# On mathematical stand models [1]

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# Contents

**Abstract**

Some concepts that may be useful for the study of forest stands through mathematical models are discussed. A set of examples to illustrate these ideas is included.

Application to stand model formulation of an approach based on the concept of dynamical system or multistage process, as used especially in Control Theory, is described. This approach involves the specification of a state vector which describes the system at a given time, and the specification of a transition function which determines the changes of state. Furthermore the need for multidimensional state vectors is pointed out.

The aim of this work is to discuss some concepts which may be useful in the study of forest stands[1] through mathematical models, illustrated with a series of examples.

This report is an extension of some ideas presented previously (García, 1968), and is based on a talk presented at Universidad Austral in January of 1972.

## Models

A system is, in general terms, a set of components and a set of relations between the components. When we are able to act upon a system affecting its behaviour, the problem of choosing the most appropriate course of action is presented. In order to choose an appropriate course of action, it is necessary to know first, at least approximately:

a) The relationship between our criterion of appropriateness and the behaviour of the system.

b) The relationship between the behaviour of the system and our actions.

We will refer only to the second point, that is, the prediction of the behaviour of a system.

The behaviour of a system can be determined directly, testing the various alternatives of action and recording the results. However, in many cases, besides the fact that the number of alternatives can be infinite, the direct experimentation can be slow, costly, or even impossible for various reasons. For example, if the system is a forest stand in which we wish to know its behaviour *vis a vis* various kinds of thinning programmes, even assuming that the number of programmes of interest were finite, the experimentation is costly and requires time. In the case of an economic or social system, the experimentation is clearly impractical.

One can resort then to the use of models. A model of a system is another system such that, having established a correspondence between aspects of both systems, it behaves in a manner similar to the original system. One can then experiment on the model (at less cost) and thus obtain information about the behaviour of the system of interest.

Models can be of various types (see for example Ackoff and Sasieni, 1968; Churchman, Ackoff and Arnoff, 1957). An example is a scale model of an aeroplane which is tested in a wind tunnel. Or a board with figures where

[1] A forest stand is an area of forest of relatively homogeneous characteristics.

military operations are simulated. Or an electric circuit which represents a suspension system of an automobile (a very close analogy exists between electrical systems and mechanical systems, with correspondence between electric potential and force, intensity of current and velocity, electrical resistance and friction, induction and inertia, condensers and springs, etc.).

In mathematical models, the system is represented by a group of symbols linked by mathematical relationships. This type of model has gradually acquired more importance with the use of digital computers, since the latter make possible the operation of complex models at low cost. It is to this type of model which we shall refer. With a mathematical model, depending on its nature and complexity, the most appropriate alternatives may be sought experimentally (simulation) or determined mathematically (optimisation).

It is important to point out that a system does not correspond to a unique mathematical model, but it is possible to construct models of distinct mathematical nature and with diverse levels of refinement. The characteristics of a model can depend, among other things, on the use for which it is constructed, on the knowledge one has of the system, on the possibilities of estimating the necessary parameters, on the available mathematical tools, on the computing facilities, etc.

In particular, a system may be considered as composed of interrelated subsystems and, in turn, these subsystems composed of other subsystems, etc. In this form, the problem of constructing a model may be subdivided into various sub-problems which may be approached separately and may involve various teams of people. It is thus that, for example, in order to develop a model of the Forest Sector one may consider as subsystems the silvicultural aspects, the forest industry, and the transportation and distribution system. Within the silvicultural area one may consider the production units (enterprises) and, within the production units, among other subsystems, the stands. Also the trees could be considered as subsystems which form a stand, etc. As already mentioned, the level of division into subsystems is a matter of convenience which depends, among other things, on the knowledge of the system and the degree of refinement sought. In the planning of forest resources management, probably the major difficulty at this time is in predicting the behaviour of a stand *vis a vis* silvicultural interventions. That is to say, the problem is in developing adequate mathematical models for the stands.

