On the forms of utility functions

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Abstract

Utility function is the most important concept in economic theory. In academic literature, utility functions can take many different forms. Human preferences are long term evolutionary products. They should provide beneficial outcomes in regular environments. From this perspective, it is unlikely that utility functions in return, standard deviation spaces are convex, as often assumed in literature. We prove that under a very general condition, utility function is the logarithmic function.

Keywords: utility function, logarithmic function

In the standard economic theory, decision makers are assumed to maximize their utility functions. So utility function is the most important concept in economic theory. In academic literature, utility functions take many different forms. But what is our own utility function? If we don't know our own utility function, how can we maximize it? In this article, we will show that utility functions cannot be convex functions in a return, standard deviation space. Then we prove that under a very general condition, utility function is the logarithmic function.

When the concept of utility was first introduced about three hundred years ago, it was a logarithmic function (Bernoulli, 1738). But in later literature, many forms of utility functions are used. More recently, there is an evolutionary argument that logarithmic functions should be the dominant preference for utility functions. Please refer to Sinn (2003) for more detailed discussion. We augment the discussion by providing a simple mathematical derivation that under a very general condition, utility function is the logarithm function. With this result, maximizing utility function is equivalent to maximizing geometric rate of return. This unifies the concept of utility, a subjective measure with the concept of return, an objective measure.

On quadratic utility functions and convex utility functions

In finance literature, quadratic utility functions are popular choices in investment theories. They take the form of

$$U = E(r) - \frac{1}{2}A\sigma^2$$

where E(r) is the expected rate of return and σ is the standard deviation of the return. A is the coefficient of risk aversion. For simplicity, we will assume A is equal to one in the following discussion. In this case, the quadratic utility function becomes

$$U = E(r) - \frac{1}{2}\sigma^2$$

Suppose there is a group of assets, whose final payoff are 1 with 50% probability and x with 50% probability. The price of all these asset is 1. So the expected returns of these assets are

$$\frac{1}{2}(x+1) - 1 = \frac{1}{2}(x-1)$$

The standard deviation of the return of these assets are

$$x - 1 - \frac{1}{2}(x-1) = \frac{1}{2}(x-1)$$

The utility of holding an asset in this group is

$$U = \frac{1}{2}(x-1) - \frac{1}{2}\left\{\frac{1}{2}(x-1)\right\}^2$$
$$= \frac{1}{2}(x-1)\{1 - \frac{1}{4}(x-1)\}$$
$$= \frac{1}{8}(x-1)(5-x)$$

This means that when x = 1 and x = 5, the investor will obtain the same level of utility as zero. However, x = 5 is a higher return than x = 1. We are choosing two assets with the same price. The first has final payment of 1. The second has final payoff of 1 with 50% probability and 5 with 50% probability. The second one dominates the first one. Yet they have the same utility. Figure 1 present the level of utility with different x. When x increases beyond 5, the utility of the asset will drop below zero. This occurs because the utility function is first order in return and second order in standard deviation. When volatility is high, standard deviation of the return will dominate. The above discussion shows that quadratic utility does not provide good description of actual investment preferences.

