On the Inconsistency between CAPM and the Utility Theory

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This version: February, 2019

Abstract

CAPM assumes investors maximize their utility. We construct a simple asset world with one risk free asset and two risky assets. We derive the expected return and variance of each asset with log utility function. The results are different from CAPM. Specifically, the expected return of asset with low (high) beta is higher (lower) than predicted by CAPM. The results obtained from the utility theory is more consistent with empirical data than CAPM.

All the calculations and tables in this paper are contained in an Excel file, which can be downloaded at http://web.unbc.ca/~chenj/papers/CAPM.xlsx

1. Introduction

Capital asset pricing model (CAPM) was developed more than half century ago (Sharpe, 1964; Lintner, 1965). It has since become the foundation of investment theory. CAPM is an important tool in investment and corporate finance. CAPM, in its original form, is very elegant. However, CAPM fits empirical data poorly (Fama and French, 2004). Many elaborate modifications of CAPM have been proposed. With increasing complexity of these modifications, it is time to reexamine the theoretical foundation of CAPM.

Return distribution of assets is ultimately determined by the trading decisions of the investors. People trade to maximize expected gains under uncertainty, which are measured by utility functions. In developing CAPM, Sharpe (1964) assumed investors maximizing their utility. He made extensive discussion on how investor preference affect return distribution of assets. But in formulating the quantitative theory of CAPM in the final section of his paper, he did not make use of investor utility. Indeed, he didn't need investor preference in formulating CAPM.

In reality, investor preferences affect asset pricing (Mehra and Prescott, 1985). We will directly examine how asset allocation decisions by investors affect asset prices when investors maximize their utility function. We construct a simple market with two risky assets and one risk free asset. The distributions of the final payoffs of the two risky assets are given. The investors are assumed to maximize logarithmic utility function. We apply the method of Lagrange multipliers to obtain the prices of different assets. With asset prices, the expected rates of return and variances can be calculated. We showed that, utility theory, by itself, can determine expected rate of returns and variances of assets.

In effect, CAPM and utility theory are two asset pricing theories that can determine the return distributions of assets. Do two models provide the same theoretical predictions? We show that the results from utility theory differ from that from CAPM. Specifically, the expected return of the asset with small (large) beta calculated from the utility theory is higher (lower) than that from CAPM, and the expected return of the asset of small (large) size is higher (lower) than that from CAPM. These theoretical calculations from utility theory are more consistent with empirical data (Black, Jensen and Scholes, 1972; Fama and French, 1993).

On the technical level, we show that the method of Lagrange multipliers, when applied to utility theory, can yield a great amount of information on asset pricing. For example, we can explicitly calculate how the amount of supply of one asset affects the pricing of all assets quantitatively. The results are consistent with intuition. Many detailed properties about the return distributions of the assets can be derived from the utility theory. These properties can be tested empirically.

The results obtained in this paper can be understood from another perspective. In Figure 6 of Sharpe (1964), many, instead of single, portfolios lie on the capital market line. In Note 18 of the paper, Sharpe further stated,

> “The area in Figure 6 representing E_R and σ_R values attained with only risky assets has been drawn at some distance from the horizontal axis for emphasis. It is likely that a more accurate representation would place it very close to the axis.” (Sharpe, 1964, p. 435)

Short term debt markets are very liquid. The short term debt instruments, and hence the risky assets, form a continuous spectrum to the risk free asset (Chen, 2010). When the efficient frontier of the risky assets form a continuous spectrum to the risky free asset, as it is the case in the actual market, the derivation of CAPM model presented in the last section of Sharpe is no more valid. Hence we should not expect it to provide good approximation to the market data.

2. The Derivation of main results

In a simple market, there are three assets. One is risk free asset. The other two assets are risky asset. The risk free asset provides a payoff of B at the end of one unit of time. The first risky asset, S, has two potential payoffs, S_1, S_2, each with probability 50%. The second risky asset, T, has two potential payoffs, T_1, T_2, each with probability 50%. We further assume the payoffs of S and T are independent. Together, there are four potential payoff outcomes, (S_1, T_1), (S_1, T_2), (S_2, T_1), (S_2, T_2), each with a probability of 25%. Suppose the prices of B, S and T are m, x and y. The goal of an investor is to maximize his expected geometric average of the investment value at the end of the time period. Equivalently, he attempts to maximize the lognormal utility of his wealth.