# Dynamical Systems

## State

We are interested in the behaviour of a system over a period of time. At any moment $t$, the system is characterised by its state, represented by a state vector $p_t$. This state vector is a point in a set or space $\mathcal{S}$, which describes adequately for our purposes the state of the system. "Adequately" signifies that if two systems have the same state vector in a given moment, we can consider them as equivalents, behaving in the same manner (for a more rigorous discussion of the concept of state see: Wiberg, 1971; Zadeh, 1964). The state vector can be a real number, a vector of finite dimension, an infinite sequence of real numbers, a function, or any other convenient mathematical entity.

For example, an even-aged stand can be characterised by its cubic volume per hectare and its age (two-dimensional vector). This is what has been used most in yield models.

In general, age in itself does not seem important as a state variable. To include it really what will be done is to consider other variables which depend on age and which have a more direct causal relationship with stand development. Especially, dominant height (mean height of dominant or dominant and codominant trees) has a very close relationship with the age for a given site, so that another state vector almost equivalent to the former can be that consisting of the volume and the dominant height.

The use of these state vectors would indicate that we are accepting that, for a given site, two stands which have equal volume and age, or equal volume and height, will behave in the same manner. However, it seems obvious that if, in spite of coincidence in these state vectors, the stands differ markedly in the number of trees per hectare, their increments can be very different.

We can then consider as state variables volume, dominant height and number of trees per hectare. In order to describe acceptably the state of an even-aged stand, at least three variables seem to be necessary which can be those indicated or others related to them, as for example, basal area, mean diameter and height. Other variables could be added such as measures of dispersion (variances) which indicate the homogeneity of the stand, measures of stem form, etc.

Another state vector used in projections by Reynolds' method is the number of trees per hectare by dbh[2] classes. This is an example of a vector of infinite dimension (sequence of real numbers), though it can be reduced

[2] Diameter at breast height (1.3 meters above ground).

to a finite-dimensional vector by limiting the number of dbh classes.

If one considers the number of trees per hectare by dbh classes and by height classes, with a finite number of classes, it follows that the state vector is a matrix (which can also be represented in other ways).

The case of a diameter distribution is an example in which the state vector is a function (vector of uncountable infinite dimension).

Upon considering a tree, its state may be described by its dbh and height (two-dimensional vector), by its dbh, height and form factor (three-dimensional vector), by its diameter as a function of height (function), etc.

It is clear then that we can take as $\mathcal{S}$ any convenient set.

### Processes

The behaviour of a system during a period of time $\mathcal{T}$ would be completely chracterised by the set of states $\{p(t) : t \in \mathcal{T}\}$. This set or sequence (countable or not) of states constitutes a process. It is possible to imagine the process as a point which moves in the state space $\mathcal{S}$. This visualisation is particularly simple when the state vector is two- or three-dimensional, enabling one then to represent $\mathcal{S}$ in a plane or in a three-dimensional space, respectively.

The processes can be deterministic or stochastic. A process is deterministic if the knowledge of an initial state and the later actions upon the system completely determine its future behaviour. Stochastic processes are characterised by containing random elements which allow only a probability distribution for future states to be determined (see for example: Cox and Miller, 1965; Barttlet, 1966). In particular, stochastic processes in which the future behaviour depends on an initial state and on the later actions upon the system and not on its previous history, are named Markov processes, being the most interesting as stochastic stand models. Here we will discuss only deterministic models.

Let us consider first the case in which we are interested only in the states at certain uniformly spaced times. In order to simplify, let us assume in addition that the spacing is of one time unit, beginning at zero. It is always possible to formulate the problem thus using an appropriate change of scale. In this case $\mathcal{T}$ is the set $\{0, 1, 2, 3, \ldots\}$. The evolution of the system will be given then by the sequence $\{p_0, p_1, p_2, p_3, \ldots\}$.

