The Theory of Judgment and its Application to Behavioral Finance

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Abstract

Returns of stock prices often exhibit cycles of momentum and reversal. During various phases of the cycles, trading volumes and trading behaviors of investors of different sizes often show distinct characteristics. It has been a long standing challenge to describe the multiple patterns simultaneously from a quantitative theory. In this paper, we present the theory of judgment, which provides a common framework to integrate behavioral and informational theories of investment. The theory of judgment provides a quantitative link between investors' judgment and their trading activities. As an application, a simple mathematical model based on the theory of judgment is constructed. The predictions derived from the model are consistent with the multiple empirical patterns of trading volumes and investor activities at the different phases of the cycle of momentum and reversal.

Initial draft. Comments welcome.

1. Introduction

Returns of stock prices often exhibit cycles of momentum and reversal. During various phases of the cycles, trading volumes and trading behaviors of investors of different sizes often show distinct characteristics. It has been a long standing challenge to describe the multiple patterns simultaneously from a quantitative theory (Lee and Swaminathan, 2000; Hvidkjaer 2006). In this paper, we present an updated version of a mathematical theory on the value and bias of judgment, which provides a common framework to integrate behavioral and informational theories of investment. The theory of judgment provides a quantitative link between investors' judgment and their trading activities. As an application, a simple mathematical model based on the theory of judgment is constructed. The predictions derived from the model are consistent with the multiple empirical patterns of trading volumes and investor activities at the different phases of the cycle of momentum and reversal.

The theory of judgment is an extension from the information theory. In real life, people have to make subjective assessment of events without possessing complete information. The theory of judgment provides a measure to value one's judgment. The valuation of a judgment is against a reference state, which is usually taken to be the maximum entropy equilibrium state (Jaynes, 1988). Since no additional information is required to determine the equilibrium state, the value of judgment from the decision making perspective can be naturally measured against the equilibrium state. However, the reference state can be a non-equilibrium steady state, such as a bubble state. Intuitively, if one buys a stock at two dollars and the equilibrium price is five dollars, then the value of your buying is three dollars. However, if the stock price can be momentarily moved to six dollars and you can take advantage of this high price, then the value of your buying is four dollars. Mathematically, the value of judgment is the average of profit or loss under different scenarios, which can be represented by a function generalized from relative entropy.

The value of judgment is always lower than or equal to the value of information with the same objective probability distribution. The value of judgment is equal to the value of information only when the subjective assessment of the probability distribution is identical to the objective probability distribution. Therefore, the concept of judgment is a generalization from the concept of information when a person does not have precise estimation of a random event, which is the case in most decision making processes. The difference between the values of judgment and information is bias, which is defined by a mathematical function called relative entropy. Entropy and relative entropy are the two most important functions in information theory and statistical mechanics (Schlögl, 1989; Cover and Thomas, 2005; Qian, 2009). Unlike the value of information, which is always positive, the value of judgment can be either positive or negative. This means that the value of active trading by investors can be either positive or negative. Trading that earn positive returns are generally attributed to information while trading that earn negative returns are generally attributed to behavioral biases. From the theory of judgment, the same judgment will have different values at different times due to changes of environmental conditions. Empirical evidences show that small individual investors often execute trades similar to those by large institutional investors but at a later stage. This could due to behavioral biases, or due to the difficulty of small investors to obtain timely information.

Under certain conditions, a judgment that is more biased may be more valuable than a less unbiased judgment. Intuitively speaking, an investor who is modestly favorable to a stock which turns out to earn very high rate of return will perform better than an investor who is modestly favorable to a stock which turns out to earn moderately high rate of return. This shows that value and bias of judgment are two distinct concepts. It will help clarify discussion in behavioral literature, which often identifies bias with low value of judgment. The theory of judgment bridges the chasm between the concept of information and cognitive bias. This will help provides a common framework for behavioral and informational perspectives in understanding financial market.