Without loss of generality, we assume the investor has one dollar to invest. The amounts the investor will invest in B, S and T are P_0, P_1 and P_2 respectively.

$$P_0+P_1+P_2 = 1$$

When the final values of S and T are S_1 and T_1, the final portfolio value of the investor will be

$$P_0\frac{B}{m} + P_1\frac{S_1}{x} + P_2\frac{T_1}{y}$$

When the final values of S and T are S_1 and T_2, the final portfolio value of the investor will be

$$P_0\frac{B}{m} + P_1\frac{S_1}{x} + P_2\frac{T_2}{y}$$

When the final values of S and T are S_2 and T_1, the final portfolio value of the investor will be

$$P_0\frac{B}{m} + P_1\frac{S_2}{x} + P_2\frac{T_1}{y}$$

When the final values of S and T are S_2 and T_2, the final portfolio value of the investor will be

$$P_0\frac{B}{m}+P_1\frac{S_2}{x}+P_2\frac{T_2}{y}$$

Since we will use above four functions repeatedly, we give them shorter symbols $1'$, $2'$, $3'$, and $4'$ to represent them. The goal of the investor is to maximize the geometric mean of his wealth, or

$$\{(P_0\frac{B}{m}+P_1\frac{S_1}{x}+P_2\frac{T_1}{y})(P_0\frac{B}{m}+P_1\frac{S_1}{x}+P_2\frac{T_2}{y})(P_0\frac{B}{m}+P_1\frac{S_2}{x}+P_2\frac{T_1}{y})(P_0\frac{B}{m}+P_1\frac{S_2}{x}+P_2\frac{T_2}{y})\}^{\frac{1}{4}}$$

To simplify the calculation, we will attempt to maximize the equivalent form by removing the ¼ power in the above expression,

$$(P_0\frac{B}{m}+P_1\frac{S_1}{x}+P_2\frac{T_1}{y})\left(P_0\frac{B}{m}+P_1\frac{S_1}{x}+P_2\frac{T_2}{y}\right)\left(P_0\frac{B}{m}+P_1\frac{S_2}{x}+P_2\frac{T_1}{y}\right)\left(P_0\frac{B}{m}+P_1\frac{S_2}{x}+P_2\frac{T_2}{y}\right) \quad (1)$$

Or a shorter form

$$1' * 2' * 3' * 4'$$

where

$$1' = P_0\frac{B}{m}+P_1\frac{S_1}{x}+P_2\frac{T_1}{y} \quad (2)$$

$$2' = P_0\frac{B}{m}+P_1\frac{S_1}{x}+P_2\frac{T_2}{y} \quad (3)$$

$$3' = P_0\frac{B}{m}+P_1\frac{S_2}{x}+P_2\frac{T_1}{y} \quad (4)$$

$$4' = P_0 \frac{B}{m} + P_1 \frac{S_2}{x} + P_2 \frac{T_2}{y} \tag{5}$$

The maximization is subject to constraints

$$P_0+P_1+P_2 = 1 \tag{6}$$

and

$$m+x+y = C \tag{7}$$

where C is a constant representing the total amount of investment money in the asset market.

Applying the Lagrange multiplier method, we need to maximize

$$f= 1' * 2' *3' *4' - u(P_0+P_1+P_2 - 1) - v(m+x+y - C) \tag{8}$$

subjecting to the constraint of (6) and (7). Here $1'$, $2'$, $3'$ and $4'$ are defined in (2), (3), (4), (5). u and v are parameters.