Since the behaviour of the system depends only on its state and our action on the system, the change of state between the times $t$ and $t+1$ can be represented by a transformation (transition function)

$$p_{t+1} = T(p_t, q_t) \ ,$$

where $q_t$ is a vector which describes our action on the system at the moment $t$ and which we will call the control vector (Bellman, 1956; Bellman and Kalaba, 1965). In other words, the state at time $t+1$ is a function $T$ of the state at time $t$ and the control vector for time $t$.

Notice that the evolution of the system is completely specified given the initial state $p_0$, the function $T$ and the control schedule $\{q_t : t = 0, 1, 2, \ldots\}$. In effect, the state at time $n$ can be calculated by applying $n$ times the function $T$:

$$\begin{aligned} p_1 &= T(p_0, q_0) \\ p_2 &= T(p_1, q_1) \\ \vdots \quad & \vdots \quad \vdots \\ p_n &= T(p_{n-1}, q_{n-1}) \; , \end{aligned}$$

that is:

$$p_n = T(T(\ldots T(T(T(p_0, q_0), q_1), q_2) \ldots, q_{n-1}), q_n) \; .$$

In the case of stand models, our actions are generally silvicultural interventions (partial cuts) which are equivalent to changes of state which we can consider as instantaneous. In order to simplify the problem, we can then consider only the evolution of the stand between two successive interventions, omitting the control vector. Our transition function, then, will be simply:

$$p_{t+1} = T(p_t) \tag{1}$$

This equation may also be written, subtracting $p_t$ from each member:

$$p_{t+1} - p_t = T(p_t) - p_t \; ,$$

or as well:

$$\Delta p = f(p) \; . \tag{2}$$

Either of the two forms, (1) or (2), is adequate for a mathematical stand model.

We go now to the continuous case. Sometimes it is of interest to consider the process as continuous in time, that is, $\{p(t)\}$ with $t$ varying continuously in an interval. $\mathcal{T}$ will be the interval $[0, \infty)$.

If, as in the case described, we consider points spaced by steps $\Delta t$ in time, the state at time $t + \Delta t$ will be a function of the state at time $t$. This function depends also on the size of the intervals considered:

$$p(t + \Delta t) = T[p(t), \Delta t] \; .$$

The function $T$ can be expanded in Taylor series around $t$, giving an expression of the form

$$p(t + \Delta t) = p(t) + f\ [p(t)]\Delta t + o(\Delta t)\ ,$$

where $o(\Delta t)$ denotes the terms of order smaller than $\Delta t$, that is such that

$$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

One has then

$$\frac{p(t + \Delta t) - p(t)}{\Delta t} = f[p(t)] + \frac{o(\Delta t)}{\Delta t}$$

If we consider decreasing intervals of $t$, that is allow $t$ to approach zero, the limit is:

$$\lim_{\Delta t \to 0} \frac{p(t + \Delta t) - p(t)}{\Delta t} = \frac{\mathrm{d}p(t)}{\mathrm{d}t} = f[p(t)]\ ,$$

or

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f(p)\ . \tag{3}$$

which is analogous to (2).

## Examples of Stand Models

### Example 1

Let us assume that we are interested only in the growth in dominant height of an even-aged stand. We take as the state vector the dominant height $H$. Then $\mathcal{S} = \Re$, the set of real numbers. The increase in dominant height is usually considered independent of tending, and thus, of other stand characteristics distinct from height, so that it is an appropriate state vector.

Let us consider a model in discrete time at intervals of a year. In accordance with (1), the height at year $t+1$ depends on the height at year $t$:

$$H_{t+1} = T(H_t)\ .$$

Let us assume that the function $T$ is linear:

$$H_{t+1} = a + bH_t\ .$$

This equation, together with the initial height $H_0$, completely determines the height development (for a discussion of this type of equations see

Goldberg, 1964):

$$\begin{aligned}
H_1 &= a + bH_0 \\
H_2 &= a + ab + b^2 H_0 \\
H_3 &= a + ab + ab^2 + b^3 H_0 \\
\vdots &\quad \vdots \quad \vdots \quad \vdots \\
H_n &= a \sum_{k=0}^{n-1} b^k + b^n H_0 \ ,
\end{aligned}$$

from where,

$$H_n = \frac{a(1-b^n)}{1-b} + b^n H_0 \ ,$$

$$H_n = \frac{a}{1-b} - (\frac{a}{1-b} - H_0) b^n \ . \tag{4}$$

It is also possible to present the model in the form $\Delta H = f(H)$:

$$\Delta H = H_{t+1} - H_t = a + (b-1) H_t \ ,$$

that is:

$$\Delta H = a - cH \ ,$$

with $c = 1 - b$.

