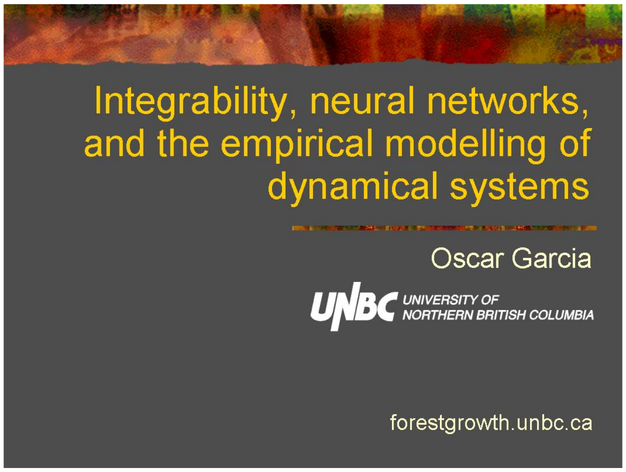
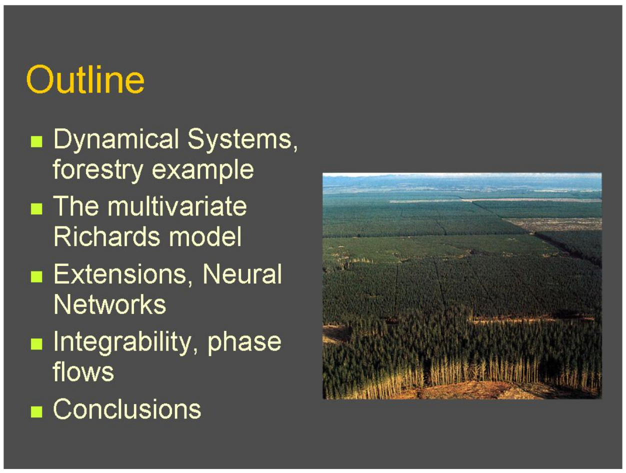


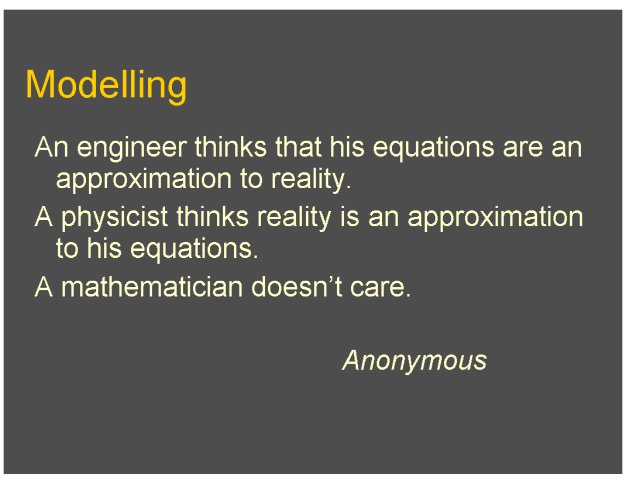
Colloque INRA "Modélisation pour les Ressources Naturelles"

Montpellier, France, 18-20 June 2008

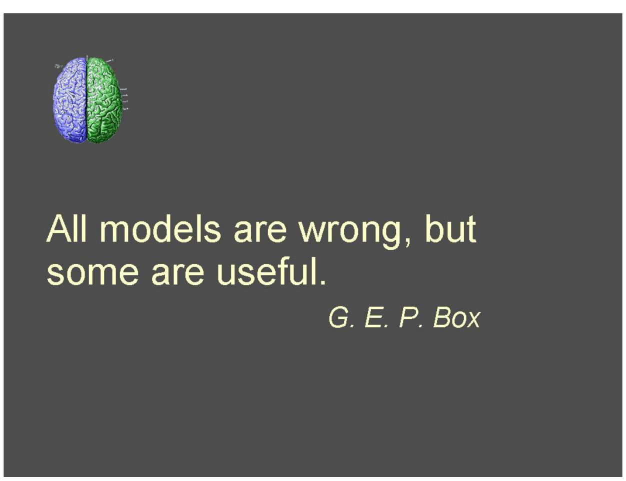
www.montpellier.inra.fr/umr_asb/colloque/colloque-juin



Review/interpretation + half-baked ideas.
Forest growth modelling examples, applicable to other systems.



