E u r o E c o n o m i c a
Issue 1(24)/2010 *ISSN: 1582-8859*

Where is the Efficient Frontier

Jing CHEN[1]

[1]*School of Business, University of Northern British Columbia, Prince George, BC, Canada, chenj@unbc.ca*

Abstract. Tremendous effort has been spent on the construction of reliable efficient frontiers. However, mean-variance efficient portfolios constructed using sample means and covariance often perform poorly out of sample. We prove that, the capital market line is the efficient frontier for the risky assets in a financial market with liquid fixed income trading. This unified understanding of riskless asset as the boundary of risky assets relieves the burden of constructing efficient frontiers in asset allocation problems, and allows for much better asset allocation decisions. It greatly simplifies the exposition on investment theory.

***Keywords*:** Capital market line; Efficient frontier; Riskless asset; Risky assets

1 Introduction

Markowitz (1952) first described the efficient frontier of the portfolios of risky assets. Sharpe (1964) and Lintner (1965) showed that under equilibrium, the capital market line, which emanates from the riskless point and is tangent to the efficient frontier at only one point, the market portfolio. In these works, the region of attainable combinations of risky assets is distinctly away from the riskless asset. The dichotomy of risky assets and riskless asset becomes the standard in subsequent works. Although it was a fairly accurate description of the capital markets some time ago, this strict dichotomy is no more consistent with today's highly liquid short term fixed income markets.

Blume and Friend (1973) observed early on that the issuance of low volatility securities such as corporate bonds would improve the efficiency of the security markets and were puzzled that "why corporations did not place more reliance on such financing". Indeed, since Blume and Friend's work, the fixed income markets have grown substantially. Currently the short term fixed income markets are extremely liquid. There are money market, corporate debt market, government debt market and exchange traded interest rate futures market. Suppose the investment horizon is three months, for example. Then a three-month money market contract can be treated as a riskless investment. An outstanding three-month money market contract from one day earlier is a risky asset, for the value of the investment three months from now is determined by the overnight rate at one day less than three months from now. The value uncertainty of the investment after three months will be very small for the short maturity of the reinvestment, i.e., one day. Because of the abundance of money market contracts and other short term fixed income securities that mature at one day less than three months, two days less than three months, and so on, the volatility of risky assets forms a continuous spectrum that converges to zero, the volatility of riskless asset. With this observation in mind, we proceed to prove that the efficient frontier of the risky assets converges to the capital market line. That is, in a well developed market with liquid short term fixed income trading, the capital market line is the efficient frontier for the risky assets.

Blume and Friend (1973) observed thirty years ago that the issuance of low volatility fixed income securities would improve the efficiency of corporate financing. A similar observation can be made that the inclusion of low volatility fixed income securities would improve the performance of portfolio investment. Our results formally prove that the construction of efficient frontier in the traditional way, a regularly practiced exercise in portfolio selection, is redundant and detrimental to investment performance. This is consistent with the observation that mean-variance efficient portfolios constructed using sample means and covariance often involve large negative weights in some assets and perform poorly out of sample (Jagannathan and Ma, 2003).

Since the development of the CAPM theory about forty years ago, riskless and risky assets are always treated separately. This dichotomy doesn't capture the evolving nature of the financial market. Specifically, it doesn't offer the insight that the development of low risk fixed income securities will improve the efficiency of the financial markets. While there is a natural tendency for the profit seeking financial markets to explore new opportunities, an insightful theory would have greatly speed up the process. Although the fixed income markets have eventually become very active over the years, current textbooks on investment still use the dichotomy of riskless asset and risky assets. A unified understanding of riskless asset as the boundary of risky assets offers much more realistic picture of today's financial markets. It also greatly simplifies the exposition on investment theory.

This work is organised as follows. In Section 2, we formally derive the results discussed above. Section 3 concludes.

2 Formal Derivation

Given the abundance and liquidity of the short term fixed income securities, we make the following assumption.

Assumption 1: The volatility of risky assets forms a continuous spectrum that converges to zero.

The second assumption is only for technical convenience. As long as the return of a risky asset converges to the return of the riskless asset when its volatility converges to zero, all the results will hold.

Assumption 2: The expected return of any risky asset is higher than that of the riskless asset.