Let us now consider a continuous model of this same type. According to eq. (3):

$$\frac{\mathrm{d}H}{\mathrm{d}t} = f(H) \ .$$

Let us assume that $f$ is linear:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \alpha - \beta H$$

This equation, together with the initial height (or the height $H_0$ at a given time $t_0$), also determines the development in height:

$$\frac{\mathrm{d}H}{\alpha - \beta H} = \mathrm{d}t$$

$$\int_{H_0}^{H} \frac{\mathrm{d}H}{\alpha - \beta H} = \int_{t_0}^{t} \mathrm{d}t$$

$$-\frac{1}{\beta}[\ln(\alpha - \beta H) - \ln(\alpha - \beta H_0)] = t - t_0$$

$$\frac{\alpha - \beta H}{\alpha - \beta H_0} = e^{-\beta(t-t_0)}$$

$$H = \frac{\alpha}{\beta}(\frac{\alpha}{\beta} - H_0)e^{-\beta(t-t_0)}$$

This equation is the same type as (4) and coincides with it when one takes:

$$b = e^{-\beta} \qquad \text{and} \qquad \frac{a}{1-b} = \frac{\alpha}{\beta}$$

### Example 2

In an even-aged stand, let us take as a state vector the 3-tuple $X = (B, D, H)$, formed by the basal area, mean diameter, and dominant height. Here $\mathcal{S} = \Re^3$. Let us consider the annual increments

$$\Delta X = f(X) \ ,$$

or, written in another manner:

$$\begin{aligned} \Delta B &= f_1(B, D, H) \\ \Delta D &= f_2(B, D, H) \\ \Delta H &= f_3(B, D, H) \ . \end{aligned}$$

Let us assume that the equations are of the following form:

$$\begin{aligned} \Delta B &= a_{11}B + a_{12}D + a_{13}H + b_1 \\ \Delta D &= a_{21}B + a_{22}D + a_{23}H + b_2 \\ \Delta B &= a_{33}H + b_3 \ . \end{aligned}$$

Considering $X$ as a column vector, this system of difference equations may be abbreviated using matrix notation:

$$\Delta X = AX + \mathbf{b} \ ,$$

where:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

It is possible to solve this equation to obtain the state after $n$ years starting from an initial state $X_0$. For this, let us make the substitution

$$X = Y - A^{-1}\mathbf{b} \ .$$

then

$$\Delta Y = AY \ , \qquad \text{or,} \qquad Y_{t+1} = (A+I)Y_t \ .$$

Applying repeatedly this recurrence relation:

$$\begin{aligned} Y_1 &= (A+I)Y_0 \\ Y_2 &= (A+I)^2 Y_0 \\ \vdots \quad & \quad \vdots \quad \vdots \\ Y_n &= (A+I)^n Y_0 \ . \end{aligned}$$

Returning to the original variable one has:

$$X_n = (A+I)^n (X_0 + A^{-1}\mathbf{b}) - A^{-1}\mathbf{b} \ .$$

Also, one can establish a continuous version of this same model:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = AX + \mathbf{b} \ , \qquad X(t_0) = X_0 \ .$$

This system of differential equations may be integrated, obtaining a formula for $X$ as a function of time, starting from any initial state $X_0$. The solution is analogous to the discrete model, coinciding with it for integer values of $t$.

It is clear that all the above is directly applicable to vectors of any finite dimension.

The Forestry Institute is working on a model of this type for *Pinus radiata*. A convenient state vector will be sought, and $A$ and $b$ will be estimated using data from experimental thinning plots.