Caricature, but much truth. Modelling views vary with background.
Math has to do with the formal manipulation of symbols, regardless of what those symbols might mean. The other two points are more relevant here.
The "physicist" view seems widespread. Talk of scientists discovering natural laws implies an independent existence of true models (laws, theories).
At least in our applications, models are artificial human constructs, approximations that neglect less important interactions. Needed for our mind to make sense of a world where everything is connected to everything.



Actually necessary: a mental model always required for understanding and prediction.
Economists have yet another view. They talk of "market failures". So it is reality which is wrong, not the model :-)
Being approximations, might as well choose convenient ones.

Dealing with Time

- Processes, systems evolving in time
- Functions of time
- Rates (Newton)
- System Theory (1960's)
- Control Theory, Nonlinear Dynamics

Over the last 200-300 years, foresters (and others) have been conditioned to think about processes through functions of time.

Radically different view, using rates of change, maybe due to Isaac Newton and taken for granted in Physics and Engineering.

Rates (state-space) approach sometimes thought of as specific to physics. System Theory separated the fundamentals from the physics. Later included with other topics.

Modern Nonlinear Dynamics preoccupied with weird behaviors (chaos, "systems pathology"). Generally, our systems are boringly well-behaved.

Dynamical systems

Instead of $\boldsymbol{y} = \boldsymbol{F}(t)$

- State: $\boldsymbol{x} = (x_1, \ldots, x_n)$
- Local transition function (rates):

  $$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

  $\boldsymbol{u}$: inputs (ODE)
- Output function:

  $$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$$

Copes with disturbances

Bold-face denotes vectors (lists of numbers).

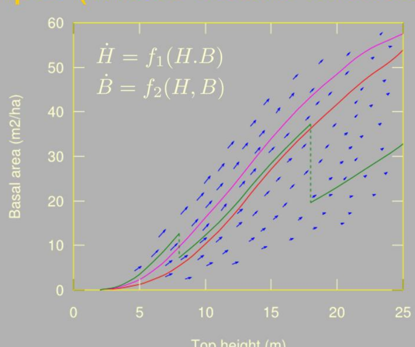
Rates: ordinary differential equations (ODEs, continuous time), or finite differences (discrete). Will mostly ignore inputs.

State description sufficiently detailed for, to a sufficient degree of approximation, (a) determine changes, and (b) estimate other variables of interest (outputs).

Disturbances due to management actions, natural causes, or "random" variation/perturbations.

Forest growth may not be smooth, but it is continuous (no intermediate size is skipped). Discrete systems can exhibit weird behavior not possible under continuity.

Example (whole-stand modelling)



Simple two-dimensional state. Different initial densities, thinning.

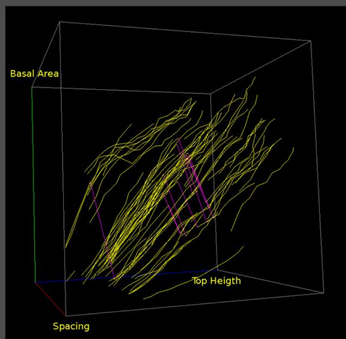
Example with thinnings is difficult to handle with functions-of-time approach. With rates, simply follow the arrows (vector field, flow), before or after disturbance.

3-D

Usually need more than two state variables.

Low-tech visualization, ca. 1980. Projections from a model of Elliott & Goulding.

3-D



Site
Eichhorn (1904)

New Zealand radiata pine permanent sample plot data. Purple lines are thinnings.

Dynamic 3-D graphics at http://web.unbc.ca/~garcia/FSTY405/threed.htm

Actually, plots are in different sites growing at different rates. The simplest assumption is that state trajectories do not change with site quality, only the speed does (Eichhorn's hypothesis). Other options if this fails.

More state variables just complicate visualization. Individual-tree models may have hundreds of state variables, but same idea. Emphasis here on whole-stand models (typically 2-5 variables); issues with individual-tree models discussed elsewhere.