Differentiating the above function respect to P_0, P_1, P_2, m, x, y, and setting them to zero, we get the following six equations

$$\frac{\partial f}{\partial P_0} = \frac{B}{m} 2'3'4' + \frac{B}{m} 1'3'4' + \frac{B}{m} 1'2'4' + \frac{B}{m} 1'2'3' - u = 0$$

$$\frac{\partial f}{\partial P_1} = \frac{S_1}{x} 2'3'4' + \frac{S_1}{x} 1'3'4' + \frac{S_2}{x} 1'2'4' + \frac{S_2}{x} 1'2'3' - u = 0$$

$$\frac{\partial f}{\partial P_2} = \frac{T_1}{y} 2'3'4' + \frac{T_2}{y} 1'3'4' + \frac{T_1}{y} 1'2'4' + \frac{T_2}{y} 1'2'3' - u = 0$$

$$\frac{\partial f}{\partial m} = -\frac{P_0 B}{m^2} 2'3'4' - \frac{P_0 B}{m^2} 1'3'4' - \frac{P_0 B}{m^2} 1'2'4' - \frac{P_0 B}{m^2} 1'2'3' - v = 0$$

$$\frac{\partial f}{\partial x} = -\frac{P_1 S_1}{x^2} 2'3'4' - \frac{P_1 S_1}{x^2} 1'3'4' - \frac{P_1 S_2}{x^2} 1'2'4' - \frac{P_1 S_2}{x^2} 1'2'3' - v = 0$$

$$\frac{\partial f}{\partial y} = -\frac{P_2 T_1}{y^2} 2'3'4' - \frac{P_2 T_2}{y^2} 1'3'4' - \frac{P_2 T_1}{y^2} 1'2'4' - \frac{P_2 T_2}{y^2} 1'2'3' - v = 0$$

The above six equations can be rearranged into

$$\frac{B}{m} 2'3'4' + \frac{B}{m} 1'3'4' + \frac{B}{m} 1'2'4' + \frac{B}{m} 1'2'3' = u \tag{9}$$

$$\frac{S_1}{x} 2'3'4' + \frac{S_1}{x} 1'3'4' + \frac{S_2}{x} 1'2'4' + \frac{S_2}{x} 1'2'3' = u \tag{10}$$

$$\frac{T_1}{y} 2'3'4' + \frac{T_2}{y} 1'3'4' + \frac{T_1}{y} 1'2'4' + \frac{T_2}{y} 1'2'3' = u \tag{11}$$

$$-\frac{P_0}{m}\left(\frac{B}{m} 2'3'4' + \frac{B}{m} 1'3'4' + \frac{B}{m} 1'2'4' + \frac{B}{m} 1'2'3'\right) = v \tag{12}$$

$$-\frac{P_1}{x}\left(\frac{S_1}{x} 2'3'4' + \frac{S_1}{x} 1'3'4' + \frac{S_2}{x} 1'2'4' + \frac{S_2}{x} 1'2'3'\right) = v \tag{13}$$

$$-\frac{P_2}{y}\left(\frac{T_1}{y} 2'3'4' + \frac{T_2}{y} 1'3'4' + \frac{T_1}{y} 1'2'4' + \frac{T_2}{y} 1'2'3'\right) = v \tag{14}$$

Dividing equation (12) to equation (9), equation (13) to equation (10), equation (14) to equation (11), we obtain,

$$\frac{P_0}{m} = \frac{P_1}{x} = \frac{P_2}{y} = -\frac{v}{u} \tag{15}$$

Adding all the denominators and numerator in the first three terms, we obtain, from formula (6) and (7)

$$\frac{P_0}{m} = \frac{P_1}{x} = \frac{P_2}{y} = \frac{P_0+P_1+P_2}{m+x+y} = \frac{1}{C} \tag{16}$$

This means that to obtain highest geometric return, the investment portfolio should be proportional to the market portfolio. This is similar but not identical to CAPM, in which proportionality only applies to risky assets.

From equation (16), formulas (2), (3), (4), (5) can be simplified into

$$1' = P_0\frac{B}{m} + P_1\frac{S_1}{x} + P_2\frac{T_1}{y} = \frac{1}{C}(B + S_1 + T_1) \tag{17}$$

$$2' = P_0\frac{B}{m} + P_1\frac{S_1}{x} + P_2\frac{T_2}{y} = \frac{1}{C}(B + S_1 + T_2) \tag{18}$$

$$3' = P_0\frac{B}{m} + P_1\frac{S_2}{x} + P_2\frac{T_1}{y} = \frac{1}{C}(B + S_2 + T_1) \tag{19}$$

$$4' = P_0\frac{B}{m} + P_1\frac{S_2}{x} + P_2\frac{T_2}{y} = \frac{1}{C}(B + S_2 + T_2) \tag{20}$$