## Example 3

Let us consider an even-aged stand described by the state vector $(V, N, H)$, where $V$ is the cubic volume per hectare, $N$ is the number of trees per hectare, and $H$ is the dominant height. Let $M$ be the total dry weight per hectare of the stand.

The annual gross increment in dry weight of the stand is equal to the photosynthetic assimilation minus the loss from respiration:

$$\Delta M = \text{assimilation} - \text{respiration}$$

Let us make the following assumptions:

(i) The gross increment in volume of the stems is proportional to the gross increment in dry weight: $\Delta V \propto \Delta M$.

(ii) The density always remains sufficiently high in order to make complete utilisation of the site. Then the assimilation per unit of surface area is constant.

(iii) Respiration can be assumed proportional to the quantity of live tissue and this proportional to the surface area of the stems. The surface area of the stems is proportional to the product of the mean diameter times the height and the number of trees.

Then:

$$\Delta V = a - bNHD \; .$$

We need to substitute for $D$ a function involving $N$, $H$, and $V$. Assuming a constant form factor, one may write $V = fBH$. Further, $B = \frac{\pi}{4}ND^2$.

Eliminating $B$ between these two formulas and solving for $D$ we obtain:

$$D = \sqrt{\frac{4}{\pi f}} \cdot \sqrt{\frac{V}{NH}} \; .$$

Substituting in the formula for the gross increment in volume one has finally

$$\Delta V = a - b'\sqrt{VNH} \; .$$

If we assume that the thinnings are frequent and heavy enough so that the natural mortality is insignificant, then $\Delta N = 0$ and the net increment of the volume is equal to the gross increment.

Assuming a linear increment for the height, as in example 1, one has the complete model:

$$\begin{aligned} \Delta V &= a - b'\sqrt{VNH} \\ \Delta N &= 0 \\ \Delta H &= c - dH \; . \end{aligned}$$

To the contrary in the former example, this system of non-linear equations apparently cannot be integrated analytically. One continuous version of this model was formulated by García (1968) and was used to simulate the effect of different thinning schedules using an analog computer. A modified version, including natural mortality, has been programmed in order to work in conversational form in the APL/360 Time-Sharing System. An example of its application is presented in the Appendix.

**Example 4**

Let us examine the Reynolds projection method with regard to the concepts which we have stated here.

As the state, one takes the number of trees per hectare by dbh class.

This can be represented by an infinite sequence of real numbers $\{x_1, x_2, x_3, \ldots\}$, where $x_i$ is the number of trees per hectare in class $i$. By making the number of dbh classes sufficiently large but finite, we may also have a finite-dimensional state vector.

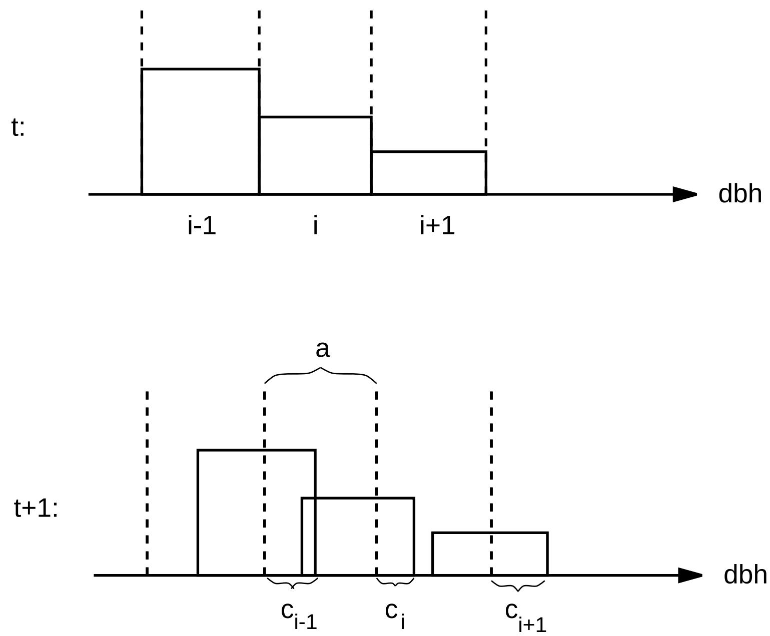


Figure 1:

Let us see how the transition function is obtained. The reasoning is illustrated graphically in figure 1. It represents first a distribution of trees by dbh classes for classes $i-1, i, i+1$, in the year $t$.

