A selection of problems

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1. (CM) The sequence $a_2, a_3, a_4, \ldots$ of real numbers is such that, for each n, $a_n > 1$ and the equation $\lfloor a_n x \rfloor = x$ has exactly n different solutions. Find $\lim_{n\to\infty} a_n$.

2. (CM) Find a finite set S of (at least two) points in the plane such that the perpendicular bisector of the segment joining any two points in S passes through exactly two points of S.

3. (CM) Find all real numbers α for which the equality $\lfloor \sqrt{n+\alpha} + \sqrt{n} \rfloor = \lfloor \sqrt{4n+1} \rfloor$ holds for every positive integer n.

4. (CM) Consider two concentric circles with radii R_1 and R ($R_1 > R$) and a triangle ABC inscribed in the inner circle. Points A_1, B_1, C_1 on the outer circle are determined by extending BC, CA, AB respectively. Prove that

$$\frac{F_1}{R_1^2} \geq \frac{F}{R^2},$$

where F_1 and F_2 are the areas of triangles $A_1B_1C_1$ and ABC respectively, with equality when ABC is equilateral.

5. (CM) The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = \frac{4}{3}$ and

$$a_{n+1} = \frac{3(5-7a_n)}{2(10a_n+17)}$$

for $n \geq 0$. Find a formula for a_n in terms of n.

6. (CM) Let $n \geq 2$ and $b_0 \in [2, 2n-1]$ be integers, and consider the recurrence

$$b_{i+1} = \begin{cases} 2b_i - 1 & \text{if} \quad b_i \leq n, \\ 2b_i - 2n & \text{if} \quad b_i > n. \end{cases}$$

Let $p = p(b_0, n)$ be the smallest positive integer such that $b_p = b_0$.

(a) Find $p(2, 2^k)$ and $p(2, 2^k + 1)$ for all $k \geq 1$.

(b) Prove that $p(b_0, n)$ divides $p(2, n)$.

7. (CM) The equation $x^3 + ax^2 + (a^2-6)x + (8-a^2) = 0$ has only positive roots. Find all possible values of a.

8. (CM) Pairs of numbers from the set $\{7, 8, ..., n\}$ are adjoined to each of the 20 different (unordered) triples of numbers from the set $\{1, 2, ..., 6\}$, to obtain twenty 5-element sets $A_1, A_2, ..., A_{20}$. Suppose that $|A_i \cap A_j| \leq 2$ for all $i \neq j$. What is the smallest n possible ?

9. (CM) (with S. Kapralov) Find all sequences $a_1 \leq a_2 \leq ... \leq a_n$ of positive integers such that

$$\begin{aligned} a_1 + a_2 + ... + a_n &= 26 \\ a_1^2 + a_2^2 + ... + a_n^2 &= 62 \\ a_1^3 + a_2^3 + ... + a_n^3 &= 164 \end{aligned}$$

10. (CM) Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers which satisfies the condition

$$3b_{n+2} \geq b_{n+1} + 2b_n$$

for every $n \geq 1$. Prove that either the sequence converges or $\lim_{n\to\infty} b_n = \infty$.

11. (CM) Let $B_1, B_2, ..., B_b$ be k-element subsets of $\{1, 2, ..., n\}$ such that $|B_i \cap B_j| \leq 1$ for all $i \neq j$. Show that

$$b \leq \left\lfloor \frac{n}{k} \left\lfloor \frac{n-1}{k-1} \right\rfloor \right\rfloor.$$

12. (CM) (with G. MacGillivray) Find a closed form expression for the n by n determinant

$$\begin{vmatrix} n & -1 & -1 & -1 & \dots & -1 & -1 & -1 \\ -1 & 3 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & \dots & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & 0 & \dots & 3 & -1 & 0 \\ -1 & 0 & 0 & 0 & \dots & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & 3 \end{vmatrix}.$$

13. (CM) Find 364 five-element subsets $A_1, A_2, ..., A_{364}$ of a 17-element set such that $|A_i \cap A_j| \leq 3$ for all $1 \leq i < j \leq 364$.

14. (CM) Prove that for every positive integer n

$$\left\lceil \frac{n^2+3n+1}{n^2+2n} \left\lceil \frac{n^2+3n}{n^2+2n-1} \cdots \left\lceil \frac{n^2+2n+2}{n^2+n+1} \right\rceil \cdots \right\rceil \right\rceil = n+1.$$

15. (CM) Let $(a_1, a_2, ..., a_n)$ be a permutation of the integers from 1 to n with the property that $a_k + a_{k+1} + ... + a_{k+s}$ is not divisible by $(n+1)$ for any choice of k and s, $k \geq 1$, $0 \leq s \leq n-k-1$. Find such a permutation

(a) for $n = 12$;

(b) for $n = 22$.

16. (AMM) Find

$$\lim_{n\to\infty} \frac{1}{n} \left\lceil \frac{3n^2+2n}{3n^2} \left\lceil \frac{3n^2+2n-1}{3n^2-1} \cdots \left\lceil \frac{3n^2+1}{3n^2-2n+1} \right\rceil \cdots \right\rceil \right\rceil .$$

17. (MM) Let $k_1, k_2, ..., k_n$ be integers such that $k_i > 2$, $i = 1, 2, ..., n$, and let

$$N = \sum_{i=1}^{n} \binom{k_i}{2}.$$

Prove that

$$\sum_{1\le i<j\le n} \binom{k_i}{2}\binom{k_j}{2} + 3\sum_{i=1}^{n} \binom{k_i+1}{4} = \binom{N}{2}.$$

18. (AMM) Let N be a positive even integer. A placement of queens on an $N \times N$ chessboard is a set of N squares on the board such that none of these squares lies on either long diagonal, and no two of these squares lie in a single row, column, or diagonal (that is, the queens are non-attacking). A cover of an $N \times N$ board is a set of $N - 2$ disjoint placements. (Thus on a 4×4 board, the placements $\{(1,3),(2,1),(3,4),(4,2)\}$ and $\{(1,2),(2,4),(3,1),(4,3)\}$ form a cover.)

(a) Show that there exists a cover of the $N \times N$ board if $N + 1$ is prime.

(b) Give an example of an even N for which $N + 1$ is not prime and for which there is no cover.

(c) Give an example of an even N, and a cover, for which $N + 1$ is not prime.

19. (MM) Prove that

$$\sum_{m=0}^{n}\sum_{r=0}^{n-m} \frac{(-2)^r}{m!r!}\binom{2n-m-r}{n} = \sum_{r=0}^{n} \frac{(-1)^r}{r!}\binom{2n-r}{n}.$$

20. (CM) Let $B = [b_{ij}]$ be an $n \times k$ matrix with entries in the set of residues modulo v, such that the k entries in each row of B are pairwise distinct. Form the $n \times [k(k-1)]$ matrix of differences $D = [d_{ip}]$, where $d_{ip} = b_{ij} - b_{is} \pmod{v}$, $j \neq s$, $1 \le j, s \le k$ and $1 \le i \le n$. Let O_q, $q = 1, 2, ..., v-1$ be the number of occurrences of the residue q in the matrix D. Show that the sum

$$\sum_{q=1}^{v-1} O_q$$

does not depend on v.