Investment decisions are made according to investors' judgment about returns of different assets. To establish a precise link between investors' judgment and investment return, we consider a simple market with only two assets: a risk free asset and a risky asset. Based on the subjective assessment of the return distribution of the risky asset, an investor can determine the optimal portion of the risky asset in the portfolio and calculate the expected rate of return of this portfolio. We prove that the first order approximation of the expected rate of return of the portfolios constructed from a judgment is equal to the value of the same judgment. Therefore, the theory of judgment provides a quantitative link between the value of a judgment and the expected rate of return of the portfolio constructed from the same judgment. In a broader sense, the theory of judgment provides a link between ideas and their monetary values.

Since the judgment about a stock determines the level of holding about the stock, the change of judgment about a stock determines the volume of trading in the market, which is considered as the key ingredient missing from the asset pricing models (Banerjee and Kremer, 2010). The theory of judgment provides a simple and intuitive tool to model trading volume in the asset market.

We will apply the theory of judgment to build a model to understand multiple empirical patterns related to the cycles of momentum and reversal. A persistent pattern in the security market is the price continuation in short to medium run and the reversal of return in the long run (DeBondt and Thaler, 1985; Jegadeesh and Titman, 1993). Several models have been developed to explain this pattern (Barberis, Shleifer and Vishny,1998; Daniel, Hirshleifer and Subrahmanyam,1998; Hong and stein,1999). However, these models could not explain other patterns related to the cycles of momentum and reversal (Lee and Swaminathan, 2000; Hvidkjaer, 2006). For example, the return patterns are often accompanied by distinct patterns of trading volume. However, "existing theories of investor behavior do not fully account for all of the evidence. … none of these models incorporate trading volume explicitly and, therefore, they cannot fully explain why trading volume is able to predict the magnitude and persistence of future price momentum." (Lee and Swaminathan, 2000, p. 2066)

The value of judgment provides a quantitative measure of the level of informedness of the investors. In our model, investors are classified into three groups as large, midsized and small investors according to their wealth. The proportion of people in each group is determined by the maximum entropy principle. The level of informedness of each group of investors are determined by or positively correlated to their level of the wealth. The judgments of the investors about the future movement of the prices of the risky assets determine the level of their holdings. During the cycles of the information processing, different investors, because of their different capacity in information processing, hold different portions of the risky asset. The changes of judgment about the risky asset by differently informed investors over time determine the trading behaviors of different groups of investors, the volume of trading and the direction and magnitude of asset price changes at different phases of cycles. The theoretical predictions generated by this quantitative model is very similar to empirical patterns recorded in Lee and Swaminathan, (2000), Hvidkjaer (2006) and others.

The theory of judgment discussed in this paper is part of the entropy theory of mind (Chen, 2003, 2004, 2005, 2007, 2008). Thermodynamics and information theory were established to understand the value and cost of engineering projects and information processing. Therefore the entropy theory of mind is naturally an economic theory of mind. Instead of developing a behavioral theory of economics directly, we propose an economic theory of behavior. Then we integrate the value and cost of information processing into the overall picture in economic decision making.

The rest of the paper is organized as follows. In Section 2, we present the theory of judgment. In Section 3, we discuss how investors' judgments determine their trading decisions and the returns of their portfolios. Section 2 and 3 are updated from an earlier work (Chen, 2008). In Section 4, we will build a quantitative model based on the theory of judgment and show the predictions derived from the model are consistent with multiple empirical patterns related to the cycles of momentum and reversal. Section 5 concludes.

2. Value and Bias of Judgment

Suppose a random variable, X, has n discrete states $\{x_1, x_2, \ldots, x_n\}$, with probability $\{p_1, \ldots p_n\}$. The subjective judgment of a person may differ from the objective probability. Suppose the subjective judgment of the probability distribution is $\{q_1, \ldots q_n\}$, then the level of uncertainty of judgment on each q_i, is

$$H(q_i) = -\ln q_i \qquad \text{for} \quad 1 \le i \le n \tag{1}$$

The total uncertainty of judgment of a random event is the average of uncertainty of judgment of each state, weighted by the objective probability distribution of the random event.