In the second part of the figure one observes the number of trees for the following year, which have been displaced by amounts equal to the annual diameter increments $\{c_i\}$. If the width of the dbh classes is $a$, one sees that the number of trees in class $i$ for the year $t+1$ is

$$x_{i,t+1} = x_{it} - \frac{c_i x_{it}}{a} + \frac{c_{i-1} x_{i-1,t}}{a}$$

Then, making $c_i/a = b_i$, the transition function is given by the equations

$$x_{i,t+1} = b_{i-1}x_{i-1,t} + (1 - b_i)x_{it} \; ; \quad i = 1, 2, 3, \ldots \tag{5}$$

or

$$\Delta x_i = b_{i-1}x_{i-1} - b_i x_i \; ; \quad i = 1, 2, 3, \ldots$$

We agree that the variables $x_{0t}$ represent the regeneration which enters the first dbh class at year $t + 1$, and that $b_0 = 1$.

The equations (5) also can be represented in matrix form using the matrix

$$\begin{bmatrix} 1-b_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_1 & 1-b_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & 1-b_3 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b_3 & 1-b_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{n-1} & 1-b_n \end{bmatrix}$$

and used to obtain expressions for the state from an initial state, as in example 2. This notation could be used for theoretical studies, but computationally it is more efficient to work directly with the equations (5).

Also, one can formulate a continuous version of this model. Let $z$ be the dbh, $c(z)$ the rate of diameter increment, $y(z, t)$ the density of trees according to diameter at time $t$ (limit of $x_i/a$ when $a$ approaches zero), $y_0(z)$ the initial density, $t_0$ the initial time, and $r(t)$ the rate of appearance of regeneration at time $t$. Then, using the same argument as in figure 1, series expansion and taking limits, one obtains the partial differential equation

$$\frac{\partial y}{\partial t} + \frac{\partial c(z)y}{\partial z} = 0 \; ,$$

with the boundary conditions

$$y(z, t_0) = y_0(z)$$

$$y(0, t) = r(t) \; .$$

The solution is:

$$y = \begin{cases} \frac{1}{c(z)} c[F^{-1}(F(z) - t + t_0)] y_0[F^{-1}(F(z) - t + t_0)] \; , & \text{for} \quad F(z) \geq t - t_0 \\ \frac{c(0)}{c(z)} r[t - F(z)] \; , & \text{for} \quad F(z) \leq t - t_0 \; , \end{cases}$$

where

$$F(z) = \int_0^z \frac{\mathrm{d}x}{c(x)} \; .$$

It is fitting to make note that this model contains two not very satisfactory assumptions. In the first place, it supposes that the behaviour of the stand depends only on the diameter of the trees. The system could be described better by adding to the state vector the heights and, in the case of mixed forests, the species. The other assumption is that the rate of increase in diameter of each tree depends only on its diameter. It is clear that the growth of a tree depends on the size and number of its neighbours, so that the rate of growth ought to be a more complicated function of the state vector.

### Example 5

We will examine briefly the structure of a type of model which has been used to simulate the development of a small portion of a stand, by way of the development of individual trees. To be specific, we will take as an example a model proposed by Newnham (1964, 1968a, 1968b) for even-aged stands.

In Newnham's model, the state of the portion of the stand (plot) whose development is to be simulated, is characterised, at a given time, by the age of the stand and by the dbh and position of each tree in the plot. This state is projected, using periods of five years, through the changes in the dbh of each tree. The periodic increment in dbh of a tree is assumed a linear function of its dbh, age, and of a "competition index", which we will discuss later. If the predicted increment is less than a certain quantity, it is assumed that the tree dies.

