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Conservation Laws for Relativistic Fluid Dynamics

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1. Introduction

In his fundamental paper of 1948, Taub [T1] derived the equations of relativistic fluid dynamics:

$$\frac{\partial}{\partial t}\left(n(1-v^2)^{-1/2}\right)+\frac{\partial}{\partial x}\left(nv(1-v^2)^{-1/2}\right)=0,$$
$$\frac{\partial}{\partial t}\left((\rho+p)\frac{v}{1-v^2}\right)+\frac{\partial}{\partial x}\left((\rho+p)\frac{v^2}{1-v^2}+p\right)=0,$$
$$\frac{\partial}{\partial t}\left((\rho+p)\frac{v^2}{1-v^2}+\rho\right)+\frac{\partial}{\partial x}\left((\rho+p)\frac{v}{1-v^2}\right)=0.$$

Here n is the rest mass density, ρ is the proper energy density, p is the pressure and v is the particle speed. Taub then obtained the Hugoniot curve of relativistic shocks, and also showed that γ, the ratio of specific heats, must be less than $\frac{5}{3}$. He gave a more systematic description of relativistic hydrodynamics in [T2]. In 1986, Thompson [Th] established several relations on the relativistic shock curves. He observed: "The relativistic shock equations are much more complicated and do not lend themselves to expressions that are both simple and general." More recently, instead of working on the full system, Smoller & Temple [ST] consider the system of conservation laws of energy and momentum in special relativity. For the equation of state $p=\sigma^2\rho$, they solved the Riemann problem and the Cauchy problem for the system. Chen [C] extended their results to the general relativistic p-system where the equation of state is $p=p(\rho)$, $p'(\rho)>0$ and $p''(\rho)\geqq 0$.

In this paper, we rigorously develop the mathematical theory of relativistic fluid dynamics. Although the system is much more complicated and the results are much harder to obtain, we establish a complete analogy between classical and relativistic hydrodynamics. We also show that the Newtonian limits of our results reduce to the classical results for which the reader is referred to [S, Chap. 18]. Because of the complexity of the relativistic system, we develop new techniques and criteria. By using the Riemann invariants

$$R = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) + \int_0^\rho \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho, \qquad S = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) - \int_0^\rho \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho$$

we introduce a simple criterion to exclude the formation of vacuum, that is,

$$R_L > S_R$$

where R_L is the value of R at $x \leqq 0$ and S_R is the value of R at $x > 0$. We notice that in relativistic p-systems, the criterion to exclude the formation of vacuum takes the same form [C]. We also note that in nonrelativistic cases, the criterion to exclude the formation of vacuum can also be written as $R_L > S_R$ as we shall see later. In fact, this criterion can be used in general hyperbolic systems to detect the loss of strict hyperbolicity. The use of the (p, v)-plane to solve the Riemann problem, following [CF], has been restricted by the seemingly paradoxical situation that there are two distinct admissible solutions [STX]. We show that the (p, v)-plane can always be used to uniquely determine the solution of the Riemann problem, as long as we keep in mind the correct entropy condition.

This paper is organized as follows. In Section 2, we give the background of the problem. In Section 3 we determine the characteristic speeds of the system, which are

$$\lambda_1 = \frac{v - \sqrt{p_\rho}}{1 - v\sqrt{p_\rho}}, \qquad \lambda_2 = v, \qquad \lambda_3 = \frac{v + \sqrt{p_\rho}}{1 + v\sqrt{p_\rho}}.$$

We can recognize that λ_1 and λ_3 are the sums of sound speed and particle speed under the Lorentz transformation. From here we obtain a simple formula

$$\sqrt{p_\rho}$$

for the speed of sound in the relativistic setting. This is much simpler than TAUB's [T2] original formula, which is given by

$$\sqrt{\frac{n}{1+i}\left(\frac{di}{dn}\right)_s}$$

where $i = (e+p)/n$ is the rest specific enthalpy. We prove below that the speed of sound is strictly less than the speed of light. We prove that the system is strictly hyperbolic with the first and third characteristic families genuinely nonlinear and the second characteristic family linearly degenerate and we find the Riemann invariants. In Section 4, we show that the Lax shock inequalities are satisfied globally and entropy is monotone along the shock curves. We also derive the condition preventing the occurrence of a vacuum and then prove the existence and uniqueness of the solution of the Riemann problem.

2. Preliminaries

This section is a brief account of Section 10, Chapter 2 of Weinberg [W]. For more details, consult that book. In the flat 2-dimensional space-time, the Minkovski metric takes the form

$$\eta^{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.1}$$

The speed of light is taken to be unity, i.e., $c = 1$. The general conservation law is

$$\nabla_i T^{ij} = 0, \qquad j = 0, 1, \tag{2.2}$$

where T^{ij} is the stress-energy tensor for a perfect fluid. In special relativity, (see [W, ST]) it is given by

$$T^{ij} = (p + \rho) U^i U^j + p\eta^{ij}, \qquad i, j = 0, 1, \tag{2.3}$$

where p is the pressure, ρ is the proper energy density, and U^i is the velocity vector, i.e.,

$$U^0 = \frac{1}{(1 - v^2)^{1/2}}, \qquad U^1 = \frac{v}{(1 - v^2)^{1/2}},$$

in which v is the velocity of the moving particle. We can write down T^{ij} explicitly as

$$\begin{pmatrix} T^{00} & T^{01} \\ T^{10} & T^{11} \end{pmatrix} = \begin{pmatrix} \frac{\rho + pv^2}{1 - v^2} & (p + \rho)\frac{v}{1 - v^2} \\ (p + \rho)\frac{v}{1 - v^2} & p + (p + \rho)\frac{v^2}{1 - v^2} \end{pmatrix}. \tag{2.4}$$

The conservation laws of energy and momentum thus become

$$\frac{\partial}{\partial t}\left((\rho + p)\frac{v^2}{1 - v^2} + \rho\right) + \frac{\partial}{\partial x}\left((\rho + p)\frac{v}{1 - v^2}\right) = 0, \tag{2.5}$$

$$\frac{\partial}{\partial t}\left((\rho + p)\frac{v}{1 - v^2}\right) + \frac{\partial}{\partial x}\left((\rho + p)\frac{v^2}{1 - v^2} + p\right) = 0. \tag{2.6}$$

Apart from energy and momentum, the rest mass is also conserved in the fluid:

$$\frac{\partial}{\partial t}\left(n(1 - v^2)^{-1/2}\right) + \frac{\partial}{\partial x}\left(nv(1 - v^2)^{-1/2}\right) = 0, \tag{2.7}$$

where n is the density of the rest mass.

As in [Th], we assume that the equation of state is given by

$$\rho - n = p/(\gamma - 1), \tag{2.8}$$

$$p = ksn^{\gamma - 1}, \tag{2.9}$$

where s is the entropy and k and $\gamma, 1 < \gamma < \frac{5}{3}$, are positive constants.

From the conservation of energy, momentum and rest mass, as in the classical case, we can deduce the entropy equation

$$\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x} = 0\,. \tag{2.10}$$

Throughout the paper we assume that the reader is familiar with the notions and terminology of conservation laws as discussed in [L, S], for example.