Integration

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

⇓ No $\boldsymbol{u}$

$$\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}_0, t - t_0)$$

(Global transition function)

Group:

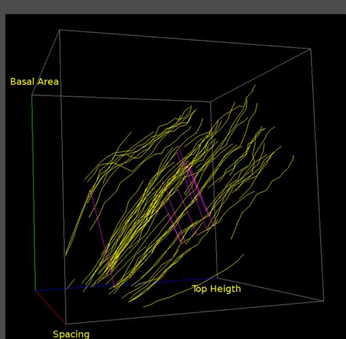
$$\boldsymbol{F}[\boldsymbol{F}(\boldsymbol{x}_0, s), t] = \boldsymbol{F}(\boldsymbol{x}_0, s + t)$$

$$\boldsymbol{F}(\boldsymbol{x}_0, 0) = \boldsymbol{x}_0$$

For simplicity, ignore inputs. Management mostly causes instantaneous state changes; inputs needed when studying climate change, etc.

Special properties of global transition function not easy to get right directly. They follow automatically from ODE.

3-D



State transitions from one time to another (without thinnings in-between).

Equation forms?

- Theoretical. Empirical. Constraints
- Simplest, linear: $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$

E.g., with $\boldsymbol{x} = (H, N, B)$

- Why not
  - Average spacing? $100/\sqrt{N}$
  - Mean diameter? $\sqrt{B/N}$
  - Volume or biomass? $\sim BH$
  - Relative spacing? $10000/(\sqrt{N}H)$
  - ... ? $H^\alpha N^\beta B^\gamma$

Ecological principles not well developed yet. Beautiful theories usually slain by ugly facts (Huxley).

New Zealand case: lots of data, aim to describe observations without preconceived ideas. For management forecasting (interpolation/extrapolation).

With less data, previous experience can be used to constrain equation forms.

H = top height, N = trees per hectare, B = basal area per hectare.

Multivariate Richards

$$\dot{\boldsymbol{x}^{\boldsymbol{C}}} = \boldsymbol{A}\boldsymbol{x}^{\boldsymbol{C}} + \boldsymbol{b}$$

where

$$\boldsymbol{x}^{\boldsymbol{C}} \equiv \exp(\boldsymbol{C} \ln \boldsymbol{x}) = (x_1^{c_{11}} x_2^{c_{12}} \cdots, \dots)$$

The (scalar) Bertalanffy-Richards:

$$\dot{x} = px - qx^m \quad \rightarrow \quad \dot{x^c} = ax^c + b$$

with

$$c = 1 - m\,, \quad a = cp\,, \quad b = -cq$$

Increased flexibility over the linear.
$\mathbf{x}^{\mathbf{C}}$ is non-standard notation
Analytically integrable, (convenient, although not such a big deal with current computing power).

Examples

$$\begin{cases} \dfrac{dH^{c_{33}}}{d\tau} = -H^{c_{33}} + b_3, \\ \dfrac{dN^{c_{22}}}{d\tau} = a_{21} B^{c_{11}} N^{c_{12}} H^{c_{13}} \\ \dfrac{dB^{c_{11}} N^{c_{12}} H^{c_{13}}}{d\tau} = a_{11} B^{c_{11}} N^{c_{12}} H^{c_{13}} + a_{12} N^{c_{22}} + a_{13} H^{c_{33}} + b_1 \end{cases}$$

Radiata pine in New Zealand (García, 1984)

t scaled by a site quality parameter

$$\frac{\mathrm{d}H^c}{\mathrm{d}t} = -\frac{1}{7}H^c + \frac{a^c}{7}, \qquad \frac{\mathrm{d}N^\gamma}{\mathrm{d}t} = -\gamma m N^\gamma.$$

$$\frac{\mathrm{d}BH}{\mathrm{d}t} = -kmBH + \beta N^\gamma + \alpha.$$

Eucalypts in Spain – closed canopy (García & Ruiz, 2002)

Special cases. Top height an independent sub-system (assumed independent of stand density, etc.)
Ideally, variables should be positive and monotonic. At least wise to avoid increasing N (credibility!)
Open question: most general conditions on the parameters for monotonic? Known for linear systems.

Parameter estimation

$$\begin{aligned} \dot{\boldsymbol{x}^{\boldsymbol{C}}} &= \boldsymbol{A}\boldsymbol{x}^{\boldsymbol{C}} + b = \boldsymbol{A}(\boldsymbol{x}^{\boldsymbol{C}} - \boldsymbol{a}) \\ &= \boldsymbol{P}^{-1}\boldsymbol{\Lambda}\boldsymbol{P}(\boldsymbol{x}^{\boldsymbol{C}} - \boldsymbol{a}) \end{aligned}$$

Let $\boldsymbol{P}(\boldsymbol{x}^{\boldsymbol{C}} - a) = \boldsymbol{z}$. Then,

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda}\boldsymbol{z} \quad \rightarrow \quad \boldsymbol{z} = \mathrm{e}^{\boldsymbol{\Lambda}(t-t_0)}\boldsymbol{z}_0$$

Stochastic differential equation:

$$\dot{\boldsymbol{z}} = \boldsymbol{\Lambda}\boldsymbol{z} + \dot{\boldsymbol{\omega}}$$

adding a Wiener (white noise) process.