We can calculate the ratio of y/x, the ratio of prices of S and T, to see how it is determined. From equations (10), (11),

$$\frac{S_1}{x}2'3'4' + \frac{S_1}{x}1'3'4' + \frac{S_2}{x}1'2'4' + \frac{S_2}{x}1'2'3' = \frac{T_1}{y}2'3'4' + \frac{T_2}{y}1'3'4' + \frac{T_1}{y}1'2'4' + \frac{T_2}{y}1'2'3'$$

Or

$$\frac{1}{x}(S_1 2'3'4' + S_1 1'3'4' + S_2 1'2'4' + S_2 1'2'3')$$

$$= \frac{1}{y}(T_1 2'3'4' + T_2 1'3'4' + T_1 1'2'4' + T_2 1'2'3')$$

So

$$\frac{y}{x} = \frac{(T_1 2'3'4' + T_2 1'3'4' + T_1 1'2'4' + T_2 1'2'3')}{(S_1 2'3'4' + S_1 1'3'4' + S_2 1'2'4' + S_2 1'2'3')} \tag{21}$$

Similarly, from equations (9), (10),

$$\frac{x}{m} = \frac{(S_1 2'3'4' + S_1 1'3'4' + S_2 1'2'4' + S_2 1'2'3')}{B(2'3'4' + 1'3'4' + 1'2'4' + 1'2'3')} \tag{22}$$

and from equations (9), (11),

$$\frac{y}{m} = \frac{T_1 2'3'4' + T_2 1'3'4' + T_1 1'2'4' + T_2 1'2'3'}{B(2'3'4' + 1'3'4' + 1'2'4' + 1'2'3')} \tag{23}$$

Next we will calculate the rate of returns of the two risky assets and check whether they are consistent with CAPM theory.

The return of the risk free asset is B/m – 1.

The expected return (arithmetic return) of the first risky asset is ½(S_1+S_2)/x – 1.

The expected return (arithmetic return) of the second risky asset is $\frac{1}{2}(T_1+T_2)/y - 1$.

The standard deviation of the first risky asset is $\frac{1}{2}(S_1-S_2)/x$.

The standard deviation of the second risky asset is $\frac{1}{2}(T_1-T_2)/y$.

The expected return (arithmetic return) of S+T, the portfolio of two risky asset is

$$\frac{x}{x+y}\left(\frac{\frac{(S_1+S_2)}{2}}{x}-1\right)+\frac{y}{x+y}\left(\frac{\frac{(T_1+T_2)}{2}}{y}-1\right)$$

$$=\frac{\frac{(S_1+S_2)}{2}+\frac{(T_1+T_2)}{2}}{x+y}-1 \tag{24}$$

The variance of S+T, the portfolio of two risky assets is

$$(\frac{x}{x+y})^2\left(\frac{\frac{(S_1-S_2)}{2}}{x}\right)^2+(\frac{y}{x+y})^2\left(\frac{\frac{(T_1-T_2)}{2}}{y}\right)^2$$

$$=\frac{\frac{1}{4}(S_1-S_2)^2+\frac{1}{4}(T_1-T_2)^2}{(x+y)^2} \tag{25}$$

Covariance of S, the first risky asset and S+T, the portfolio of two risky assets is

$$\frac{x}{x+y}\left(\frac{\frac{(S_1-S_2)}{2}}{x}\right)^2=\frac{(\frac{S_1-S_2}{2})^2}{x(x+y)} \tag{26}$$

Covariance of T, the second risky asset and S+T, the portfolio of two risky assets is

$$\frac{y}{x+y}\left(\frac{\frac{(T_1-T_2)}{2}}{y}\right)^2=\frac{(\frac{T_1-T_2}{2})^2}{y(x+y)} \tag{27}$$

β of S and T can be calculated from (25), (26) and (27) as

$$\beta_S = \frac{cov(S,S+T)}{var(S+T)} = \frac{\frac{(\frac{S_1-S_2}{2})^2}{x(x+y)}}{\frac{\frac{1}{4}(S_1-S_2)^2+\frac{1}{4}(T_1-T_2)^2}{(x+y)^2}} = \frac{x+y}{x}\frac{(S_1-S_2)^2}{(S_1-S_2)^2+(T_1-T_2)^2} \tag{28}$$

$$\beta_T = \frac{cov(T,S+T)}{var(S+T)} = \frac{\frac{(\frac{T_1-T_2}{2})^2}{y(x+y)}}{\frac{\frac{1}{4}(S_1-S_2)^2+\frac{1}{4}(T_1-T_2)^2}{(x+y)^2}}$$

$$= \frac{x+y}{y}\frac{(T_1-T_2)^2}{(S_1-S_2)^2+(T_1-T_2)^2} \tag{29}$$

where the ratio x/y can be determined from formula (21).