The competition index is intended to represent the effect of size and proximity of neighbouring trees upon the growth of a tree. That used by Newnham is based on a circle of competition within which a tree would compete for site factors, and which is assumed equivalent to the projection of the crown of a tree of the same size growing on open ground. Experimentally, it has been determined that such crown diameter is an approximately linear function of dbh The competition index for a given tree is defined according to the overlapping of the circles of competition in the form described in figure 2 (for other indices of competition see: Opie, 1968; Mitchell, 1969; Van Laar, 1969).

In order to advance a plot to the following period, the procedure consists of calculating for each tree its competition index and, with this, its diameter increment.

These types of models can be interpreted in different manners making use of the concepts discussed here. The most direct form is to consider a single system: the complete plot. Then the state vector is composed of the

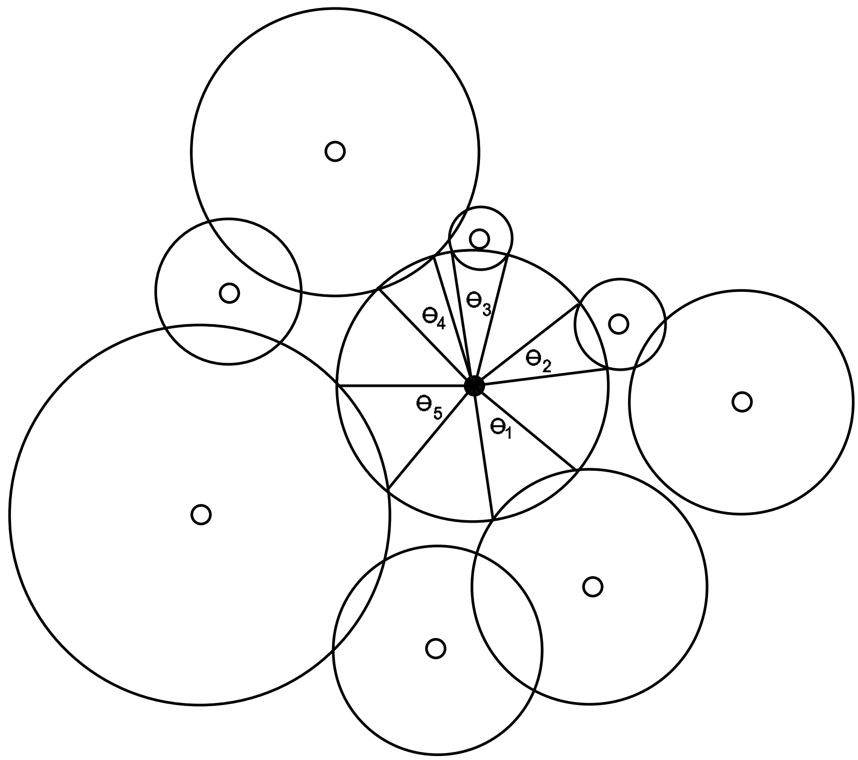


Figure 2: Competition index $= \sum_{i=1}^{5} \theta_i / 2\pi$

positions of all the trees (specified, for example, by rectangular coordinates), one or more state variables for each tree and, optionally, one or more global state variables for the stand. In the model which we have described, the state variables for each tree should be its dbh and its age, if age is considered as the age of each tree. Age could be considered alternatively as a global variable for the stand. If in place of age we used the dominant height, it would be necessarily a global variable. The transition function would be the procedure which permits the advancement from the present state to the state of the following period. It is fitting to make note that the transition function does not need to have a closed analytical expression, but it is any procedure which determines uniquely a state as a function of the previous state (and of the corresponding control vector, if it exists). In this case the transition function is an algorithm more or less complicated, programmed for its execution in a computer.