Then get the prob. distribution (likelihood function), and maximize over the parameters

Cap-lambda is diagonal matrix of eigenvalues. Assume real eigenvalues (complex imply unacceptable oscillations), and distinct (not restrictive with empirical estimates).
Estimate the transformed parameters.
Actually, height sub-model is fitted separately first. Includes additional complications: plot-specific site-dependent parameters, and measurement/sampling errors.

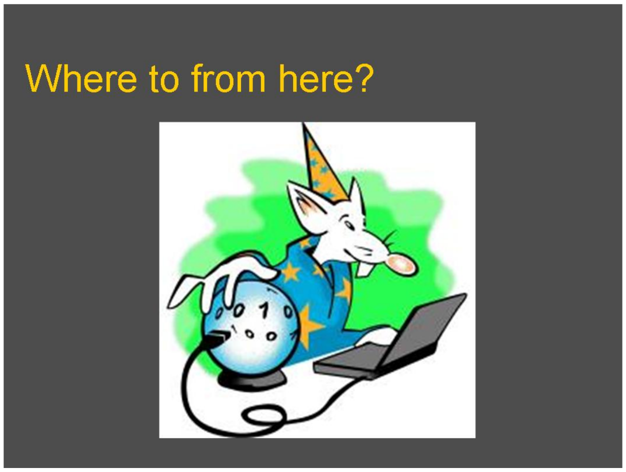
Variations / extensions

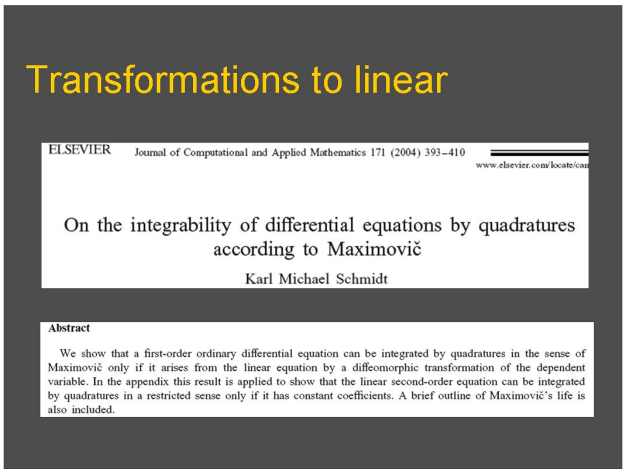
- Multipliers for site, genetic improvement
- Additional state variables: relative closure, phosphorus concentration
- Those variables in multipliers:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \varphi(R)\boldsymbol{f}(\boldsymbol{x}) \quad \rightarrow \quad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = \boldsymbol{f}(\boldsymbol{x})$$

with a “physiological time” such that

$$\varphi(R)\,\mathrm{d}t = \mathrm{d}\tau$$

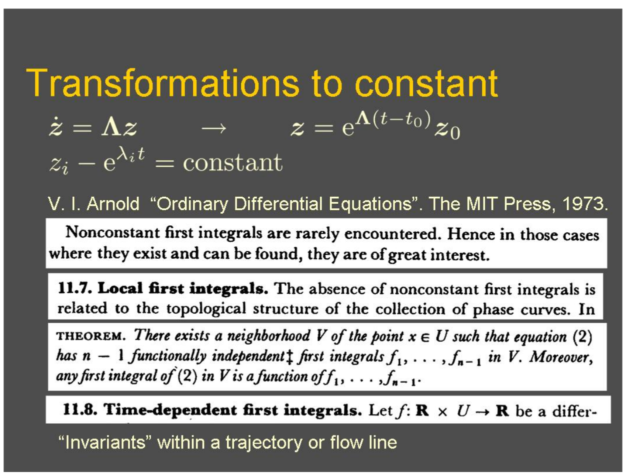




Numerical integration technically not a big problem, but "ugly". Avoid if not necessary.

