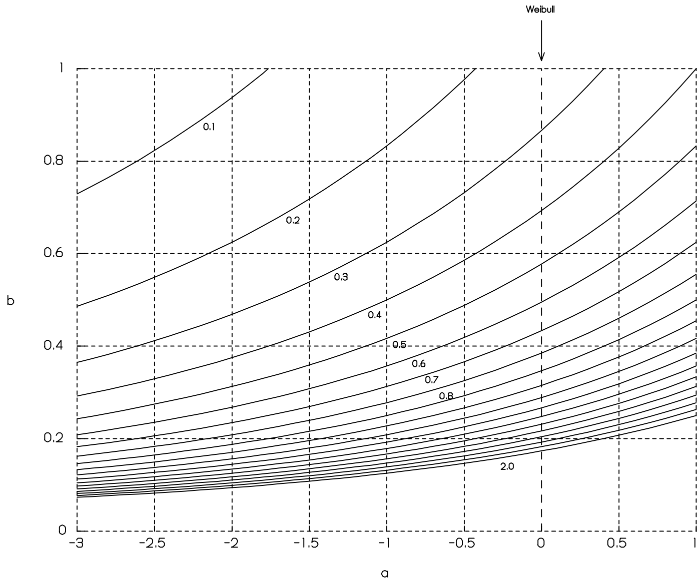

Figure 3: Steepness for $G_{-}(a, b)$. The range $b \leq 0$ where the curves do not go through the origin is omitted.

For $G_{-}(a, b)$ we have

$$\text{steepness} = (1/2)^{1-a} B(1/2, a)^{1-b} [B(B(1, a), b) - B(B(1/2, a), b)]$$

(Figure 3.)

References

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