$$\sum_{j=1}^{n} p_j(-\ln q_j) \qquad (2)$$

We will compare the uncertainty of the judgment against that of a reference probability distribution of a reference state. The reference state can be the equilibrium state, or a non-equilibrium steady state. Take a simple example of binary states of up and down in the stock market. Let {p, 1-p} represent the probability of up and down of market in the next period. If on average, stocks are up 55% of the time and down 45% of the time, {0.55, 0.45} represent the equilibrium state and {0.3, 0.7} represent a non-equilibrium state, which has a higher probability to go down than to go up in the next period. Suppose the reference probability distribution of a random event is $\{r_1, \ldots r_n\}$. Then the total level of uncertainty of the reference state is

$$\sum_{j=1}^{n} p_j(-\ln r_j)$$

The value of judgment can be defined as the reduction of uncertainty from the reference state, which is

$$\mathrm{V(p,q,r)} = \sum_{j=1}^{n} p_j(-\ln r_j) - \sum_{j=1}^{n} p_j(-\ln q_j) = \sum_{j=1}^{n} p_j(\ln \frac{q_j}{r_i}) \qquad (3)$$

The right hand side of Formula (3) is a function generalized from relative entropy. We will call it generalized relative entropy. When each

$$q_j = p_j, \qquad 1 \le j \le n$$

The value of judgment becomes the value of information.

$$\sum_{j=1}^{n} p_j (\ln \frac{p_j}{r_i}) \qquad (4)$$

From Gibbs inequality (Gibbs, 1902),

$$\sum_{j=1}^{n} p_j \ln(p_j) \geq \sum_{j=1}^{n} p_j \ln(q_j)$$

Therefore, the value of judgment is always less than or equal to the value of information with the same probability distribution and reference distribution.

In practice the reference probability distribution $\{r_1, \ldots r_n\}$ is often understood as the maximum entropy distribution under known constraints. When there is no known constraints, the maximum entropy distribution is {1/n, … 1/n} and Formula (3) becomes

$$V(p,q) = \sum_{j=1}^{n} p_j \ln q_j \; + \ln n \qquad (5)$$

The value of judgment can be positive or negative. This means that active trading by investors can increase or decrease the value of their investment portfolios. The distance between the objective distribution and one's judgment is the measure of bias, which can be defined as

$$B(p,q) = \sum_{j=1}^{n} p_j \ln(p_j) - \sum_{j=1}^{n} p_j \ln(q_j) \; = \; \sum_{j=1}^{n} p_j \ln \frac{p_j}{q_j} \tag{6}$$

which is the relative entropy function. It is always nonnegative and is zero if and only if each

$$q_j = p_j, \qquad\qquad 1 \le j \le n$$

In general, the bias will be smaller when q_j is closer to p_j.

For the simplicity of exposition, we will only consider events with two possible outcomes, state 1 and state 2 in remaining part of the paper. We will apply some simple calculations to illustrate the properties of the value of judgment and the measure of bias. First, we will assume the reference state is the equilibrium state, which is assumed to be {0.55, 0.45}. We begin with the calculation of the value of a judgment that is the same as the equilibrium state. From (3), the value of the judgment that agrees with the equilibrium state is

$$p \ln \frac{0.55}{0.55} + (1 - p) \ln \frac{0.45}{0.45} = 0$$

Hence the value of the judgment that agrees with the equilibrium state is zero, regardless of the actual probability distributions of the states. Intuitively speaking, an investor who agrees with the market does not believe he possess valuable information and put his money into an index fund.

Now consider two random events with different probability distributions. Assume in the first event, the objective probability of state 1 is 60% and the probability of state 2 is 40%. Someone estimates the probability of state 1 is 57.5% and the probability of state 2 is 42.5%. We assume the equilibrium state remains {0.55, 0.45}. From (3), the value of his judgment is

$$0.6\ln\frac{0.575}{0.55}+0.4\ln\frac{0.425}{0.45}=0.0038$$

From (6), the bias of this judgment is

$$0.6\ln\frac{0.6}{0.575}+0.4\ln\frac{0.4}{0.425}=0.0013$$

In the second event, the probability of state 1 is 57.5% and the probability of state 2 is 42.5%. Someone estimates the probability of state 1 is 57.5% and the probability of state 2 is 42.5%. From (3), the value of his judgment is

$$0.575\ln\frac{0.575}{0.55}+0.425\ln\frac{0.425}{0.45}=0.0013$$

while the bias of this judgment is zero. From the above calculation, we find that, under certain conditions, the judgment that is more biased turned out to be more valuable than a less biased judgment. Intuitively speaking, an investor who is modestly favorable to a stock which turns out to earn very high rate of return will perform better than an investor who is modestly favorable to a stock which turns out to earn moderately rate of return. By separating value and bias of judgment, we will be able to perform more precise analysis to investor behaviors, which we will discuss later.