S+T forms the whole market of risky assets. According to CAPM,

$$R(S) - R(B) - \frac{cov(S,S+T)}{var(S+T)}\big(R(S+T) - R(B)\big) \tag{30}$$

and

$$R(T) - R(B) - \frac{cov(T,S+T)}{var(S+T)}\big(R(S+T) - R(B)\big) \tag{31}$$

should be zero, where R() is the expected rate of return of an asset. We will calculate the values of (30) and (31).

$$R(S) - R(B) - \frac{cov(S,S+T)}{var(S+T)}\big(R(S+T) - R(B)\big)$$

$$= \left(\frac{\frac{(S_1 + S_2)}{2}}{x} - 1 \right) - \left(\frac{B}{m} - 1 \right)$$

$$- \frac{\frac{x}{x+y} \left(\frac{\frac{(S_1 - S_2)}{2}}{x} \right)^2}{\frac{\frac{1}{4}(S_1 - S_2)^2 + \frac{1}{4}(T_1 - T_2)^2}{(x+y)^2}} \{ \left(\frac{\frac{(S_1 + S_2)}{2} + \frac{(T_1 + T_2)}{2}}{x+y} - 1 \right)$$

$$- \left(\frac{B}{m} - 1 \right) \}$$

$$= \left(\frac{\frac{(S_1 + S_2)}{2}}{x} \right) - \left(\frac{B}{m} \right) - \frac{\frac{x+y}{x} \left(\frac{S_1 - S_2}{2} \right)^2}{\frac{1}{4}(S_1 - S_2)^2 + \frac{1}{4}(T_1 - T_2)^2} \{ \left(\frac{\frac{(S_1 + S_2)}{2} + \frac{(T_1 + T_2)}{2}}{x+y} \right) - \left(\frac{B}{m} \right) \}$$

$$= \left(\frac{\frac{(S_1 + S_2)}{2}}{x} \right) - \left(\frac{B}{m} \right) - \frac{\left(\frac{S_1 - S_2}{2} \right)^2}{\frac{1}{4}(S_1 - S_2)^2 + \frac{1}{4}(T_1 - T_2)^2} \{ \left(\frac{\frac{(S_1 + S_2)}{2} + \frac{(T_1 + T_2)}{2}}{x} \right) - \left(\frac{B}{m} \frac{x+y}{x} \right) \}$$

$$= \frac{1}{x} \left\{ \frac{(S_1 + S_2)}{2} - \left(\frac{Bx}{m} \right) \right.$$

$$- \frac{\left(\frac{S_1 - S_2}{2} \right)^2}{\frac{1}{4}(S_1 - S_2)^2 + \frac{1}{4}(T_1 - T_2)^2} \left[\left(\frac{(S_1 + S_2)}{2} + \frac{(T_1 + T_2)}{2} \right) \right.$$

$$\left. \left. - \left(\frac{B}{m}(x+y) \right) \right] \right\} \qquad (32)$$

To keep the length of the formulas manageable, we will introduce four more short representations.

$$1^{\circ} = 2'3'4', \;\; 2^{\circ} = 1'3'4', \;\; 3^{\circ} = 1'2'4', \;\; 4^{\circ} = 1'2'3'$$