3. Eigenvalues and Eigenvectors of the System

Our system is

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(n(1-v^2)^{-1/2}\right) + \frac{\partial}{\partial x}\left(nv(1-v^2)^{-1/2}\right) = 0,\\
&\frac{\partial}{\partial t}\left((\rho+p)\frac{v}{1-v^2}\right) + \frac{\partial}{\partial x}\left((\rho+p)\frac{v^2}{1-v^2}+p\right) = 0,\\
&\frac{\partial}{\partial t}\left((\rho+p)\frac{v^2}{1-v^2}+\rho\right) + \frac{\partial}{\partial x}\left((\rho+p)\frac{v}{1-v^2}\right) = 0
\end{aligned} \tag{3.1}$$

where the pressure $p = (\gamma-1)(\rho-n)$.

First we will find the characteristic speeds of the system. Let

$$A = \begin{pmatrix} \frac{n}{\sqrt{1-v^2}} \\ (\rho+p)\frac{v}{1-v^2} \\ (\rho+p)\frac{v^2}{1-v^2}+\rho \end{pmatrix}, \quad B = \begin{pmatrix} \frac{nv}{\sqrt{1-v^2}} \\ (\rho+p)\frac{v^2}{1-v^2}+p \\ (\rho+p)\frac{v}{1-v^2} \end{pmatrix}.$$

The Jacobian of A with respect to (n, v, ρ) is

$$\begin{aligned}
JA &= \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{nv}{(1-v^2)^{3/2}} & 0 \\ \frac{-(\gamma-1)v}{1-v^2} & \frac{(\rho+p)}{1-v^2} + 2\frac{(\rho+p)v^2}{(1-v^2)^2} & \frac{\gamma v}{1-v^2}+1 \\ -\frac{(\gamma-1)v^2}{1-v^2} & 2\frac{(\rho+p)v}{1-v^2} + 2\frac{(\rho+p)v^3}{(1-v^2)^2} & \frac{gv^2}{1-v^2}+1 \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{nv}{(1-v^2)^{3/2}} & 0 \\ \frac{-(\gamma-1)v}{1-v^2} & \frac{(\rho+p)(1+v^2)}{(1-v^2)^2} & \frac{\gamma v}{1-v^2}+1 \\ -\frac{(\gamma-1)v^2}{1-v^2} & 2\frac{(\rho+p)v}{(1-v^2)^2} & \frac{gv^2}{1-v^2}+1 \end{pmatrix}.
\end{aligned} \tag{3.2}$$

The Jacobian of B with respect to (n, v, ρ) is

$$JB = \begin{pmatrix} \frac{v}{\sqrt{1-v^2}} & \frac{n}{(1-v^2)^{3/2}} & 0 \\ -\frac{\gamma-1}{1-v^2} & 2\frac{v(\rho+p)}{(1-v^2)^2} & \frac{\gamma-1+v^2}{1-v^2} \\ -\frac{(\gamma-1)v}{1-v^2} & \frac{(\rho+p)(1+v^2)}{(1-v^2)^2} & \frac{\gamma v}{1-v^2} \end{pmatrix}. \tag{3.3}$$

Thus the Jacobian of the system is

$$J = JB \cdot JA^{-1}$$

(3.4)
$$= \begin{pmatrix} \frac{(-(\gamma-1)(\rho+p)v^2+\gamma\rho)v}{\rho+p-\gamma pv^2} & \frac{n((\gamma-1)v^2+1)(1-v^2)^{1/2}}{\rho+p-\gamma pv^2} & -\frac{\gamma nv(1-v^2)^{1/2}}{-\gamma pv^2+\rho+p} \\ \frac{(\gamma-1)(\rho+p)(1-v^2)^{3/2}}{-\gamma pv^2+\rho+p} & \frac{[(\gamma-1)nv^2+2\gamma(\rho-p)-3(\gamma-1)n]v}{p+\rho-\gamma pv^2} & \frac{(\gamma-1)(p+\rho)-\gamma\rho v^2}{p+\rho-\gamma pv^2} \\ 0 & 1 & 0 \end{pmatrix}.$$

Simple calculations yield the three eigenvalues of J:

(3.5)
$$\lambda_1 = \frac{(\rho-(\gamma-1)p)v - \sqrt{\gamma p(p+\rho)}(1-v^2)}{p+\rho-\gamma pv^2},$$
$$\lambda_2 = v,$$
$$\lambda_3 = \frac{(\rho-(\gamma-1)p)v + \sqrt{\gamma p(p+\rho)}(1-v^2)}{p+\rho+\gamma pv^2}.$$

We can simplify λ_1 and λ_3: Let $\theta = \sqrt{\frac{\gamma p}{p+\rho}}$. Then

(3.6)
$$\lambda_1 = \frac{(\rho-(\gamma-1)p)v - \sqrt{\gamma p(p+\rho)}(1-v^2)}{p+\rho-\gamma pv^2} = \frac{v-\theta}{1-v\theta}.$$

From $p = ksn^\gamma$ and $n = \rho - \frac{p}{\gamma-1}$, we have

$$p = ks\left(\rho - \frac{p}{\gamma-1}\right)^\gamma.$$

Thus

$$\frac{\partial}{\partial\rho}p(\rho,s) = ks\gamma\left(\rho - \frac{p}{\gamma-1}\right)^{\gamma-1}\left[1 - \frac{\partial}{\partial\rho}p(\rho,s)/(\gamma-1)\right]$$
$$= \frac{\gamma p}{\rho - p/(\gamma-1)} - \frac{\gamma p}{\rho-p/(\gamma-1)}\frac{\partial}{\partial\rho}p(\rho,s)/(\gamma-1).$$

Hence we have

$$\frac{(\gamma-1)(p+\rho)}{(\rho-p/(\gamma-1))(\gamma-1)}\frac{\partial}{\partial\rho}p(\rho,s) = \frac{\gamma p}{\rho-p/(\gamma-1)}.$$

Then

(3.7)
$$\frac{\partial}{\partial\rho}p(\rho,s) = \frac{\gamma p}{p+\rho}.$$

Thus, from (3.6),

(3.8)
$$\lambda_1 = \frac{v-\sqrt{p_\rho}}{1-v\sqrt{p_\rho}}.$$

Similarly,

(3.9)
$$\lambda_3 = \frac{v+\sqrt{p_\rho}}{1+v\sqrt{p_\rho}}.$$

If v, the speed of the fluid, is zero, then

$$\lambda_3 = \sqrt{p_\rho}.$$

This shows that the sound speed is $\sqrt{p_\rho}$. Now we prove that the sound speed is less than the speed of light. We have

$$p_\rho = \frac{\gamma p}{p+\rho} = \frac{\gamma}{1+\frac{\rho}{p}}. \tag{3.10}$$

Since $\rho - n = p/(\gamma-1)$,

$$\rho > p/(\gamma - 1)$$

and then

$$\frac{\rho}{p} > 1/\gamma - 1.$$

Thus, from (3.10),

$$p_\rho < \frac{\gamma}{1+1/(\gamma-1)} = \gamma - 1.$$

TAUB [T1, T2] showed that the kinetic theory of gases yields $\gamma \leqq \frac{5}{3}$. Then

$$p_\rho \leqq \tfrac{2}{3},$$

that is, all sound speeds are bounded by $\sqrt{\frac{2}{3}}$ of the speed of light.

Remarks. 1. In the classical case, the sound speed is $\sqrt{p_n}$. Since $\rho - n = e$, and e is very small compared with the rest mass in non-relativistic case,

$$\rho \approxeq n \ .$$

So $\sqrt{p_n}$ is the Newtonian limit of $\sqrt{p_\rho}$. We see that in the relativistic case, the eigenvalues are the sums of the sound speed and the particle speed under the Lorentz transformation, and their Newtonian limits are exactly what we have in the classical situation.
2. In [T2], the sound speed c is given by the complicated formula

$$c^2 = \frac{n}{1+i}\left(\frac{di}{dn}\right)_s$$

where $i = e + p/n$ is the rest specific enthalpy.