Next, we will consider values of judgments when reference states are equilibrium and non-equilibrium states respectively. Intuitively, we are comparing values of investment decisions when market settles down in equilibrium state or moves into a bubble state. Suppose an investor spot a good stock with high growth potential. Assume the objective probability of this stock to move up and down is {0.6, 0.4}. The investor's own assessment of the stock is {0.575, 0.425}. We will calculate the value of his assessment if the stock settles into the equilibrium state of {0.55, 0.45} or a bubble state {0.40, 0.60}, which means that the stock will have 40% chance going up and 60% chance going down the next time period.

When the stock will settles into the equilibrium state, the value of judgment is

$$0.6\ln(\frac{0.575}{0.55}) + 0.1\ln(\frac{0.425}{0.45}) = 0.0038$$

When the stock will move into the bubble state, the value of the same judgment is

$$0.6\ln(\frac{0.575}{0.4}) + 0.4\ln(\frac{0.425}{0.6}) = 0.0798$$

The value of the judgment in a bubble state is much higher the value of the same judgment in the equilibrium state. Intuitively, investors holding shares of a stock benefit from the high stock price.

More systematic discussion on the properties of the theory of judgment can be found in Chen (2008). In the next section, we will discuss the link between investors' judgment and their trading decisions.

3. Value of Judgment and the Expected Rate of Return of Portfolios

Investment decisions are made according to investors' judgment about stocks. In this section, we will investigate the quantitative relation between judgment and the portfolios constructed from the judgment. We will consider a simple market with a risk free asset and a risky asset. The payoff of one unit risk free asset is 1+ r at the end of each time period. The payoffs of one unit risky asset can be either (1 + r)(1 + d) with probability p or (1+ r)(1– d) with probability 1-p. Investors can only assess the probabilities subjectively.

Investors aim at maximize expected geometric return (Latane and Tuttle, 1967; Fernholz, 2002; Sinn, 2003). Based on the subjective assessment of the return distribution of the risky asset, an investor determines the optimal combination of the risk free asset and the risky asset in the portfolio. Then he can calculate the expected rate of return of this portfolio. Suppose an investor assesses the return distribution of the risky asset to be {q, 1-q}. Assume the portfolio he constructed contains a portion x of risky asset and the remaining portion of 1 – x is risk free asset. The expected geometric return over the risk free rate is

$$\begin{aligned} &((1-x)(1+r)+x(1+r)(1+d))^{q}((1-x)(1+r)+x(1+r)(1-d))^{1-q}-(1+r) \\ &=((1+xd)^{q}(1-xd)^{1-q}-1\ )(1+\mathrm{r}) \qquad (7) \end{aligned}$$

To determine the value of x at which the portfolio will have the maximal rate of return, we differentiate the above formula with respect to x.

$$
\begin{aligned}
&\frac{d}{dx}((1+xd)^{q}(1-xd)^{1-q}-1) \\
&= d(1+xd)^{q-1}(1-xd)^{-q}(2q-1-xd)
\end{aligned}
$$

The above differentiation equals zero when

$$
x = \frac{2q-1}{d} \tag{8}
$$

At this value of x, the portfolio obtains the highest expected geometric return. Plug the value of x into (7), the expected rate of return is

$$
\begin{aligned}
&((1+2q-1)^{q}(1-(2q-1))^{1-q}-1)(1+r) \\
&= (2q^{q}(1-q)^{1-q}-1)(1+r)
\end{aligned}
$$

If the objective return distribution of the risky asset is {p, 1-p} instead of the subjectively assessed {q, 1-q}, the expected rate of return of the portfolio over the risk free rate is

$$
2q^{p}(1-q)^{1-p}-1 \tag{9}
$$

The first order approximation of (9) is

$$
\begin{aligned}
&\ln(2q^{p}(1-q)^{1-p}) \\
&= p\ln q + (1-p)\ln(1-q) + \ln 2
\end{aligned}
$$

Comparing the above result with (5), we find that the first order approximation of the expected rate of return of the portfolio constructed from a certain judgment is exactly equal to the value of the judgment. As the value of judgment provides a good approximation to the rate of return on investment, it can be conveniently used to understand the relation between human judgment and patterns in investment returns and stock market.