From (22), (23), formula (32) will be

$$= \frac{1}{x}\{\frac{(S_1 + S_2)}{2} - \left(\frac{S_1 1^{\circ} + S_1 2^{\circ} + S_2 3^{\circ} + S_2 4^{\circ}}{1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ}}\right)$$

$$- \frac{\left(\frac{S_1 - S_2}{2}\right)^2}{\frac{1}{4}(S_1 - S_2)^2 + \frac{1}{4}(T_1 - T_2)^2}[\left(\frac{(S_1 + S_2)}{2} + \frac{(T_1 + T_2)}{2}\right)$$

$$- \frac{(S_1 + T_1)1^{\circ} + (S_1 + T_2)2^{\circ} + (S_2 + T_1)3^{\circ} + (S_2 + T_2)4^{\circ}}{1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ}}]\}$$

$$= \frac{1}{x}\frac{1}{1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ}}\frac{1}{(S_1 - S_2)^2 + (T_1 - T_2)^2}\{\frac{(S_1 + S_2)}{2}(1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ})((S_1 - S_2)^2$$

$$+ (T_1 - T_2)^2) - (S_1 1^{\circ} + S_1 2^{\circ} + S_2 3^{\circ} + S_2 4^{\circ})((S_1 - S_2)^2 + (T_1 - T_2)^2)$$

$$- (S_1 - S_2)^2[\left(\frac{(S_1 + S_2)}{2} + \frac{(T_1 + T_2)}{2}\right)(1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ})$$

$$- ((S_1 + T_1)1^{\circ} + (S_1 + T_2)2^{\circ} + (S_2 + T_1)3^{\circ} + (S_2 + T_2)4^{\circ})]\}$$

$$= \frac{1}{x}\frac{1}{1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ}}\frac{1}{(S_1 - S_2)^2 + (T_1 - T_2)^2}\{(T_1 - T_2)^2\left[\frac{1}{2}(S_1 + S_2)(1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ})\right.$$

$$\left. - (S_1 1^{\circ} + S_1 2^{\circ} + S_2 3^{\circ} + S_2 4^{\circ})\right] - (S_1 - S_2)^2[\frac{1}{2}(T_1 + T_2)(1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ})$$

$$+ (S_1 1^{\circ} + S_1 2^{\circ} + S_2 3^{\circ} + S_2 4^{\circ})$$

$$- ((S_1 + T_1)1^{\circ} + (S_1 + T_2)2^{\circ} + (S_2 + T_1)3^{\circ} + (S_2 + T_2)4^{\circ})$$

$$= \frac{1}{x} \frac{1}{1^\circ + 2^\circ + 3^\circ + 4^\circ} \frac{1}{(S_1 - S_2)^2 + (T_1 - T_2)^2} \{(T_1 - T_2)^2 \Big[\frac{1}{2}(S_1 + S_2)(1^\circ + 2^\circ + 3^\circ + 4^\circ)$$

$$- (S_1 1^\circ + S_1 2^\circ + S_2 3^\circ + S_2 4^\circ)\Big]$$

$$- (S_1 - S_2)^2 \Big[\frac{1}{2}(T_1 + T_2)(1^\circ + 2^\circ + 3^\circ + 4^\circ) - (T_1 1^\circ + T_2 2^\circ + T_1 3^\circ + T_2 4^\circ\Big)]\}$$

$$= \frac{1}{x} \frac{1}{1^\circ + 2^\circ + 3^\circ + 4^\circ} \frac{1}{(S_1 - S_2)^2 + (T_1 - T_2)^2} \{\frac{1}{2}(T_1 - T_2)^2 [-(S_1 - S_2)1^\circ - (S_1 - S_2)2^\circ$$

$$+ (S_1 - S_2)3^\circ + (S_1 - S_2)4^\circ]$$

$$- \frac{1}{2}(S_1 - S_2)^2 [-(T_1 - T_2)1^\circ + (T_1 - T_2)2^\circ - (T_1 - T_2)3^\circ + (T_1 - T_2)4^\circ]\}$$

$$= \frac{1}{2} \frac{1}{x} \frac{1}{1^\circ + 2^\circ + 3^\circ + 4^\circ} \frac{(S_1 - S_2)(T_1 - T_2)}{(S_1 - S_2)^2 + (T_1 - T_2)^2} \{(T_1 - T_2)[-1^\circ - 2^\circ + 3^\circ + 4^\circ]$$