First, we construct a mean-variance efficient locus from two assets, one the market portfolio, *M*, and another asset with very low volatility, *A*. Suppose the covariance matrix of the two assets is

$$\sum = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

where σ_{11} and σ_{22} are the variances of asset *A* and market portfolio *M* respectively. The expected return vector of *A* and *M* is

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Let σ and μ be the standard deviations and means of the weighted portfolios of *A* and *M*. The mean-variance efficient locus of *A* and *M* is, from Ingersoll(1987) for example,

$$\sigma^2 = \frac{a\mu^2 - 2b\mu + c}{\Delta} \qquad (1)$$

where

$$a \equiv 1'\Sigma^{-1}1 > 0,\ b \equiv 1'\Sigma^{-1}Z,$$
$$c \equiv Z'\Sigma^{-1}Z,\ \ \Delta \equiv ac - b^2 > 0$$

Equation (1) is a parabola connecting *M* and *A* in the mean-variance space.

Proposition 1: Suppose $\sigma_{11} < \varepsilon^2$, where σ_{11} is the variance of asset *A*. Then in the mean-standard deviation space, the distance from any point on the efficient locus of asset *A* and market portfolio *M* to the capital market line is less than ε.

Proof: See Figure 1. *RM* is the capital market line. In the mean-standard deviation space, *AB*, the distance from *A* to the mean axis, is the standard deviation of asset A and hence less than ε. From Assumption 2, *AC*, the distance from *A* to the capital market line, is less than AB and hence less than ε as well. *RM* is tangent to the mean-standard deviation efficient locus at *M* and the distance from the efficient locus to the capital market line is a decreasing function from *A* to *M*. So the distance from any point on the efficient locus of asset *A* and market portfolio *M* to the capital market line is less than ε.

From Proposition 1, we can see that the introduction of lower volatility securities makes the efficient frontier of risky assets closer to the capital market line and hence improves the efficiency of the capital markets. So there is a natural tendency for the financial markets to create lower volatility securities.

The following is the main result of the paper.

Proposition 2: In a well developed financial market with liquid short term fixed income trading, the efficient frontier of all risky assets is the capital market line.

Proof: The efficient frontier of all risky assets must be between the capital market line and the mean-standard deviation efficient locus of two risky assets. Apply Proposition 1 and let ε approach zero.

The above result shows that the attainable combinations of risky assets contain the whole region under the capital market line. The inclusion of riskless asset merely adds the border to the original set. This result is actually very natural after one presents it. From here, it becomes self-evident that the construction of the efficient frontiers in the traditional way is detrimental to investment performance. (Figure 2)

3 Conclusion

Traditionally, risky assets and riskless asset are treated as two distinct classes. Observing the volatility of risky assets forms a continuous spectrum that converges to zero, we view riskless asset as the natural boundary of risky assets. This unifying approach is not only theoretically appealing, but also practically important. It relieves the burden of constructing efficient frontiers in asset allocation problems, removes the restriction posed by the traditional efficient frontiers, and allows for much better asset allocation decisions. It also offers a simple and intuitive theory of portfolio investment.

4 REFERENCES

Blume, Marshall and Irwin Friend, (1973), A new look at the capital asset pricing model, *Journal of Finance*, 28, 19 – 34.
Ingersoll, Jonathan E., (1987), *Theory of Financial Decision Making*, (Rowman & Littlefield, Totowa, N.J.)
Lintner, John, (1965), The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, *Review of economics and statistics*, 47, 13-37
Jagannathan, Ravi and Ma, Tongshu, (2003), Risk reductions in large portfolios: Why imposing the wrong constraints helps, *Journal of Finance*, 58, 1651-1683.
Markowitz, Harry, (1952), Portfolio Selection, *Journal of Finance*, 7, 77-91
Sharpe, William, (1964), Capital asset prices: A theory of market equilibrium under conditions of risk, *Journal of Finance*, 19, 425-442

FIGURE LEGENDS:

Figure 1: **The distance from the efficient locus to the capital market line**

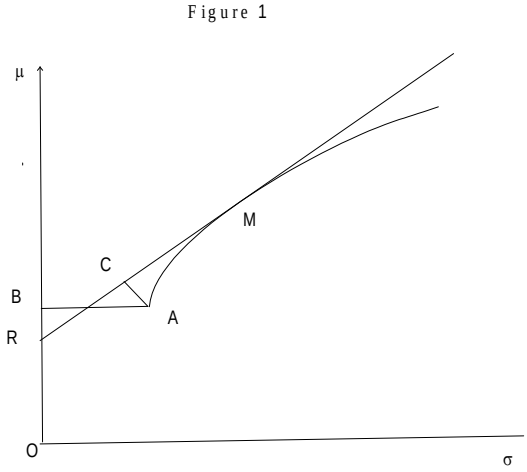

Figure 2: **Constructing efficient frontier is detrimental to investment performance** Points on DP are attainable combinations of risky assets which have the same risk, but higher expected return than P, the global minimum variance point with respect to a certain efficient frontier.

Figure 2