Another form for interpreting these models is to consider the plot as a system and the trees as subsystems. Each subsystem is characterised by its state vector (dbh and age in Newnham's model) and by its transition function.

The transition function of a subsystem determines its change of state as a function of its state and of a control vector which depends on the state of the remaining subsystems (for example, a competition index). The state and the changes of state of the system (plot) are determined then by the states and changes of state of all the subsystems which comprise it.

## Discussion

The principal idea which has been set forth here is the application to the formulation of stand models of a scheme based on the concept of a dynamic system, as utilised especially in Control Theory. According to this scheme, the formulation of a model contains two parts: the specification of a state vector, which describes the system at a given time, and the specification of a transition function, which describes the changes of state. Although many authors have presented stand models which agree with this, the explicit establishment of this conceptual model can help to clarify and guide the study of forest stands. It is fitting to make note that these concepts of state and transition function are valid not only for stand models, but also for many other systems in which time plays a fundamental role.

Another important aspect is the necessity, in the great majority of the cases, of using multidimensional state vectors to describe a stand adequately. Thus, in the case of an even-aged stand, we saw that at least three state variables are necessary for a satisfactory description. Other variables of interest, like various types of volumes, cash returns etc., can be calculated as functions of the state variables, with regression equations if necessary. This contrasts with a large part of the work performed on yield tables, where the volume per hectare has been used as the only state variable, including in some cases also an index of density. The system of yield tables is unsatisfactory for predicting the development of managed stands, especially with species of rapid growth, for two reasons: the use of inadequate state vectors, and the attempt of determining directly the state-age relationship instead of using transition functions.

it is possible to use models in discrete time or in continuous time. The discrete models are more adequate for working in digital computers and generally are easier to understand for people with less mathematical training.

The continuous models are convenient for simulation in analog computers and, in some cases, can be more convenient for the mathematical study of the system. The stand models can be used as a guide for management or for research in silviculture, logging, genetic improvement, etc. (see for example: Smith *et al.*, 1965; Newnham, 1966; Sakai *et al.*, 1968).

In forest management applications, the stand model normally constitutes a submodel within a model of a larger system, which can contain other stands, price forecasting models, other subsystems, and the relationship between the state variables and an index of the efficiency of the system (García, 1968). It is then possible to attempt to determine the values of the control variables which maximise the index of efficiency, using appropriate optimisation techniques. If this is not possible one must turn to simulation techniques in order to arrive at a "good" solution. In formulating a stand model with these objectives, it is necessary then to keep in mind the computational difficulties which may arise.

Besides the former, the type and complexity of the model determines also the difficulties of determining the parameters which must be estimated from experimental data. It also should be kept in mind that the state vector used determines the information on the initial state of the stand necessary for predicting its growth. Consequently, in the formulation of a stand model for management purposes, one ought to arrive at a compromise between the level of refinement of the model and the information required for its use, the possibilities of estimating the parameters with the available information and the optimisation difficulties.

For the management of pure even-aged stands the most adequate models seem to be those similar to examples 2 and 3. The ideal would be a model based on eco-physiological considerations, as in example 3. For a model for *Pinus radiata* of the Instituto Forestal, however, the lack of an adequate theoretical expression for natural mortality and difficulties of estimation of the parameters caused by deficiencies in the experimental data make preferable the use of an empirical model similar to example 2. The linearity of that model will facilitate its use as well.

For uneven-aged stands it is necessary to resort to more detailed models, analogous to example 4 although more complex. The development of adequate models of this type still requires much research.

The basic research necessary for the development of the models mentioned in the previous paragraph, and for improvement of the other models mentioned for even-aged stands, can be guided partly by means of the study of even more detailed models, as those in example 5, even if these are not directly useful for forest management. These models at individual tree level, which we could call structural models, can be useful, as has been mentioned before, for research in silviculture, logging, genetics, pathology, etc. The study of stochastic models of this type may also be useful.

## Appendix — Example of Simulation of the Growth of a Stand

The simulation example has been omitted in this translation, but may be obtained from pages 23–26 of the original paper.

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