It is easy to see that $\lambda_1 < \lambda_2 < \lambda_3$. So the system (3.1) is strictly hyperbolic. Next we show that λ_1 and λ_3 are genuinely nonlinear while λ_2 is linearly degenerate.

We find the eigenvector of each eigenvalue. For simplicity, instead of using the conservation of rest mass, we use the equation of entropy (2.10):

$$\frac{\partial s}{\partial t} + v\frac{\partial s}{\partial x} = 0,$$

to replace the equation of the conservation of mass. Then the system becomes

$$(3.11)\qquad \begin{aligned} &\frac{\partial}{\partial t}\left((\rho+p)\frac{v}{1-v^2}\right)+\frac{\partial}{\partial x}\left((\rho+p)\frac{v^2}{1-v^2}+p\right)=0,\\ &\frac{\partial}{\partial t}\left((\rho+p)\frac{v^2}{1-v^2}+\rho\right)+\frac{\partial}{\partial x}\left((\rho+p)\frac{v}{1-v^2}\right)=0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\frac{\partial s}{\partial t}+v\frac{\partial s}{\partial x}=0 \end{aligned}$$

where $p = p(\rho, s)$. This system is equivalent to (3.1) when we have smooth solutions. In a way like that for the system (3.1), we can find the Jacobian of the system with respect to (ρ, v, s) as

$$(3.12)\qquad \bar{J} = \begin{pmatrix} \frac{(-1+p_\rho)v}{v^2p_\rho-1} & -\frac{p+\rho}{v^2p_\rho-1} & \frac{vp_s(1-v^2)}{v^2p_\rho-1} \\ -\frac{p_\rho(1-v^2)^2}{(p+\rho)(v^2p_\rho-1)} & \frac{(-1+p_\rho)v}{v^2p_\rho-1} & -\frac{(1-v^2)^2p_s}{(p+\rho)(v^2p_\rho-1)} \\ 0 & 0 & v \end{pmatrix}$$

with eigenvalues

$$(3.13)\qquad \lambda_1 = \frac{v-\sqrt{p_\rho(\rho,s)}}{1-v\sqrt{p_\rho(\rho,s)}}, \quad \lambda_2 = v, \quad \lambda_3 = \frac{v+\sqrt{p_\rho(\rho,s)}}{1+v\sqrt{p_\rho(\rho,s)}}.$$

Let

$$\bar{J_2} = \bar{J} - \lambda_2 \cdot I = \begin{pmatrix} \frac{(1-p_\rho)v}{1-v^2p_\rho} - v & \frac{\rho+p}{1-v^2p_\rho} & -\frac{vp_s(1-v^2)}{1-v^2p_\rho} \\ \frac{p_\rho(1-v^2)^2}{(p+\rho)(1-v^2p_\rho)} & \frac{(1-p_\rho)v}{1-v^2p_\rho} - v & \frac{(1-v^2)^2p_s}{(p+\rho)(1-v^2p_\rho)} \\ 0 & 0 & 0 \end{pmatrix}$$

$$(3.14)\qquad = \begin{pmatrix} \frac{(-vp_\rho)(1-v^2)}{1-v^2p_\rho} & \frac{\rho+p}{1-v^2p_\rho} & -\frac{vp_s(1-v^2)}{1-v^2p_\rho} \\ \frac{p_\rho(1-v^2)^2}{(p+\rho)(1-v^2p_\rho)} & \frac{(-vp_\rho)(1-v^2)}{1-v^2p_\rho} & \frac{(1-v^2)^2p_s}{(p+\rho)(1-v^2p_\rho)} \\ 0 & 0 & 0 \end{pmatrix}.$$

If r_2 is the eigenvector corresponding to λ_2, then r_2 satisfies

$$\bar{J_2} \cdot r_2 = 0$$

with solution

$$r_2 = (-p_s, 0, p_\rho).$$

Moreover, $\nabla\lambda_2$, the gradient of λ_2 with respect to (ρ, v, s), is

$$\nabla\lambda_2 = (\partial_\rho\lambda_2, \partial_v\lambda_2, \partial_s\lambda_2) = (\partial_\rho v, \partial_v v, \partial_s v) = (0, 1, 0).$$

Thus

$$(3.15)\qquad \nabla\lambda_2 \cdot r_2 = (0, 1, 0) \cdot (-p_s, 0, p_\rho) = 0,$$

i.e., $\lambda_2 = v_2$ is linearly degenerate.

Now we prove that λ_1 is genuinely nonlinear. Let

$$\bar{J}_1 = \bar{J} - \lambda_1 \cdot I = \begin{pmatrix} \frac{\sqrt{p_\rho}(1-v^2)}{1-v^2 p_\rho} & \frac{\rho+p}{1-v^2 p_\rho} & -\frac{v p_s (1-v^2)}{1-v^2 p_\rho} \\ \frac{p_\rho (1-v^2)^2}{(p+\rho)(1-v^2 p_\rho)} & \frac{\sqrt{p_\rho}(1-v^2)}{1-v^2 p_\rho} & \frac{(1-v^2)^2 p_s}{(p+\rho)(1-v^2 p_\rho)} \\ 0 & 0 & \frac{\sqrt{p_\rho}(1-v^2)}{1-v\sqrt{p_\rho}} \end{pmatrix}.$$

Thus r_1, the eigenvector of $\lambda_1 = \frac{v-\sqrt{p(\rho,s)_\rho}}{1-v\sqrt{p(\rho,s)_\rho}}$, satisfies

$$\bar{J}_1 \cdot r_1 = 0$$

with solution

$$r_1 = \left(1, -\frac{\sqrt{p_\rho}(1-v^2)}{p+\rho}, 0\right).$$

We find that

$$\begin{aligned} \nabla \lambda_1 &= (\partial_\rho \lambda_2, \partial_v \lambda_2, \partial_s \lambda_2) \\ &= \left(-\frac{1}{2}\frac{\partial^2 p}{\partial \rho^2}(1-v^2)(v\sqrt{p_\rho}-1)^2\sqrt{p_\rho}, \frac{1-p_\rho}{(1-v\sqrt{p_\rho})^2}, -\frac{1}{2}\frac{\frac{\partial^2 p}{\partial s \partial r}(1-v^2)}{(1-v\sqrt{p_\rho})^2\sqrt{p_\rho}}\right), \end{aligned}$$

so

$$r_1 \cdot \nabla \lambda_1 = -\frac{1}{2}(1-v^2)\frac{\frac{\partial^2 p}{\partial \rho^2}(p+\rho) + 2p_\rho(1-p_\rho)}{(1-v\sqrt{p_\rho}\,)^2\sqrt{p_\rho}(p+\rho)}. \tag{3.16}$$

From (3.7), $\frac{\partial p}{\partial \rho} = \frac{\gamma p}{p+\rho}$, and so

$$\frac{\partial^2 p}{\partial \rho^2} = \frac{\partial}{\partial \rho}\frac{\gamma p}{p+\rho} = \frac{\gamma(\gamma-1)p}{(p+\rho)^3} n > 0.$$

Since $\frac{\partial^2 p}{\partial \rho^2} > 0$, $r_1 \cdot \nabla \lambda_1$ is always negative. Thus λ_1 is genuinely nonlinear. Similarly, we can prove that λ_3 is genuinely nonlinear.