From (8), the judgment about a stock determines the level of holding about the stock. The change of judgment about a stock determines the volume of trading in the market, which is considered as the key ingredient missing from the asset pricing models (Banerjee and Kremer, 2010). The theory of judgment provides a link between investors' judgment and trading volume in the asset market, which will be applied to understand cycles of trading in the next section. In the following, we will apply the theory to calculate several numerical examples.

When interest rate is measured on the inflation adjusted basis, the risk free interest rate can be set to be zero as a good approximation to reality. The payoffs of one unit risky asset can be either 1 + d with probability p or 1– d with probability 1-p.   We can calibrate the equilibrium value of *p* and *d* with the empirical data on return and standard deviation. The arithmetic mean rate of return of the risky asset is

$$pd + (1 - p)(-d) = (2p - 1)d \tag{10}$$

and the standard deviation of the risky asset is

$$\{p[pd - (2p-1)d]^2 + (1-p)[(1-p)(-d) - (2p-1)d]^2\}^{1/2} = 2d\sqrt{p(1-p)} \tag{11}$$

respectively. Setting p = 0.55 and d = 0.25 results mean rate of return to be 2.5% and standard deviation to be 24.87%, which are similar to actual real rate of return and standard deviation of many stock markets in the world in the last ten years. With p = 0.55 and d = 0.25, the proportion of risky asset in the equilibrium portfolio, following formula (8), is

$$\frac{2p-1}{d} = \frac{2\times 0.55-1}{0.25} = 0.4$$

These numbers will be regarded as default values of an equilibrium portfolio in the next section.

Different people at different times in different places may have different opinions about the future of the stock markets. We will calculate the proportions of asset to allocate to the risky assets with different expectations. When p is equal to 57.5%, 60%, 62.5% while keeping d fixed at 0.25, the optimal allocations to risky asset are

$$\frac{2\times 0.575-1}{0.25} = 0.6$$
$$\frac{2\times 0.60-1}{0.25} = 0.8$$
$$\frac{2\times 0.625-1}{0.25} = 1.0$$

while the arithmetic means of the risky asset, following formula (10), are

$$(2\times 0.575-1)\times 0.25 = 3.75\%$$
$$(2\times 0.6-1)\times 0.25 = 5\%$$
$$(2\times 0.625-1)\times 0.25 = 6.25\%$$

The real returns of the best performing stock markets in the worlds, such as United States, over the second half of the last century are close to or above 6.25%. This justifies the standard statement of high risk, high return and the common practice of allocating most or all assets in risky securities in long term investments. However, if the future expected real returns of risky assets are lower, as some researchers have suggested, the proportions of risky assets in investment portfolios should be lower as well to achieve higher expected returns.

4. Investor Heterogeneity in Information Processing and Related Market Patterns

Empirical and theoretical investigations suggest that the heterogeneity in information processing by investors of different sizes is the main cause of many market patterns. In this section, we will apply the theory of judgment to build a simple model to study trading behaviors of the heterogeneous investors and the resulting market patterns.

First we will determine the statistical distribution of investors with different levels of wealth. From earlier studies, such as Silva and Yakovenko (2005), wealth distribution follows exponential law as a first approximation. Investors can be classified based on their wealth. Suppose each investor in group *i* has *i* unit of wealth. Since the number of investors in each group of wealth follows the exponential law, the proportion of investors with *i* unit of wealth is

$$\frac{1}{2^i}$$

Since

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$$

the proportion of investor population is normalized. The total wealth of the economy is

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

Because the investor population is normalized, the average wealth of an investor is 2.