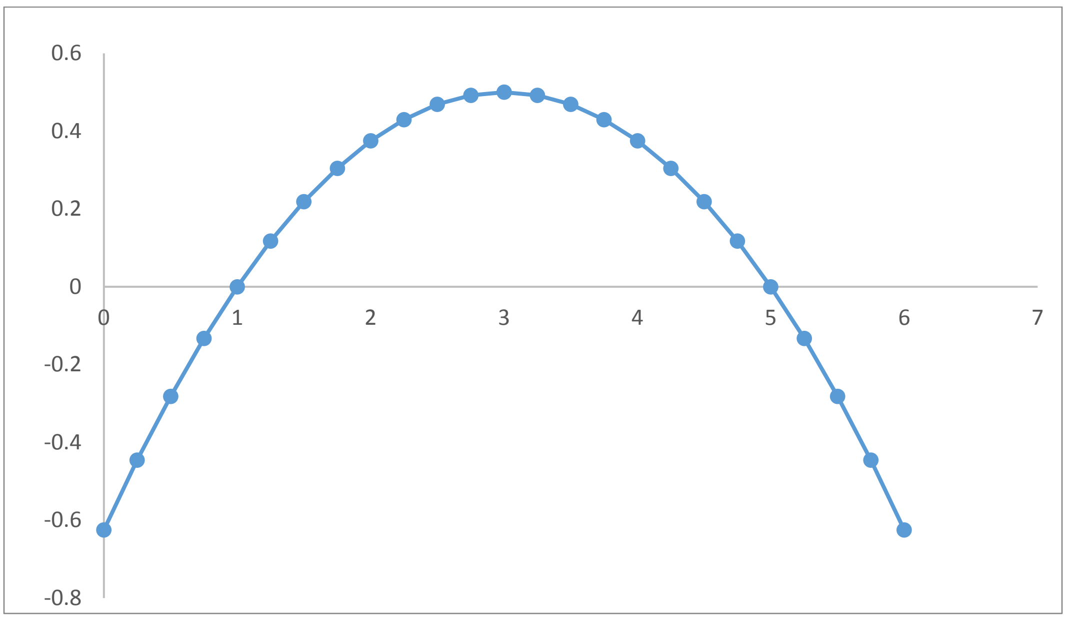

Figure 1

In general, a convex utility function in expected return, standard deviation space looks like Figure 2. Let A, B represent the return and standard deviation of two assets on the utility curve. If the returns of the two assets are perfectly correlated, the return and standard deviation of the weighted combinations of these two assets form the straight line AB, which is above the utility curve. These weighted combinations of assets A and B offers better return, volatility trade off than the utility curve. If A and B are not perfectly correlated, their weighted combinations will offer even better return, volatility trade off. The efficient frontiers of assets are concave curves (Markowitz, 1952; Chen, 2010). Human preferences are long term evolutionary products. While the types of decisions human take in the past may not be identical to modern portfolio selections, the benefits from diversity should be similar. For example, eating different types of foods may provide more complete nutrients humans need. Therefore, human utility functions should be similar to curves of efficient frontiers, which are concave. It is unlikely that many people will have convex utility functions assumed in investment theory (Sharpe, 1964). In view of this, utility functions suggested in investment literature and textbooks should be revised.

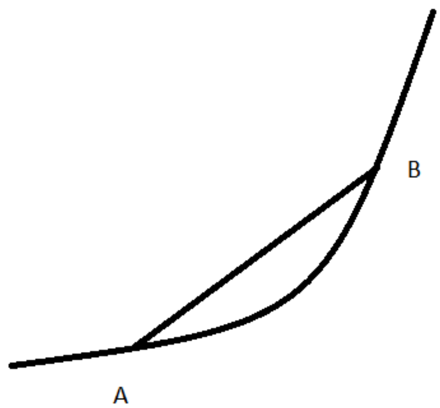

Figure 2.

Logarithm utility functions

Suppose we go to a store to buy two things, one battery and one book. If the battery and the book are independent from each other, then the utility of battery and book considered together should be the same as considered separately. (Does your utility function satisfy this condition?) Mathematically, for a utility function f, if two variables x, y, are independent from each other, then f(xy) = f(x) + f(y). It can be proved (see appendix) by a simple method that if f(xy) = f(x) + f(y), then f(x) must be a logarithmic function (Applebaum,1996). So under a very general condition, our utility function is a logarithmic function. When the utility function is a logarithm function, it is equivalent to geometric rate of return. Subjective utility becomes equivalent to objective return.

The above derivation is very simple. Yet the method can be used to establish some of the most profound results in science. A simple formula was engraved on Boltzmann's tombstone. It was S = k logW. The formula linked entropy, S, to the number of microstates, W, with a logarithmic function. In Shannon's information theory, the amount of information, S, is related to the probability of a signal, P, by a logarithmic function as well, S = -logP (Shannon, 1948). The similarities among physical entropy, information and utility functions, are not accidental. Rather, it indicates that human mind, including mathematical capabilities, is an adaptation to the physical world (Chen, 2016).

Traditionally, differences in human behaviors are often attributed to differences of utility functions. For example, differences in investment patterns between young people and old people are often attributed to their differences in preferences. If the utility functions are the same, differences in investment patterns between young people and old people can be attributed to their differences in income and expense. Old people generally have higher expense than income. So their investment portfolio often have higher proportion of fixed income assets to compensate for cash outflow. Young people generally contribute to their pension funds regularly. So their investment portfolio often have lower proportion of fixed income assets to compensate for cash inflow.

In standard economic theory, utility functions can take infinitely many forms. There can only be finitely many empirical data. The standard economic theories can be difficult to test empirically. We show that utility functions for most people cannot be convex. In many cases, the utility functions should be the logarithm function. When utility functions narrow down, many economic theories become easier to test empirically and can be improved from empirical testing.

Appendix

If for all variables x, y, $f(xy) = f(x) + f(y)$, then $f(x) = k \log(x)$.

Proof： Differentiate both side of $f(xy) = f(x) + f(y)$ with respect to x，we obtain

$$y f'(xy) = f'(x)$$

Let $y = 1/x$, the above equality becomes

$$1/x \, f'(1) = f'(x).$$

Therefore

$$f(x) = k \log(x) + C,$$

Substitute the above equality into $f(xy) = f(x) + f(y)$, we obtain $C = 0$. Hence

$$f(x) = k \log(x)$$

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