We now find the Riemann invariants of each eigenvalue. By definition, a Riemann invariant w of λ_j satisfies

$$r_j \cdot (w_\rho, w_v, w_s) = 0.$$

For λ_1, $r_1 = \left(1, -\frac{\sqrt{p_\rho}(1-v^2)}{p+\rho}, 0\right)$, and the equation $r_1 \cdot (w_\rho, w_v, w_s) = 0$ is equivalent to

$$w_\rho - w_v \frac{\sqrt{p_\rho}(1-v^2)}{p+\rho} = 0.$$

We can easily see that s is a Riemann invariant. The other one is constant along the curve determined by the differential equation

$$\frac{d\rho}{dv} = -\frac{p+\rho}{\sqrt{p_\rho}(1-v^2)},$$

which we solve to get

$$\frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)+\int\frac{\sqrt{p_\rho}}{p+\rho}\,d\rho = \text{ constant.} \tag{3.17}$$

Thus $\frac{1}{2}\ln(\frac{1+v}{1-v})+\int\frac{\sqrt{p_\rho}}{p+\rho}d\rho$ is another Riemann invariant of λ_1. Similarly, the Riemann invariants of λ_3 can be found to be

$$\text{s and } \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right)-\int\frac{\sqrt{p_\rho}}{p+\rho}\,d\rho. \tag{3.18}$$

The eigenvector of λ_2 is $(-p_s, 0, p_\rho)$. A Riemann invariant must satisfy

$$-p_s \cdot w_\rho + p_\rho \cdot w_s = 0$$

which has two easy solutions v and p. So the two Riemann invariants of λ_2 are

$$v \quad \text{and} \quad p. \tag{3.19}$$

4. The Riemann Problem

In this section, we discuss the Riemann problem of system (3.1), namely the initial-value problem with initial data $U_0(x) \equiv (\rho_0(x), v_0(x), p_0(x))$ consisting of a pair of constant states $U_L \equiv (\rho_L, v_L, p_L)$ *and* $U_R \equiv (\rho_R, v_R, p_R)$ separated by a jump discontinuity at $x = 0$, that is,

$$U_0(x) = \begin{cases} U_L & \text{if} \quad x \leqq 0, \\ U_R & \text{if} \quad x > 0. \end{cases}$$

To study the Riemann problem, we must first investigate the properties of the shock curves. In [T1], Taub found the Hugoniot curve of relativistic shocks, namely,

Lemma 4.1. [T1]. *The Hugoniot curve of relativistic shocks is*

$$\frac{\rho+p}{n^2}(\rho+p_L) = \frac{\rho_L+p_L}{n_L^2}(\rho_L+p), \tag{4.1}$$

where (n, ρ, p) represent the rest mass density, energy density and pressure at right.

Proof. From (3.1), our system is

$$\begin{aligned}
\frac{\partial}{\partial t}\left(n(1-v^2)^{-1/2}\right)+\frac{\partial}{\partial x}\left(nv(1-v^2)^{-1/2}\right) &= 0,\\
\frac{\partial}{\partial t}\left((\rho+p)\frac{v}{1-v^2}\right)+\frac{\partial}{\partial x}\left((\rho+p)\frac{v^2}{1-v^2}+p\right) &= 0,\\
\frac{\partial}{\partial t}\left((\rho+p)\frac{v^2}{1-v^2}+\rho\right)+\frac{\partial}{\partial x}\left((\rho+p)\frac{v}{1-v^2}\right) &= 0,
\end{aligned}$$

where the pressure $p = (\gamma-1)(\rho-n)$.

Take a coordinate system in which the shock speed is zero. The Rankine-Hugoniot conditions are

$$[F] = s[U] = 0, \tag{4.2}$$

that is,

$$\frac{nv}{\sqrt{1-v^2}} = \frac{n_L v_L}{\sqrt{1-v_L^2}}, \tag{4.3}$$

$$(\rho + p)\frac{v^2}{1-v^2} + p = (\rho_L + p_L)\frac{v_L^2}{1-v_L^2} + p_L, \tag{4.4}$$

$$(\rho + p)\frac{v}{1-v^2} = (\rho_L + p_L)\frac{v_L}{1-v_L^2}. \tag{4.5}$$

Define

$$\frac{nv}{\sqrt{1-v^2}} = \frac{n_L v_L}{\sqrt{1-v_L^2}} = M. \tag{4.6}$$

If $v = 0$, then $M = 0$. So $v_L = 0$, that is, the speed of both sides are equal. This is a contact discontinuity. In shock waves, we can assume that $M \neq 0$.

From (4.4), we have

$$\frac{nv}{\sqrt{1-v^2}} \cdot \frac{\rho + p}{n} \frac{v}{\sqrt{1-v^2}} - \frac{n_L v_L}{\sqrt{1-v_L^2}} \cdot \frac{\rho_L + p_L}{n_L} \frac{v_L}{\sqrt{1-v_L^2}} = -(p - p_L),$$

that is,

$$M\left[\frac{\rho + p}{n} \frac{v}{\sqrt{1-v^2}} - \frac{\rho_L + p_L}{n_L} \frac{v_L}{\sqrt{1-v_L^2}}\right] = -(p - p_L). \tag{4.7}$$

From (4.5), we deduce that

$$M\left[\frac{\rho + p}{n} \frac{1}{\sqrt{1-v^2}} - \frac{\rho_L + p_L}{n_L} \frac{1}{\sqrt{1-v_L^2}}\right] = 0. \tag{4.8}$$

From (4.7), (4.8), (4.6) we obtain

$$\begin{aligned} &M\left[\frac{\rho + p}{n} \frac{1}{\sqrt{1-v^2}} - \frac{\rho_L + p_L}{n_L} \frac{1}{\sqrt{1-v_L^2}}\right] * \frac{1}{\sqrt{1-v^2}} \\ &\quad - M\left[\frac{\rho + p}{n} \frac{v}{\sqrt{1-v^2}} - \frac{\rho_L + p_L}{n_L} \frac{v_L}{\sqrt{1-v_L^2}}\right] * \frac{v}{\sqrt{1-v^2}} \\ &\quad = (p - p_L)\frac{v}{\sqrt{1-v^2}} = (p - p_L) \cdot \frac{M}{n}, \end{aligned}$$

that is,

$$\frac{\rho + p}{n} - \frac{\rho_L + p_L}{n_L}\left(\frac{1 - v v_L}{\sqrt{1-v_L^2}\sqrt{1-v^2}}\right) = (p - p_L) \cdot \frac{1}{n}; \tag{4.9}$$

similarly,

$$\frac{\rho+p}{n}\frac{1-vv_L}{\sqrt{1-v^2}\sqrt{1-v_L^2}}-\frac{\rho_L+p_L}{n_L}=(p-p_L)\frac{1}{n_L}. \tag{4.10}$$

From (4.9), (4.10), it follows that

$$\begin{aligned}&\left(\frac{\rho+p}{n}-\frac{\rho_L+p_L}{n_L}\left(\frac{1-vv_L}{\sqrt{1-v_L^2}\sqrt{1-v^2}}\right)\right)*\frac{\rho+p}{n}\\ &+\left(\frac{\rho+p}{n}\frac{1-vv_L}{\sqrt{1-v^2}\sqrt{1-v_L^2}}-\frac{\rho_L+p_L}{n_L}\right)*\frac{\rho_L+p_L}{n_L}\\ &=\left(\frac{\rho+p}{n}\right)^2-\left(\frac{\rho_L+p_L}{n_L}\right)^2=(p-p_L)\left[\frac{\rho+p}{n^2}+\frac{\rho_L+p_L}{n_L^2}\right].\end{aligned}$$

We rearrange terms to get

$$\left(\frac{\rho+p}{n}\right)^2-\frac{(\rho+p)(p-p_L)}{n^2}=\left(\frac{\rho_L+p_L}{n_L}\right)^2+\frac{(\rho_L+p_L)(p-p_L)}{n_L^2}, \tag{4.11}$$

which can be simplified to

$$\frac{\rho+p}{n^2}(\rho+p_L)=\frac{\rho_L+p_L}{n_L^2}(\rho_L+p). \tag{4.12}$$

This defines the Hugoniot curve of the relativistic shocks. □

Next we show that the Lax entropy conditions hold globally along the shock curves. For the non-relativistic case, see [We, S].