From calculation, the Gini coefficient of this model economy is 33.3. The Gini coefficients of industrialized economies are roughly between 25 and 45. So the wealth distribution in this model economy is representative of real economies.

To further simplify discussion, we lump investors into three groups. The wealth for each member of the three groups of investors are 1, 4 and 9 respectively. The first group is the uninformed small investors; the second group is moderately informed midsized investors and the third group is the highly informed large investors. The proportions of three groups of investors are determined by maximum entropy principle (Jaynes, 1957) with the constraints on total wealth

$$\begin{aligned} p_1 + p_2 + p_3 &= 1 \\ p_1 + 4p_2 + 9p_3 &= 2 \end{aligned} \tag{12}$$

Solving the maximum entropy problem

$$\max\{-p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3)$$

subjecting to the constraints (12) gives the following answer

$$p_1 = 0.73$$
$$p_2 = 0.23$$
$$p_3 = 0.04$$

The total wealth for each group of investors are

$$w_1 = 1p_1 = 0.73$$
$$w_2 = 4p_2 = 0.95$$
$$w_3 = 9p_3 = 0.33$$

Roughly speaking, small investors with 1 unit of wealth represent individual investors. Empirical evidences show that individual investors as a group lose money from their trading activities. So we will assume this group of investors does not possess information of positive value. Midsized investors with 4 unit of wealth possess information of moderate value. Large investors with 9 unit of wealth possess information of high value. The specific values of information possessed by investors will be quantified later.

From calculations performed in the last section, the equilibrium levels of *p*, *d* and the proportion of risky asset are set to be 0.55, 0.24 and 0.4 respectively. From time to time, the price movement of the risky asset will deviate from the equilibrium level because of various reasons. For simplicity, we will assume *d* to be constant while *p* may change over time. Investors with higher wealth level can detect more valuable information. Specifically, we assume large investors with 9 unit of wealth can detect all information {p, 1-p} with *p* up to 0.6 and midsized investors with 4 unit of wealth can detect all information {p, 1-p} with *p* up to 0.575.

We follow the standard literature on the assumptions of price movement of securities. The price of the risk free asset is assumed to be constant. The price movement of the risky asset is proportional to net active trading by the investors.

Now we consider a trading process that lasts for four time periods. At the beginning of period one, the firm underlying the risky security starts a project. Investing in this security will generate payoff either 1 + d with probability 0.6 or 1– d with probability 0.4 at the end of period two, at which time the earning from the project becomes publicly known. Large investors with 9 unit of wealth detect this information and purchase additional shares of the risky security. The proportion of wealth they invest in the risky security after the purchasing, according to formula (8), is

$$\frac{2p-1}{d} = \frac{2\times 0.6-1}{0.25} = 0.8$$

Since the total wealth of this group of investors is

$$w_3 = 0.33$$

The total volume of buying, which is the new holding minus the equilibrium holding at 40%, is

$$0.33\times(0.8-0.4) = 0.13$$

The purchasing by large investor increases the price of the shares and reduces the future expected returns. When the price increases to a certain level, this security will generate payoff that is equivalent to either 1 + d with probability 0.575 or 1– d with probability 0.425 at the end of period two. This is the end of period one and the beginning of period two. In period two, midsized investors with 4 unit of wealth detect this information and purchase shares of the risky security. The proportion of wealth they invest in the risky security, according to formula (8), is

$$\frac{2p-1}{d} = \frac{2\times 0.575-1}{0.25} = 0.6$$

Since the total wealth of the midsized investors is

$$w_2 = 0.95$$

The total volume of their buying, which is the new holding minus the equilibrium holding at 40%, is

$$0.95\times(0.6-0.4) = 0.19$$

In period two, large investors with 9 unit of wealth will also keep sixty percent of their wealth in the risky asset. As a result, they will reduce the original holding. The total volume of their selling is