$$- (S_1 - S_2)[-1^\circ + 2^\circ - 3^\circ + 4^\circ]\}$$

$$\frac{1}{2} \frac{1}{x} \frac{1}{1^\circ + 2^\circ + 3^\circ + 4^\circ} \frac{(S_1 - S_2)(T_1 - T_2)}{(S_1 - S_2)^2 + (T_1 - T_2)^2} \{(T_1 - T_2)[(S_1 - S_2)2'4' + (S_1 - S_2)1'3']$$

$$- (S_1 - S_2)[(T_1 - T_2)3'4' + (T_1 - T_2)1'2']\}$$

$$= \frac{1}{2} \frac{1}{x} \frac{1}{1^\circ + 2^\circ + 3^\circ + 4^\circ} \frac{(S_1 - S_2)^2 (T_1 - T_2)^2}{(S_1 - S_2)^2 + (T_1 - T_2)^2} \{[2'4' + 1'3'] - [3'4' + 1'2']\}$$

$$= \frac{1}{2} \frac{1}{x} \frac{1}{1^\circ + 2^\circ + 3^\circ + 4^\circ} \frac{(S_1 - S_2)^2 (T_1 - T_2)^2}{(S_1 - S_2)^2 + (T_1 - T_2)^2} \frac{1}{C} \{[(B + S_1 + T_2)(B + S_2 + T_2)$$

$$+ (B + S_1 + T_1)(B + S_2 + T_1)] - \quad [(B + S_2 + T_1)(B + S_2 + T_2)$$

$$+ (B + S_1 + T_1)(B + S_1 + T_2)]\}$$

$$= \frac{1}{2}\frac{1}{x}\frac{1}{1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ}}\frac{(S_1 - S_2)^2(T_1 - T_2)^2}{(S_1 - S_2)^2 + (T_1 - T_2)^2}\frac{1}{C}\{(B + S_2 + T_2)(S_1 - S_2 + T_2 - T_1)$$

$$- (B + S_1 + T_1)(S_1 - S_2 + T_2 - T_1)\}$$

$$= \frac{1}{2}\frac{1}{x}\frac{1}{1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ}}\frac{(S_1 - S_2)^2(T_1 - T_2)^2}{(S_1 - S_2)^2 + (T_1 - T_2)^2}\frac{1}{C}\{(S_2 + T_2 - S_1 - T_1)(S_1 - S_2 + T_2 - T_1)\}$$

$$= \frac{1}{2}\frac{1}{x}\frac{1}{1^{\circ} + 2^{\circ} + 3^{\circ} + 4^{\circ}}\frac{(S_1 - S_2)^2(T_1 - T_2)^2}{(S_1 - S_2)^2 + (T_1 - T_2)^2}\frac{1}{C}\{(T_1 - T_2)^2 - (S_1 - S_2)^2\}$$

$$= \frac{1}{2}\frac{1}{x}\frac{1}{C^4}\frac{1}{\#}\frac{(S_1 - S_2)^2(T_1 - T_2)^2}{(S_1 - S_2)^2 + (T_1 - T_2)^2}\{(T_1 - T_2)^2$$

$$- (S_1 - S_2)^2\} \tag{33}$$

Where # is

$$(B + S_1 + T_2)(B + S_2 + T_1)(B + S_2 + T_2) + (B + S_1 + T_1)(B + S_2 + T_1)(B + S_2 + T_2) +$$
$$(B + S_1 + T_1)(B + S_1 + T_2)(B + S_2 + T_2) + (B + S_1 + T_1)(B + S_1 + T_2)(B + S_2 + T_1)$$

In general, the function (33) is not equal to zero. When

$$(T_1 - T_2)^2 > (S_1 - S_2)^2$$

The function is larger than zero. This means the expected return of the risky asset S is higher than CAPM predicted when the variance of the payoff of S is less than that of T. This is roughly (but not exactly) equivalent to low beta stocks have higher expected return than predicted by CAPM (Black, Jensen and Scholes, 1972).

3. Numerical calculations

We will calculate some numerical values of some results derived above to gain more concrete understanding. First, we will examine how the amount of risk free asset affect the prices of different risky assets. The prices of the two risky assets are x and y. We will calculate some numerical values of y/x, the ratio of two prices from (21). Assume $(S_1, S_2) = (1, 2)$, $(T_1, T_2) = (1, 3)$. Let B, the terminal value of the risk free asset, take different values. The following table list the calculated values of y/x with different values of B.