Theorem 4.1. *For system* (3.1), *the Lax entropy conditions are satisfied everywhere along the shock curves.*

Proof. We only consider 1-shocks. The case of 3-shocks can be treated similarly. Without loss of generality, we choose a coordinate system in which $v_L = 0$. If we write the system (3.1) in the form $U_t + F(U)_x = 0$, the jump condition is $\sigma[U] = [F]$, where $[f]$ denotes the jump of f across the shock. We assume that the jump conditions define the shock curve $U = U(\varepsilon)$ with shock speed $\sigma = \sigma(\varepsilon), \varepsilon \leqq 0$. From the general theory of conservation laws, we know that for ε negative and small, $\lambda_1(\varepsilon) < \sigma(\varepsilon) < \lambda_1(0)$. We show that this inequality holds everywhere along the shock curve.

Thus, suppose that ε_1 is the first point where $\lambda_1(\varepsilon) = \sigma(\varepsilon)$, $\varepsilon_1 < 0$. Since

$$\sigma'[U] + \sigma U' = dFU', \tag{4.13}$$

if we consider this at $\varepsilon = \varepsilon_1$, and multiply it by the left eigenvector $l_1(\varepsilon_1)$, we get $\sigma' l_1 \cdot [U] = 0$. Suppose now that

$$l_1 \cdot [U] \neq 0; \tag{4.14}$$

then $\sigma'(\varepsilon_1) = 0$; hence at ε_1, $\lambda_1 U' = dFU'$, so that $U' = r_1$. Then

$$\frac{d}{dp}(\sigma - \lambda_1)\bigg|_{\varepsilon=\varepsilon_1} = -\frac{d\lambda_1}{dp}\bigg|_{\varepsilon=\varepsilon_1} = -\nabla\lambda_1 \cdot r_1 = -1.$$

Thus $\lambda_1 = \sigma$ for some ε_2, $\varepsilon_1 < \varepsilon_2 < 0$; this contradicts the definition of ε_1. We conclude that if (4.14) holds, then $\sigma(\varepsilon) > \lambda_1(\varepsilon)$ for all ε. Furthermore, if $\sigma(\varepsilon_1) = \lambda_1(0)$ for some $\varepsilon_1 < 0$, then there is an ε_2 with $\varepsilon_1 < \varepsilon_2 < 0$ so that $\sigma'(\varepsilon_2) = 0$. Now $\sigma(\varepsilon_2) > \lambda_1(\varepsilon_2)$ so (4.13) at ε_2 gives $\sigma U' = dFU'$. Thus $\sigma'(\varepsilon)$ does not change sign. Hence $\sigma' > 0$ so $\sigma(\varepsilon) < \lambda_1$, if $\varepsilon < 0$.

To finish the proof, we need to show that (4.14) holds. From (3.4),

$$J_1 = J - \lambda_1 * I = \begin{pmatrix} a_{11} & \frac{n[(\gamma-1)v^2+1]\sqrt{1-v^2}}{p+\rho-\gamma p v^2} & \frac{-\gamma n v\sqrt{1-v^2}}{p+\rho-\gamma p v^2} \\ \frac{(\gamma-1)(p+\rho)(1-v^2)^{3/2}}{p+\rho-\gamma p v^2} & a_{22} & \frac{(\gamma-1)(p+\rho)-\gamma\rho v^2}{p+\rho-\gamma p v^2} \\ 0 & 1 & -\frac{v-\sqrt{p_\rho}}{1-v\sqrt{p_\rho}} \end{pmatrix}$$

(4.15)

where

$$a_{11} = \frac{[(\gamma-1)(p+\rho)(1-v^2)]v + (1-v^2)\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2},$$
$$a_{22} = \frac{[(\gamma-1)nv^2 + (2-\gamma)(p+\rho)]v + (1-v^2)\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2}.$$

The left eigenvector l_1 of J_1 satisfies

$$l_1 \cdot J_1 = 0. \tag{4.16}$$

Since the third element of the first column is zero, we can set

$$l_1[1] = J_1[2,1] = \frac{(\gamma-1)(p+\rho)(1-v^2)^{3/2}}{p+\rho-\gamma p v^2}, \tag{4.17}$$

$$l_1[2] = -J_1[1,1] = \frac{(\gamma-1)(p+\rho)(1-v^2)v + (1-v^2)\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2}. \tag{4.18}$$

Since the third element of the second column is 1,

$$\begin{aligned} l_1[3] &= -\left(l_1[1]\cdot J_1[1,2] + l_1[2]\cdot J_1[2,2]\right) \\ &= -\frac{(\gamma-1)(p+\rho)(1-v^2)^{3/2}}{p+\rho-\gamma p v^2}\frac{n[(\gamma-1)v^2+1]\sqrt{1-v^2}}{p+\rho-\gamma p v^2} \\ &\quad + \frac{(\gamma-1)(p+\rho)(1-v^2)v + (1-v^2)\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2} \\ &\quad * \frac{[(\gamma-1)nv^2 + (2-\gamma)(p+\rho)]v + (1-v^2)\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2} \\ &= -\frac{(1-v^2)[-(\gamma-1)(p+\rho) + v\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2}. \end{aligned} \tag{4.19}$$

We also have

$$(4.20)\quad [U] = \left(\frac{n}{\sqrt{1-v^2}} - n_L,\ (\rho+p)\cdot\frac{v}{1-v^2},\ (\rho+p)\frac{v^2}{1-v^2}+\rho-\rho_L\right).$$