19th-Century mathematician, somewhere in Eastern Europe.

Multivariate Richards: transformation to linear. Apparently, any analytically integrable system is transformable to linear.



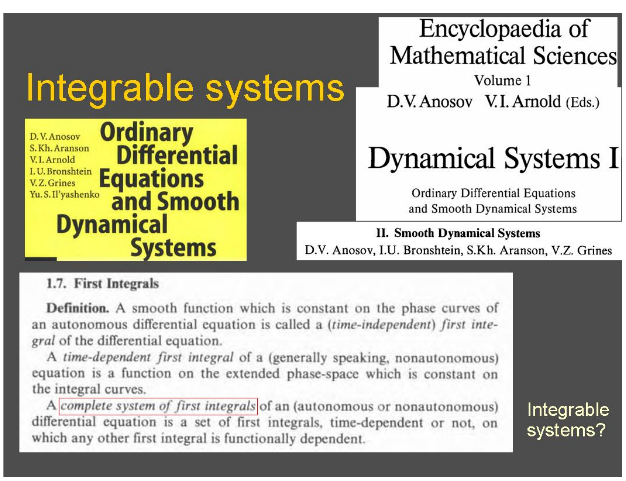
And linear (at least if "well-behaved") is transformable to constants.

Out of my depth here!

Constants: (local, time-dependent?) first integrals. Call them "invariants".

t in linear can be eliminated, to get n-1 time-invariant and one time-variant (local?) first integrals.

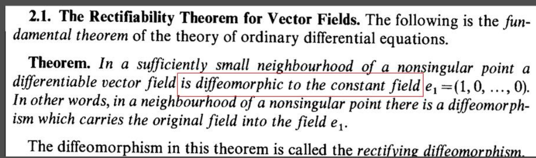
More on Arnold's book later. Like better the older MIT edition; the newer one by Springer contains extra material apparently added due to curriculum requirements.



More on definitions, etc.

(Book essentially the same as a chapter in collection).

Integrability


2.1. The Rectifiability Theorem for Vector Fields. The following is the *fundamental theorem* of the theory of ordinary differential equations.

Theorem. *In a sufficiently small neighbourhood of a nonsingular point a differentiable vector field is diffeomorphic to the constant field* $e_1 = (1, 0, \ldots, 0)$. *In other words, in a neighbourhood of a nonsingular point there is a diffeomorphism which carries the original field into the field* e_1.

The diffeomorphism in this theorem is called the *rectifying diffeomorphism.*

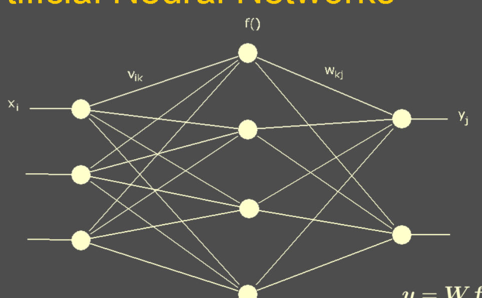
Diffeomorphic to a constant field <=> Integrable?

Modelling

- Assumption: For a "wide enough" class of systems there exists a smooth one-to-one transformation of the n state variables into n independent invariants
- Model (approximate) these transformations
- "Automatic" ways of doing this?

Modelling by empirical mapping into invariants?

Artificial Neural Networks



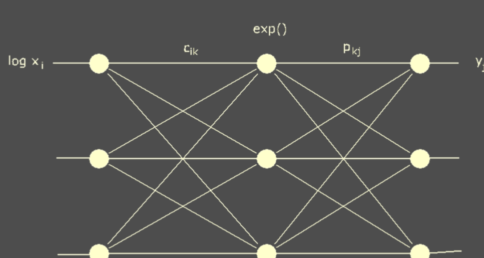
$$y = Wf(Vx)$$

Problem: Not one-to-one

Popular way of modelling empirical transformations. Much hyped, picturesque language and heuristics. Can be interpreted in more conventional ways, and some features may actually be useful.

f() is some nonlinear scalar function, commonly a logistic. V and W matrices of parameters to be estimated.

The multivariate Richards network

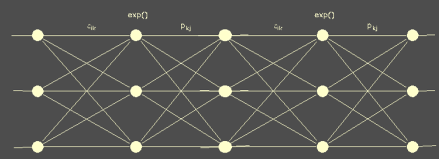


$$y = P\exp(C \ln x) = Px^C$$

One-to-one if the number of nodes is the same in all layers. Of course, that way we loose flexibility.