B	0	1	2	3	4	5	100	1000
y/x	1.20	1.23	1.25	1.26	1.27	1.28	1.33	1.33

From the above table, we can find the value of y/x is a function of B. As the amount of risk free asset increases, the valuation of T, the more risky asset, also increases. This is consistent with our intuition. When B is zero, the value of y/x reaches minimum. When B approaches infinity, the value of y/x approaches its maximum at $(T_1+T_2)/(S_1+S_2)$. This means when the amount of risk free asset is larger, investors value risky asset higher. When the amount of risk free asset is really large, risk premium disappears. Overall, it means that the valuation of risky assets is dependent on the amount of risk free assets.

Next, we will calculate β of the two risky assets, *S* and *T* with different values of risk free asset, *B*. Assume $(S_1, S_2) = (1, 2)$, and $(T_1, T_2) = (1, 3)$. From formulas (28) and (29),

B	0	1	2	3	4	5	100	1000
beta S	0.440	0.446	0.450	0.453	0.455	0.456	0.466	0.467
beta T	1.466	1.450	1.440	1.433	1.428	1.425	1.402	1.400

Asset T, which is more risky, has higher value of β. The value of β is also dependent on the amount of risk free asset in the market. When the amount of risk free asset increases, β of the more risky asset declines and β of the less risky asset increases.

Third, we will discuss the relation between size and return. To focus on the issue of size, we look into two asset *S*, *T* with the following payoff distribution, *S*: (1, 2), *T*: (2, 4). Their only difference is their size. We calculate the price differentials of *S* and *T*, as well as their β, with different amount of risk free asset in the market. The results, calculated from formulas (21), (28) and (29), are displayed in the following table.

B	0	1	2	3	4	5	100	1000
y/x	1.927	1.940	1.949	1.955	1.961	1.965	1.997	2.000
beta S	0.585	0.588	0.590	0.591	0.592	0.593	0.599	0.600
beta T	1.215	1.212	1.211	1.209	1.208	1.207	1.201	1.200

The expected payoff of T is two times the expected payoff of S. However, the price is T is less than two times of the price of S. This means that the expected return of T, the larger firm, is higher than the expected return of S, the smaller firm. Intuitively, this is easy to understand. A smaller firm has higher diversification value. So it is over weighted in value relative to its payoff. This causes its expected rate of return to be lower. But why in Fama-French three factor model, small firms has positive loading? We need to look at the values of β. The β of S, the smaller firm, is much smaller than that of T, the larger firm. So according to CAPM, the expected rate of return of S is much lower than that of T. The positive loading of firm size in Fama-French three factor model partially offset this problem. But it does not mean smaller firms provide higher rate of return. Currently, empirical evidence on size and return in inconclusive (Van Dijk, 2011).

Finally, we will test whether CAPM holds when investors make investment decision by maximizing logarithm utility. To calculate the rate of returns of different securities, we assume the total return of all the securities are ten percent. Or m+x+y = (B +(S1+S2)/2+(T1+T2)/2))/1.1. The following table displays the calculated results of various returns when B is equal to 1, S= (1, 2), T= (2, 4).

Prices of securities	
m+x+y	5
m	0.9494
x	1.3780
y	2.6726
Returns of securities	
B	0.0533
S	0.0886

T	0.1225
S+T	0.1109
Risk premiums	
S+T-B	0.0577
S-B	0.0353
T-B	0.0692
Test of CAPM	
S-B- βS*(S+T-B)	0.0014
T-B- βT*(S+T-B)	-0.0007

If CAPM model is valid, the last two terms should be zero. But among these two terms, the first, representing the smaller firm, is positive, the second, representing the larger firm, is negative. This means small firms has higher return and large firms has lower return than predicted by CAPM. This is consistent with statistical results obtained by Fama and French (1993).

4. Concluding remarks

In CAPM, it is assumed that investors maximize their utility. We construct a very simple asset market. Then we apply the utility theory to calculate the return and variance of different assets. The results are different from that from CAPM. This shows that the theory of CAPM is internally inconsistent. Furthermore, the theoretical results derived from the utility theory are more consistent with market data. The methodology in this work can be extended to general asset markets. The same methodology can be applied to other utility functions as well.

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