From (4.12) the Hugoniot curve is

$$\frac{(p+\rho)(p_L+\rho)}{n^2} = \frac{(\rho_L+p_L)(\rho_L+p)}{n_L^2}.$$

From [C], the shock curves satisfy

$$(4.21)\qquad v^2 = \frac{(p-p_L)(\rho-\rho_L)}{(p+\rho_L)(p_L+p)}.$$

Since $p > p_L$ in the 1-shock, we conclude that

$$(4.22)\qquad \rho > \rho_L.$$

Thus

$$(p+\rho)(p_L+\rho) > (p+\rho_L)(p_L+\rho_L).$$

Then from the Hugoniot curve, we obtain

$$(4.23)\qquad n > n_L.$$

Now

$$\begin{aligned}
l_1\cdot[U] &= \frac{(\gamma-1)(p+\rho)(1-v^2)^{3/2}}{p+\rho-\gamma p v^2}\cdot\left(\frac{n}{\sqrt{1-v^2}}-n_L\right)\\
&\quad -\frac{(\gamma-1)(p+\rho)(1-v^2)v+(1-v^2)\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2}\cdot(\rho+p)\cdot\frac{v}{1-v^2}\\
&\quad -\frac{(1-v^2)[-(\gamma-1)(p+\rho)+v\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2}\cdot\left[(\rho+p)\frac{v^2}{1-v^2}+\rho-\rho_L\right]\\
&= \frac{(\gamma-1)(p+\rho)(1-v^2)^{3/2}}{p+\rho-\gamma p v^2}\cdot\left(\frac{n}{\sqrt{1-v^2}}-n_L\right)\\
&\quad -\frac{\sqrt{\gamma p(p+\rho)}}{p+\rho-\gamma p v^2}\cdot(\rho+p)v-\frac{[v\sqrt{\gamma p(p+\rho)}]}{p+\rho-\gamma p v^2}\cdot(\rho+p)v^2\\
&\quad -\frac{(1-v^2)[-(\gamma-1)(p+\rho)+v\sqrt{\gamma p(p+\rho)}]}{p+\rho-\gamma p v^2}\cdot(\rho-\rho_L)\\
&> 0
\end{aligned}$$

since each of the four terms is bigger than zero. Thus

$$l_1\cdot[U] \neq 0.$$

This finishes the proof. □

To show that the shock curves are physically relevant, we need to prove that the entropy changes monotonically along the shock curves.

Theorem 4.2. *The entropy increases monotonically along* 1-*shock curves*, *and decreases monotonically along* 3-*shock curves*.

Proof. We consider 1-shocks first. From $p = ksn^{\gamma}$, we get

$$s = k^{-1} p n^{-\gamma}, \tag{4.24}$$

so

$$\frac{ds}{dp} = k^{-1} n^{-\gamma} + k^{-1} p(-\gamma) n^{-\gamma-1} \frac{dn}{dp}. \tag{4.25}$$

We need $ds/dp > 0$, whence

$$\frac{dn}{dp} < \frac{n}{\gamma p}. \tag{4.26}$$

Let us compute dn/dp along the Hugoniot curve

$$\frac{p+\rho}{n^2}(p_L + \rho) = \frac{p_L + \rho_L}{n_L^2}(p + \rho_L)$$

where $\rho = n + p/(\gamma - 1)$. We can rewrite this as

$$\left(\frac{\gamma}{\gamma-1} p + n\right)\left(p_L + n + \frac{p}{\gamma-1}\right) n_L^2 = (p_L + \rho_L)(p + \rho_L) n^2.$$

Differentiating both sides with respect to p, we get, after a straightforward calculation that

$$\begin{aligned}
&\left[(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2\right] \frac{dn}{dp} \\
&\quad = \frac{\gamma}{\gamma-1}(p_L + \rho)n_L^2 + (p + \rho)n_L^2 \frac{1}{\gamma-1} - (p_L + \rho_L)n^2,
\end{aligned}$$

where we replace $n + p/(\gamma - 1)$ by ρ in the above equation. We want to prove that the coefficient of dn/dp is positive; i.e., that

$$(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2 > 0. \tag{4.27}$$

It suffices to show that

$$(p_L + \rho_L)(p + \rho_L)n - (p + \rho)n_L^2 > 0 \tag{4.28}$$

since $(p + \rho)n_L^2 > (p_L + \rho)n_L^2$. But from the Hugoniot jump condition (4.12),

$$(p_L + \rho_L)(p + \rho_L)n - (p + \rho)n_L^2 = \frac{(p + \rho)n_L^2}{n}[p_l + \rho - n] > 0.$$

Thus

$$\frac{dn}{dp} = \frac{\frac{\gamma}{\gamma-1}(p_L + \rho)n_L^2 + (p + \rho)n_L^2 \frac{1}{\gamma-1} - (p_L + \rho_L)n^2}{(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2}.$$

We need to show, from (4.26), that

$$\frac{dn}{dp} < \frac{n}{\gamma p},$$

that is,

$$\frac{\frac{\gamma}{\gamma-1}(p_L+\rho)n_L^2+(p+\rho)n_L^2\frac{1}{\gamma-1}-(p_L+\rho_L)n^2}{(p_L+\rho_L)(p+\rho_L)2n-(p_L+\rho)n_L^2-(p+\rho)n_L^2}<\frac{n}{\gamma p}.$$

After a straightforward calculation, using (4.12), we get

$$\begin{aligned}&\left[\frac{\gamma}{\gamma-1}(p_L+\rho)n_L^2+(p+\rho)n_L^2\frac{1}{\gamma-1}-(p_L+\rho_L)n^2\right]\gamma p\\&\quad-\left[(p_L+\rho_L)(p+\rho_L)2n-(p_L+\rho)n_L^2-(p+\rho)n_L^2\right]n\\&=n_L^2\left\{\frac{p_L+\rho}{p+\rho_L}\gamma p[\rho_L-\rho]+(p+\rho)(p-p_L)\right\}.\end{aligned}$$

Thus we need to show that

$$\frac{p_L+\rho}{p+\rho_L}\frac{\gamma p}{p+\rho}>\frac{p-p_L}{\rho-\rho_L}. \tag{4.29}$$

First we prove that

$$\frac{\gamma p}{p+\rho}>\frac{p-p_L}{\rho-\rho_L}. \tag{4.30}$$

We fix the point (p,ρ) and let (p_L,ρ_L) move along the Hugoniot curve:

$$\frac{p+\rho}{n^2}(p_L+\rho)=\frac{p_L+\rho_L}{n_L^2}(p+\rho_L).$$

Differentiating both sides of this equation with respect to ρ_L and rearranging terms we get

$$\frac{dp_L}{d\rho_L}=\frac{2n_L(p+\rho)(p_L+\rho)-(p+\rho_L)n^2-(p_L+\rho_L)n^2}{(p+\rho_L)n^2-(p+\rho)n_L^2+2/(\gamma-1)(p+\rho)(p_L+\rho)n_L}.$$

We have

$$(p+\rho_L)n^2-(p+\rho)n_L^2>0 \tag{4.31}$$

since, from (4.22),

$$p_L+\rho_L<p_L+\rho$$

and since the Hugoniot jump condition

$$(p+\rho_L)n^2(p_L+\rho_L)=(p+\rho)n_L^2(p_L+\rho)$$

holds. Thus the denominator is bigger than zero. By the intermediate value theorem,

$$\frac{p-p_L}{\rho-\rho_L}=\frac{dp(\xi)}{d\rho(\xi)}$$

for some $\rho_L<\rho(\xi)<\rho$. For convenience, we write $p_0=p(\xi),\ \rho_0=\rho(\xi)$. So it is sufficient to prove that

$$\frac{dp_0}{d\rho_0}<\frac{\gamma p}{p+\rho},$$

that is,

$$\frac{2n_0(p+\rho)(p_0+\rho)-(p+\rho_0)n^2-(p_0+\rho_0)n^2}{(p+\rho_0)n^2-(p+\rho)n_0^2+2/(\gamma-1)(p+\rho)(p_0+\rho)n_0}-\frac{\gamma p}{p+\rho}<0. \tag{4.32}$$