$$0.33\times(0.6-0.8) = -0.066$$

At the end of period two, the earning from the project becomes publicly known and the share price of the risky asset fully reflects the underlying fundamentals. Small investors with 1 unit of wealth do not possess private information. Instead, they observe the share price movement in the last two time periods and the earning announcement at the end of period two. They extrapolate the past results to the future and invest accordingly. Because the share prices have moved up steadily over the last two time periods, it will be natural for small investors to base the trading decisions on the best trading decisions from two periods earlier. Specifically, in period three, on average, small investors will allocate eighty percent of their assets in the risky security. Since total wealth for the small investors is

$$w_1 = 0.73$$

The total volume of buying by the small investors, which is the new holding minus the equilibrium holding at 40%, is

$$0.73 \times (0.8 - 0.4) = 0.29$$

Now we will consider the trading activities of large and midsized investors. Their trading decisions are based on the information they received. By default, we assume no new information in the future. In this case, the movement of stock price will return to its equilibrium condition. As a result, the holdings of the risky asset by large and midsized investors will return to the equilibrium state of forty percent. The total amount they will sell is

$$(0.33 + 0.95) \times (0.4 - 0.6) = -0.25$$

The net active trading by all investors is

$$0.29 - 0.25 = 0.04$$

Because of the small net active trading in period three, the price movement in this period is moderate. However, trading at the beginning of period three could be dominated by small investors who mainly depend on easy to understand information, such as earning data, which is distributed widely to the general public at very narrow time frames. So trading by small investors is highly correlated (Barber, Odean and Zhu, 2009b). Trading decisions by large and mid size investors depend more on intangible information, which is the main determinant of future returns (Daniel and Titman, 2006). But intangible information is less precisely defined and trading activities generated by intangible information is less concentrated. This means that the beginning of the period three is marked by rise of asset prices while prices decline over the rest of period three. This is consistent with the empirical evidence (Hvidkjaer 2008; Barber, Odean, and Zhu, 2009a, 2009b).

In period four, most relevant information has been acted upon and share price will finally reach equilibrium. Since share price at the end of period two has already fully reflect the fundamentals, the expected price level at the end of period four will be equal to the share price at the end of period two. Therefore, the combined net active trading of period three and four should be zero, which means that the net active trading in period four should be -0.04. Since large and midsized investors already balanced their portfolio to equilibrium state in period three, the active trading is mainly generated by small investors who are reducing their holding from last period's buying. As there is little new information to generate extra trading, the total trading can be approximated by the net active trading.

We can summarize the trading activities in the four time periods into the following table:

	Period one	Period two	Period three	Period four
Net trading	0.13	0.12	0.04	-0.04
Trading volume	0.13	0.19	0.29	0.04

The average net trading of the four periods is

$$\frac{1}{4}(0.13+0.12+0.04-0.04)=0.06$$

Since the net trading of the first two periods are higher than the average, share prices increase in the first two periods are higher than the average. They are the winner periods. In the last two periods, the net trading is lower than the average. Share prices change in the last two periods are lower than the average. They are the loser periods. Among the winner periods, the trading volume of the first period is lower than that of the second period. Among the loser periods, the trading volume of the fourth period is lower than that of the third period. The four trading periods can be summarized as

Period one	Period two	Period three	Period four
Low volume winner	High volume winner	High volume loser	Low volume loser

This is exactly the same as the empirical pattern documented in Lee and Swaminathan (2000), which they call momentum life cycle.

Hvidkjaer (2006) examined the trading behaviors of investors of different sizes at the stages of low volume winner, high volume winner, high volume loser and low volume loser. He inferred the background of investors from the sizes of the trades. In his classification, large trades are two times or more as large as small trades. In our model, the midsized and large investors are four and nine times larger than the small investors. So it is natural to merge the midsized and large investors into one group as large investors when comparing our theoretical predictions to the empirical results discussed in Hvidkjaer (2006). We will examine how trading patterns predicted from our model correspond to empirical patterns. The clearest resemblance between the predictions of model and the empirical patterns occur in high volume loser stage. From Figure 2 of Hvidkjaer (2006), small investors are active buyers while large investors are active sellers in this stage, which is exactly what the model has predicted. Our results are also consistent with Feng and Seasholes (2004), who showed that informed investors are selling while uninformed investors are buying after information release. In the low volume loser stage, from Figure 3 of Hvidkjaer (2006), small investors are more active sellers than large investors. In the low volume and high volume winner stages, from Figure 2 and 3 of Hvidkjaer (2006), large investors are more active buyers than small investors. If we interpret trading activities calculated from our model as dominate activities instead of all activities, the predictions of our model during these stages are consistent with the empirical patterns.