The multi-Richards happens to correspond to one of these.

The multivariate Richards network



- Estimation
  - Regularization, penalize overparameterization
  - “Pruning”

Flexibility could be restored by increasing the number of layers (e.g., iterated multi-Richards).

AAN techniques to fight overparameterization: (a) Regularization, similar to penalized maximum-likelihood in Statistics. (b) Better but more difficult, force sparsity, i.e., zero coefficients.

Integration

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

⇓ No $\boldsymbol{u}$

$$\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}_0, t - t_0)$$

(Global transition function)

Group:

$$\boldsymbol{F}[\boldsymbol{F}(\boldsymbol{x}_0, s), t] = \boldsymbol{F}(\boldsymbol{x}_0, s + t)$$

$$\boldsymbol{F}(\boldsymbol{x}_0, 0) = \boldsymbol{x}_0$$

Conventional ODE modelling. Transition function with the required properties obtained by integration.

Modelling the global T.F. (flow)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

⇑ No $\boldsymbol{u}$

$$\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}_0, t - t_0)$$

(Global transition function)

Group:

$$\boldsymbol{F}[\boldsymbol{F}(\boldsymbol{x}_0, s), t] = \boldsymbol{F}(\boldsymbol{x}_0, s + t)$$

$$\boldsymbol{F}(\boldsymbol{x}_0, 0) = \boldsymbol{x}_0$$

Possible alternative: model the transition. Obtain the ODE (if needed) by differentiation.

Arnold

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$$

⇑ No $\boldsymbol{u}$

$$\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}_0, t - t_0)$$

“Phase flow”

(Global transition function)

Group:

$$\boldsymbol{F}[\boldsymbol{F}(\boldsymbol{x}_0, s), t] = \boldsymbol{F}(\boldsymbol{x}_0, s + t)$$

$$\boldsymbol{F}(\boldsymbol{x}_0, 0) = \boldsymbol{x}_0$$

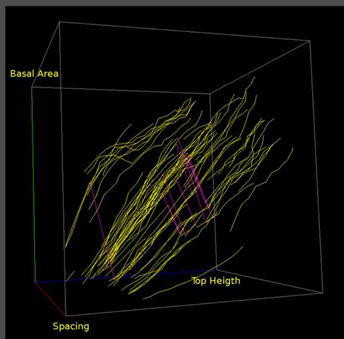
“one-parameter group of transformations”

In fact, Arnold’s book starts with “phase flows” as the more primitive concept (phase is physicists jargon for state).

A proper state flow (one-parameter group of transformations) always generates an ODE system. On the other hand, only ODES satisfying certain conditions generate flows.

Lie groups?

3-D



Stand trajectories as streamlines in a flow.

In forest modelling...

- “Algebraic difference equations”, “Self-referencing functions”
- Examples (*A* = age) :

$\ln N_2 = \ln N_1 - 0.005602(A_2^{1.3334} - A_1^{1.3334})$

$\rightarrow \quad \ln N + 0.005602\, A^{1.3334} = \text{constant}$

Clutter *et al* (1983)

$H_2 = a(H_1/a)^{(A_1/A_2)^m}$

$\rightarrow \quad A^m \ln(H/a) = \text{constant}$

Tomé *et al* (2006)

- 1-D. Often confuse integration constants with site-dependent parameters
- But, perhaps it makes sense, after all?

A somewhat mysterious paradigm that keeps popping up in forest models. Apparently introduced in the late 60’s by J.L.Clutter (my supervisor at UGA).

In two variables, usually one of them age.

Frequently used in site index (height growth) models, where much confusion has arisen from taking as the equivalent of integration constants what are really site-dependent parameters.

Their derivation is never very clear, involving more or less ingenious manipulations to ensure “path-invariance” (the group properties). Basically, equate an invariant evaluated at two points in time.

An extension to more than two variables would be similar to the invariant mapping approach.

Conclusions / Summary

- Dynamical modelling with ODEs seem natural, although it is rare in forestry
- Multivariate Richards, an example of transformation to linear ODEs, or to invariants
- More general empirical transformations to invariants: ANN, etc.
- Modelling the invariants themselves, rather than ODEs