After a straightforward calculation we find

$$\begin{aligned}
&\left[2n_0(p+\rho)(p_0+\rho)-(p+\rho_0)n^2-(p_0+\rho_0)n^2\right](p+\rho)\\
&\qquad-\left[(p+\rho_0)n^2-(p+\rho)n_0^2+2/(\gamma-1)(p+\rho)(p_0+\rho)n_0\right]\gamma p\\
&=2n_0(p+\rho)(p_0+\rho)n-(p+\rho)(p+\rho_0)n^2-(p+\rho)(p_0+\rho_0)n^2\\
&\qquad-\left[(p+\rho_0)n^2-(p+\rho)n_0^2\right]\gamma p\\
&<2n_0(p+\rho)(p_0+\rho)n-(p+\rho)(p+\rho_0)n^2-(p+\rho)(p_0+\rho_0)n^2\\
&=(p+\rho)n[2n(p_0+\rho)-(p+\rho_0)n-(p_0+\rho_0)n].
\end{aligned}$$

We need to prove that

$$2n_0(p_0+\rho)-(p+\rho_0)n-(p_0+\rho_0)n<0. \tag{4.33}$$

Since

$$(p+\rho_0)n+(p_0+\rho_0)n>2\sqrt{(p+\rho_0)(p_0+\rho_0)}\,n,$$

we only have to show

$$\sqrt{(p+\rho_0)(p_0+\rho_0)}\,n>n_0(p_0+\rho)\ .$$

But from (4.12), we have

$$(p+\rho_0)(p_0+\rho_0)n^2=(p+\rho)(p_0+\rho)n_0^2>n_0^2(p_0+\rho)^2.$$

Thus

$$\frac{dp_0}{d\rho_0}<\frac{\gamma p}{p+\rho}$$

and hence

$$\frac{p-p_L}{\rho-\rho_L}<\frac{\gamma p}{p+\rho}. \tag{4.34}$$

Since the speed of sound $\sqrt{\frac{\gamma p}{p+\rho}}$ is less than 1, we have

$$\frac{p-p_L}{\rho-\rho_L}<1.$$

Hence

$$\frac{\rho+p_L}{\rho_L+p}>1.$$

By virtue of (4.34), we arrive at (4.29). Thus

$$\frac{ds}{dp}>0$$

along the Hugoniot curve. The entropy increases monotonically along 1-shock curves. Similarly, we can prove that the entropy decreases monotonically along 3-shock curves. □

Now we give a condition to prevent a vacuum. Since along 2-waves, the pressure p is constant, and $p = sn^\gamma$, a vacuum does not occur when a 1-wave interacts with a 2-wave or when a 3-wave interacts with a 2-wave. It is easy to see that only the interaction of a 1-rarefaction and a 3-rarefaction wave may possibly create vacuum.

From (3.17), (3.18), it follows that

$$R = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) + \int_0^\rho \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho, \quad S = \frac{1}{2}\ln\left(\frac{1+v}{1-v}\right) - \int_0^\rho \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho \tag{4.35}$$

are two Riemann invariants. Let the initial condition be

$$U(x,0) = \begin{cases} (\rho_L, v_L, p_L), & x < 0, \\ (\rho_R, v_R, p_R), & x \geqq 0. \end{cases}$$

We can thus determine (R_L, S_L) and (R_R, S_R) from (4.35), where (R_L, S_L) and (R_R, S_R) are the Riemann invariants at $x < 0$ and $x \geqq 0$. Along 1-rarefaction waves, R and the entropy s are constant; along 3-rarefaction waves, S and the entropy s are constant; along 2-waves the particle speed v and the pressure p are constant. Let us denote the state at the left of the 2-wave as state 1, and the state at the right of the 2-wave as state 2. Then

$$R_1 = R_L, \qquad S_2 = S_R, \qquad v_1 = v_2.$$

Then

$$\begin{aligned} R_L - S_R &= R_1 - S_2 \\ &= \frac{1}{2}\ln\left(\frac{1+v_1}{1-v_1}\right) + \int_0^{\rho_1} \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho - \left(\frac{1}{2}\ln\left(\frac{1+v_2}{1-v_2}\right) - \int_0^{\rho_2} \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho\right) \\ &= \int_0^{\rho_1} \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho + \int_0^{\rho_2} \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho. \end{aligned} \tag{4.36}$$

If $R_L \leqq S_R$, then

$$\int_0^{\rho_1} \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho + \int_0^{\rho_2} \frac{\sqrt{p_\rho}}{p+\rho}\,d\rho \leqq 0,$$

i.e., $\rho \leqq 0$. But since $n = \rho - p/(\gamma - 1) \leqq \rho$,

$$n \leqq 0. \tag{4.37}$$

Hence a vacuum occurs. Thus we have proved

Theorem 4.3. *The condition for a vacuum to occur is*

$$R_L \leqq S_R.$$

Remark. This approach to derive the condition for vacuum to occur is general. When two eigenvalues coalesce and the system loses strict hyperbolicity, the corresponding Riemann invariants should also coalesce. Consider classical gas dynamics for example. The condition for a vacuum to occur is (see [S, p. 355])

$$\frac{2}{\gamma-1}(c_L+c_R) \leqq v_R - v_L$$

where c is the sound speed and v is the particle speed. The pressure for the polytropic gas is

$$p = k^2(s)\rho^{\gamma},$$

where ρ is the density and $k(s) > 0$ is a function of entropy s. The sound speed is

$$c = \sqrt{p_\rho} = k(s)\sqrt{\gamma}\rho^{(\gamma-1)/2}.$$

The two Riemann invariants are

$$R = v + \frac{2}{\gamma-1}k(s)\sqrt{\gamma}\rho^{(\gamma-1)/2} = v + \frac{2}{\gamma-1}c,$$

$$S = v - \frac{2}{\gamma-1}k(s)\sqrt{\gamma}\rho^{(\gamma-1)/2} = v - \frac{2}{\gamma-1}c.$$

So $R_L \leqq S_R$ in this case reduces, after rearrangement, to

$$\frac{2}{\gamma-1}(c_L+c_R) \leqq v_R - v_L,$$

as before. □

We now work on the existence of solutions of the Riemann problem.

Lemma 4.2.

$$\frac{dv}{dp} < 0 \textit{ on } 1\textit{-shock curves}, \quad \frac{dv}{dp} > \ 0 \textit{ on } 3\textit{-shock curves}.$$

Proof. We only consider the 1-shock case. The 3-shock case is similar. From (4.21),

$$v^2 = \frac{(p-p_L)(\rho-\rho_L)}{(p+\rho)(p_L+\rho)}.$$

We have

$$\frac{dv^2}{dp} = 2v\frac{dv}{dp} = \frac{d}{dp}\frac{(p-p_L)(\rho-\rho_L)}{(p+\rho_L)(p_L+\rho)}.$$

Since $v < 0$ on shock curves, $dv/dp < 0$ is equivalent to $dv^2/dp > 0$. But

$$\frac{dv^2}{dp} = \frac{d}{dp}\frac{(p - p_L)(\rho - \rho_L)}{(p + \rho)(p_L + \rho)}$$
$$= \frac{p_L + \rho_L}{(p + \rho_L)^2 (p_L + \rho)^2}\left[(\rho - \rho_L)(p_L + \rho) + (p - p_L)(p + \rho_L)\frac{d\rho}{dp}\right].$$

We need to prove that

$$(\rho - \rho_L)(p_L + \rho) + (p - p_L)(p + \rho_L)\frac{d\rho}{dp} > 0. \tag{4.38}$$

First we find the expression for $d\rho/dp$. The Hugoniot jump condition is

$$(p + \rho)(p_L + \rho)n_L^2 = (p_L + \rho_L)(p + \rho_L)n^2$$

where $n = \rho - p/(\gamma - 1)$.