Alternatively, we can refine the model to make it more realistic. We had assumed the level of informedness of an investor is determined only by his wealth. To be more consistent with reality, we now assume the level of informedness of an investor is highly correlated but not determined by his wealth. Specifically, the correlation between wealth and level of informedness is represented by the following matrix

	0.55	0.575	0.60
1	0.7	0.2	0.1
2	0.15	0.7	0.15
3	0.1	0.2	0.7

This means that among investors with one unit of wealth, 70%, are uninformed, as 0.55 is the equilibrium state, 20% are informed at the level of 0.575 and 10% are informed at the level of 0.60. The level of informedness of group two and three investors can be understood similarly.

We make a further refinement about the informedness of investors who can detect the information {p, 1-p} with *p* up to 0.6. We will assume these investors detect the information but do not interpret the information precisely. To be more specific, these investors make a judgment that *p* is equal to 0.575 instead of 0.6. This is very natural since most investors underestimate the significant of new information. Note that the judgment of this group of investors is still more valuable than the group of investors who estimate *p* to be 0.575 when it is actually 0.575. With the refined model, we can recalculate the trading activities following the same procedure as before. But this time we will measure the trading activities of large investors and small investors separately. The calculation of net trading by small and large investors at period four is determined by the proportional holdings of small and large investors at the end of period three. The results are shown in the following table.

	Period 1	Period 2	Period 3	Period 4
small investor net trading	0.0145	0.0291	0.1600	-0.0076
large investor net trading	0.0742	0.1455	-0.1498	-0.0026

total net trading	0.0887	0.1746	0.0102	-0.0102
trading volume	0.0887	0.1746	0.2735	0.0102

We can find that the trading patterns of small and large investors calculated in each period are qualitatively similar to the empirical patterns recorded in Figure 2 and 3 of Hvidkjaer (2006). However, we would not expect the patterns predicted from our model to be identical to empirical patterns collected in the literature. Our model presents an investment cycle initiated by a positive information signal. The empirical patterns are combinations of all kinds of cycle and non-cycle activities. For example, when the news is negative, a similar pattern exists at opposite directions. Different cycles have different amplitudes and length. In the future, we may conduct empirical investigations by filtering out different cycle and non-cycle components. This could help detect investment strategies with high level of returns.

This theory of judgment based model captured many stylized patterns of trading activities during the momentum reversal cycle. The mathematics involved is vey simple and the intuition from the model is very clear. However, it is still in an early stage of development. Many refinements can be made in the future, some of which are listed as follows.

First, relation between earning momentum and price momentum can be added into the model. Empirical evidence shows strong relation between earning and price movement (Lee and Swaminathan, 2000; Chordia and Shivakumar, 2006; Chen, Moise and Zhao, 2009). By modeling earning process over several periods of time, we can further clarify the trading mechanisms of small investors. If earning trends last longer, small investors, as well as other investors, will be more confident that the momentum will continue.

Second, informed investors can anticipate and influence the trading activities of uninformed investors. When informed investors can anticipate the trading behaviors of uninformed investors, they may base their trading decisions not only on the fundamental information, but also on the trading activities of uninformed investors. Informed investors, who usually have strong track records and are major stakeholders of publicly listed companies, can also influence or inform uninformed investors in certain ways to alter the trajectories of price movement to benefit themselves. These investor activities may be captured by more refined models.

5. Concluding Remarks

Many competing theories have been proposed to understand financial anomalies (Brav and Heaton, 2002). Comparing with other quantitative theories in behavioral finance, the theory of judgment is simple, natural and intuitive. The mathematical tools of the theory of judgment only involve simple algebraic functions such as logarithm functions and occasional use of calculus. The theory of judgment is a natural extension from the entropy theory of information and statistical mechanics. The links between investors' information processing and trading decisions are very intuitive under the theory of judgment.

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