Differentiating both sides with respect to p, we have

$$\left(1 + \frac{d\rho}{dp}\right)(p_L + \rho)n_L^2 + (p + \rho)\frac{d\rho}{dp}n_L^2$$
$$= (p_L + \rho_L)n^2 + (p_L + \rho_L)(p + \rho_L)2n\left(\frac{d\rho}{dp} - \frac{1}{\gamma - 1}\right),$$

which yields

$$\frac{d\rho}{dp} = \frac{(p_L + \rho)n_L^2 - (p_L + \rho_L)n^2 + (p_L + \rho_L)(p + \rho_L)2n\frac{1}{\gamma-1}}{(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2}. \tag{4.39}$$

Thus we have

$$(\rho - \rho_L)(p_L + \rho) + (p - p_L)(p + \rho_L)\frac{d\rho}{dp}$$
$$= (\rho - \rho_L)(p_L + \rho)$$
$$+ (p - p_L)(p + \rho_L)\frac{(p_L + \rho)n_L^2 - (p_L + \rho_L)n^2 + (p_L + \rho_L)(p + \rho_L)2n\frac{1}{\gamma-1}}{(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2}$$
$$= \frac{(\rho - \rho_L)(p_L + \rho)[(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2]}{(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2}$$
$$+ \frac{(p - p_L)(p + \rho_L)[(p_L + \rho)n_L^2 - (p_L + \rho_L)n^2 + (p_L + \rho_L)(p + \rho_L)2n\frac{1}{\gamma-1}]}{(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2}.$$

From (4.27), the denominator of this above expression is bigger than zero, and the numerator is

$$(\rho - \rho_L)(p_L + \rho)\left[(p_L + \rho_L)(p + \rho_L)2n - (p_L + \rho)n_L^2 - (p + \rho)n_L^2\right]$$
$$+ (p - p_L)(p + \rho_L)\left[(p_L + \rho)n_L^2 - (p_L + \rho_L)n^2 + (p_L + \rho_L)(p + \rho_L)2n\frac{1}{\gamma - 1}\right]$$

$$= 2(\rho - \rho_L)(p_L + \rho)\big[(p + \rho_L)(p_L + \rho_L)n - (p + \rho)n_L^2\big]$$
$$+ (p + \rho_L)(p_L + \rho_L)2n(p + \rho_L)(p - p_L)\frac{1}{\gamma - 1}$$
$$> 0,$$

since from (4.28),

$$(p + \rho_L)(p_L + \rho_L)n - (p + \rho)n_L^2 > 0.$$

So (4.38) is true and hence $dv/dp < 0$ on the 1-shock curves. □

Next we prove a simple property of rarefaction waves.

Lemma 4.3. *$dv/dp < 0$ on 1-rarefaction waves and $dv/dp > 0$ on 3-rarefaction waves.*

Proof. On 1-rarefaction waves, we have

$$R = \frac{1}{2}\ln\left(\frac{1 + v}{1 - v}\right) + \int_0^\rho \frac{\sqrt{p_\rho}}{p + \rho} = \text{constant}.$$

Differentiating both sides with respect to p gives

$$\frac{dv}{dp} = -(1 - v^2)\frac{\sqrt{p_\rho}}{p + \rho}\frac{d\rho}{dp}.$$

But when $s = \text{constant}$, which is the case on rarefaction waves, we have, from (3.7), that

$$\frac{d\rho}{dp} = \frac{p + \rho}{\gamma p} > 0.$$

Thus

$$\frac{dv}{dp} < 0.$$

Similarly, $dv/dp > 0$ on 3-rarefaction waves. □

As in [S], for $\overline{U} \in R^3$, we define

$$S_i(\overline{U}) = \{(p, v, s) : (p, v, s) \text{ can be connected from the left by an } i\text{-shock wave from } \overline{U}\}, \quad i = 1, 3,$$
$$R_i(\overline{U}) = \{(p, v, s) : (p, v, s) \text{ can be connected from the left by an } i\text{-rarefaction wave from } \overline{U}\}, \quad i = 1, 3.$$

Let

$$S_i^p(\overline{U}) = \{(p, v) : (p, v) \in \text{projection of } S_i(\overline{U}) \text{ on the } (p,v)\text{-plane}\}, \quad i = 1, 3,$$
$$R_i^p(\overline{U}) = \{(p, v) : (p, v) \in \text{projection of } R_i(\overline{U}) \text{ on the } (p,v)\text{-plane}\}, \quad i = 1, 3,$$
$$T_i^p(\overline{U}) = S_i^p(\overline{U}) \cup R_i^p(\overline{U}), \quad i = 1, 3.$$

From Lemma 4.2 and Lemma 4.3, we know that $dv/dp < 0$ on T_1^p and $\frac{dv}{dp} > 0$ on T_3^p. So we can define T_1^p by

$$v = f_1(p; p_L,\ v_L, s_L). \tag{4.40}$$

The T_1^p curve divides the (p, v)-plane into two parts. If (p_R, v_R) is below T_1^p, i.e., if

$$v_R < f_1(p_R; p_L,\ v_L, s_L),$$

then the 3-wave of the Riemann problem should be a shock because $dv/dp > 0$ on 3-waves, and on a shock curve, the particle speed at the left is always bigger than the particle speed at the right. On a rarefaction curve, the particle speed at the left is always smaller than the particle speed at that right. If

$$v_R > f_1(p_R; p_L,\ v_L, s_L),$$

the 3-wave of the Riemann problem is a rarefaction wave.

Having determined the 3-wave, we can investigate the 1-wave. We define the inverse 3-wave as

$$v = f_3(p; p_R,\ v_R, s_R), \tag{4.41}$$

where $f_3(p; p_R, v_R, s_R)$ is the inverse 3-rarefaction wave if the third wave is a rarefaction wave or the inverse 3-shock wave if the third wave is 3-shock. If

$$v_L < f_3(p_L; p_R, v_R, s_R),$$

the 1-wave is a rarefaction wave. Otherwise it is a 1-shock.

Solving

$$v = f_1(p; p_L,\ v_L, s_L), \quad v = f_3(p; p_R,\ v_R, s_R) \tag{4.42}$$

we find a unique solution (p_M, v_M) because $dv/dp < 0$ on the 1-wave curve and $dv/dp > 0$ on the 3-wave curve. To complete the solution of the Riemann problem, we need to find the entropy level at the left and right on the contact waves. But since the entropy is uniquely determined by the value of p on shocks and is constant on rarefaction waves, we can obtain s_{M1} by the the value of p_M on 1-curves and s_{M2} by the value of p_M on 3-curves. Thus we deduce

Theorem 4.4. *Consider the system of gas dynamics* (3.1) *for an ideal polytropic gas whose equation of state is* $\rho = n + p/(\gamma - 1)$ *and* $p = ksn^{\gamma-1}$. *Let* U_L *and* U_R *be any two states. Then there is a unique solution to the Riemann problem with these initial states, if and only if*

$$r_L < s_R. \tag{4.43}$$

If (4.43) *is violated, then a vacuum occurs in the solution. The* 3*-component of the solution is a rarefaction if*

$$v_R > f_1(p_R; p_L,\ v_L, s_L),$$

and is a shock otherwise. The 1*-component is a rarefaction if*

$$v_L < f_3(p_L; p_R, v_R, s_R),$$

and is a shock otherwise, where f_1, f_3 *are defined in* (4.40), (4.41). *The solution is unique in the class of constant states separated by rarefaction waves and shock waves.*

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